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Generalized Fibonacci Polynomials and Some Fundamental Properties

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Abstract. Various sequences of polynomials by the names of Fibonacci and Lucas polynomials occur in the literature over a century. Generalization of the Fibonacci polynomial has been done using various approaches. One usually found in the literature that the generalization is done by varying the initial conditions. In this paper we study the so-called generalized Fibonacci polynomials: $u_n(x) = xu_{n-1}(x) + u_{n-2}(x), n \geq 2$ with $u_0(x) = a$ and $u_1(x) = 2a+1$, where a is any integer. Further we give some fundamental properties about the generalized Fibonacci polynomials.

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1. Introduction

Various sequences of polynomials by the names of Fibonacci and Lucas polynomials occur in the literature over a century. Many works dealt with different properties of these polynomials and their applications. Fibonacci polynomials appear in different frameworks. Fibonacci polynomials are special cases of Chebyshev polynomials and have been studied on a more advanced level by many mathematicians. Fibonacci polynomials were studied in 1883 by the Belgian mathematician Eugene Charles Catalan and the German mathematician E. Jacobsthal. The polynomials studied by Catalan are defined by the recurrence relation

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x),\ n \geq 3 \text{ with } F_1(x) = 1,\ F_2(x) = x. \tag{1.1}$$

The Fibonacci polynomials studied by Jocobstral are defined by

$$J_n(x) = J_{n-1}(x) + xJ_{n-2}(x),\ n \ge 3 \text{ with } J_1(x) = 1,\ J_2(x) = 1. \quad (1.2)$$

He first gave the name "Fibonacci polynomials".

The Fibonacci polynomials studied by P. F. Byrd are defined by

$$\varphi_n(x) = 2x\varphi_{n-1}(x) + \varphi_{n-2}(x),\ n \ge 2 \quad \text{with } \varphi_0(x) = 0,,\ \varphi_1(x) = 1. \quad (1.3)$$

The Lucas polynomials originally studied in 1970 by Bicknell and they are defined by

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x),\ n \ge 2 \text{ with } L_0(x) = 2,\ L_1(x) = x \quad [4]. \quad (1.4)$$

Basin show that Q matrix generates a set of Fibonacci Polynomials satisfying the recurrence relation

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x),\ n \ge 2 \text{ with } f_0(x) = 0,\ f_1(x) = 1. \quad [18] \quad (1.5)$$

He derives the explicit forms and generating function by matrix method. Equation (1.5) is now the accepted form of Fibonacci polynomials. The first few polynomials of (1.5) are

$$f_1(x) = 1, \quad f_2(x) = x, \quad f_3(x) = x^2 + 1, \quad f_4(x) = x^3 + 2x,$$
$$f_5(x) = x^4 + 3x^2 + 1, \quad f_6(x) = x^5 + 4x^3 + 3x \textit{ and so on.}$$

The Fibonacci numbers are recovered by evaluating the polynomials at x=1

$f_n(1) = F_n$, where F_n is Fibonacci numbers (1.6)

Generating function of Fibonacci polynomials is

$$\sum_{n=0}^{\infty} f_n(x)t^n = t\left(1 - xt - t^2\right)^{-1}. \quad (1.7)$$

Explicit sum formula for Fibonacci polynomials is

$$f_n(x) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-k-1}{k} x^{n-1-2k}, \quad (1.8)$$

The Lucas polynomials [3] are defined by the recurrence formula:

$$l_{n+1}(x) = xl_n(x) + L_{n-1}(x),\ n \ge 2 \text{ with } l_0(x) = 2,\ l_1(x) = x \quad (1.9)$$

Generating function of Lucas polynomials is

$$\sum_{n=0}^{\infty} l_n(x)t^n = (2 - xt)\left(1 - xt - t^2\right)^{-1}. \quad (1.10)$$

Explicit sum formula for Lucas polynomials is given by

$$l_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k}, \quad (1.11)$$

where $\binom{n}{k}$ a binomial coefficient and $[x]$ is define as the greatest integer less than or equal to x.

The Fibonacci and Lucas polynomials possess many fascinating properties which have been studied in [2-8], [11], [13-19] and [21-22].

Generalized Fibonacci polynomials have been intensively studied for many years and have become an interesting topic in Applied Mathematics. Hoggatt and Bicknell introduced a generalized Fibonacci polynomials and their relationship to diagonals of Pascal's triangle [20]. Also after investigating the generalized Q-matrix, Ivie introduced a special case [12]. Nalli and Haukkanen introduced h(x) Fibonacci polynomials that generalize both Catalan's Fibonacci polynomials and Byrd's Fibonacci Polynomials and the k-Fibonacci number. Also they provided properties for these h(x) Fibonacci polynomials where h(x) is a polynomial with real coefficients [4]. In this paper we study some properties of generalized Fibonacci polynomials.

2. Generalized Fibonacci Polynomials

Generalization of the Fibonacci polynomial has been done using various approaches. Mainly, two certain recursive schemes are observed, which associated with generalization of Fibonacci polynomials: (1) By changing the recurrence relation while preserving the initial terms, (2) By changing the initial terms but recurrence relation is preserved. Generalized Fibonacci polynomials $u_n(x)$ are defined as

$u_n(x) = xu_{n-1}(x) + u_{n-2}(x),\ n \ge 2.$ with $u_0(x) = a$ and

$u_1(x) = 2a + 1$, where a is integer. (2.1)

The first few terms of (2.1) are as follows:

$u_0(x) = a,$

$u_1(x) = 2a+1,$

$u_2(x) = (2a+1)x + a,$

$u_3(x) = (2a+1)x^2 + ax + (2a+1),$

$u_4(x) = (2a+1)x^3 + ax^2 + 2(2a+1)x + a,$

$u_5(x) = (2a+1)x^4 + ax^3 + 3(2a+1)x^2 + 2ax + (2a+1),$ and so on.

The characteristic equation of (2.1) is $\lambda^2 - x\lambda - 1 = 0,$ which has two real roots

$\alpha = \dfrac{x+\sqrt{x^2+4}}{2}$ and $\beta = \dfrac{x-\sqrt{x^2+4}}{2}$

Also,

$$\alpha\beta = -1,\ \alpha+\beta = x,\ \alpha-\beta = \sqrt{x^2+4},\ \alpha^2+\beta^2 = x^2+2. \tag{2.2}$$

Binet's formula

$$u_n(x) = A\alpha^n + B\beta^n = A\left(\frac{x+\sqrt{x^2+4}}{2}\right)^n + B\left(\frac{x-\sqrt{x^2+4}}{2}\right)^n. \tag{2.3}$$

Here, $A = \dfrac{(2a+1)-a\beta}{\alpha-\beta}$ and $B = \dfrac{a\alpha-(2a+1)}{\alpha-\beta}.$

Also, $AB = \dfrac{-(a^2+3a+1)}{(\alpha-\beta)^2}$ and $A+B = u_0(x) = a.$

(2.4)

Generating function

$$\sum_{n=0}^{\infty} u_n(x)t^n = \frac{a+(2a+1-ax)t}{1-xt-t^2}. \tag{2.5}$$

Hypergeometric representation of generating function

By generating function (2.5), we have

$$\sum_{n=0}^{\infty} u_n(x)t^n = \frac{a+(2a+1-ax)t}{1-xt-t^2}$$

$$= [a+(2a+1-ax)t][1-(x+t)t]^{-1}$$

$$= [a+(2a+1-ax)t]\sum_{n=0}^{\infty}(x+t)^n t^n$$

$$= [a+(2a+1-ax)t]\sum_{n=0}^{\infty} t^n \sum_{k=0}^{n}\binom{n}{k} x^{n-k}t^k$$

$$= [a+(2a+1-ax)t]\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{n!}{k!\,n-k!}x^{n-k}t^{n+k}$$

$$= [a+(2a+1-ax)t]\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{n+k!}{k!\,n!}x^n t^{n+2k}$$

$$= [a+(2a+1-ax)t]\sum_{n=0}^{\infty}\frac{(xt)^n}{n!}\sum_{k=0}^{\infty}\frac{n+k!}{k!}t^{2k}$$

$$\sum_{n=0}^{\infty}\frac{u_n(x)}{n!}t^n = [a+(2a+1-ax)t]e^{xt}\sum_{k=0}^{\infty}\frac{n+k!}{k!}(t^2)^k$$

$$= [a+(2a+1-ax)t]e^{xt}\sum_{k=0}^{\infty}\frac{n+k!}{n!}\frac{(t^2)^k}{k!},$$

$$= [a+(2a+1-ax)t]e^{xt}\sum_{k=0}^{\infty}\frac{\overline{|n+k+1}}{\overline{|n+1}}\frac{(t^2)^k}{k!},$$

$$= [a+(2a+1-ax)t]e^{xt}\sum_{k=0}^{\infty}(n+1)_k\frac{(1)_k}{(1)_k}\frac{(t^2)^k}{k!}.$$

Hence,

$$\sum_{n=0}^{\infty} u_n(x)\frac{t^n}{n!} = [a+(2a+1-ax)t]e^{xt}\,{}_2F_1\left(n+1,\ 1;\ 1;\ t^2\right). \tag{2.6}$$

3. Some Fundamental Properties of Generalized Fibonacci Polynomials

In this section we present some fundamental properties of generalized Fibonacci polynomials which are described in [8] and [14]. Generating function, binet's formula and explicit sum formula have been used to establish the results.

Theorem 3.1. Prove that

$$u_{n+1}(x) - u_{n-1}(x) = xu_n(x),\ n \ge 1. \tag{3.1}$$

Proof. By generating function of generalized Fibonacci polynomials, we have

$$\sum_{n=0}^{\infty} u_n(x)t^n = [a+(2a+1-ax)t]\left(1-xt-t^2\right)^{-1}.$$

Differentiating both sides with respect to t, we get

$$\sum_{n=0}^{\infty} nu_n(x)t^{n-1}$$

$$= [a+(2a+1-ax)t](x+2t)\left(1-xt-t^2\right)^{-2}$$

$$+(2a+1-ax)\left(1-xt-t^2\right)^{-1}.$$

$$\left(1-xt-t^2\right)\sum_{n=0}^{\infty} nu_n(x)t^{n-1}$$
$$=\left[a+(2a+1-ax)t\right]\left(x+2t\right)\left(1-xt-t^2\right)^{-1}+(2a+1-ax),$$
$$=\left(x+2t\right)\sum_{n=0}^{\infty} u_n(x)t^n+(2a+1-ax),$$

$$\sum_{n=0}^{\infty} nu_n(x)t^{n-1}-\sum_{n=0}^{\infty} nxu_n(x)t^n-\sum_{n=0}^{\infty} nu_n(x)t^{n+1}$$
$$=\sum_{n=0}^{\infty} xu_n(x)t^n-\sum_{n=0}^{\infty} 2u_n(x)t^{n+1}+(2a+1-ax).$$

Now equating the coefficient of t^n on both sides we get,

$$(n+1)u_{n+1}(x)-nxu_n(x)-(n-1)u_{n-1}(x)=xu_n(x)+2u_{n-1}(x),$$
$$(n+1)u_{n+1}(x)-(n+1)u_{n-1}(x)=(n+1)xu_n(x),$$
$$u_{n+1}(x)-u_{n-1}(x)=xu_n(x).$$

Theorem 3.2. Prove that

$$u'_{n+1}(x)=xu'_n(x)+u_n(x)+u'_{n-1}(x),\ n\geq 1. \tag{3.2}$$

Proof. By (3.1), we have

$$u_{n+1}(x)-u_{n-1}(x)=xu_n(x),\ n\geq 1.$$

Differentiating both sides with respect to x, obtain

$$u'_{n+1}(x)-u'_{n-1}(x)=xu'_n(x)+u_n(x),$$
$$u'_{n+1}(x)=xu'_n(x)+u_n(x)+u'_{n-1}(x).$$

Theorem 3.3. Prove that

$nu_n(x)=xu'_n(x)-2u'_{n-1}(x),\ n\geq 1$ and

$xu'_{n+1}(x)=(n+1)u_{n+1}(x)-2u'_n(x),\ n\geq 1.$

Proof. By generating function of generalized Fibonacci-Like polynomials, we have

$$\sum_{n=0}^{\infty} u_n(x)t^n=\left[a+(2a+1-ax)t\right]\left(1-xt-t^2\right)^{-1}.$$

Differentiating both sides with respect to t, we get

$$\sum_{n=0}^{\infty} nu_n(x)t^{n-1}$$
$$=(2a+1-ax)\left(1-xt-t^2\right)^{-1} \tag{3.3}$$
$$+\left[a+(2a+1-ax)t\right]\left(x+2t\right)\left(1-xt-t^2\right)^{-2}.$$

Differentiating both sides with respect to x, we get

$$\sum_{n=0}^{\infty} u'_n(x)t^n=\left[a+(2a+1-ax)t\right]\left(1-xt-t^2\right)^{-2}t-at\left(1-xt-t^2\right)^{-1},$$

$$\sum_{n=0}^{\infty} u'_n(x)t^{n-1}=\left[a+(2a+1-ax)t\right]\left(1-xt-t^2\right)^{-2}-a\left(1-xt-t^2\right)^{-1},$$

$$\sum_{n=0}^{\infty} u'_n(x)t^{n-1}+a\left(1-xt-t^2\right)^{-1}=\left[a+(2a+1-ax)t\right]\left(1-xt-t^2\right)^{-2}. \tag{3.4}$$

Using (3.4) in (3.3), we get

$$\sum_{n=0}^{\infty} nu_n(x)t^{n-1}$$
$$=(2a+1-ax)\left(1-xt-t^2\right)^{-1}$$
$$+\left(x+2t\right)\left\{\sum_{n=0}^{\infty} u'_n(x)t^{n-1}+a(1-xt-t^2)^{-1}\right\},$$
$$=(2a+1-ax)\left(1-xt-t^2\right)^{-1}$$
$$+\left(x+2t\right)\sum_{n=0}^{\infty} u'_n(x)t^{n-1}+a\left(x+2t\right)(1-xt-t^2)^{-1}.$$

Now equating the coefficient of t^{n-1} on both sides, we get

$$nu_n(x)=xu'_n(x)+2u'_{n-1}(x). \tag{3.5}$$

Again equating the coefficient of t^n on both sides, we get

$$(n+1)u_{n+1}(x)=xu'_{n+1}(x)+2u'_n(x),$$
$$xu'_{n+1}(x)=(n+1)u_{n+1}(x)-2u'_n(x). \tag{3.6}$$

Theorem 3.4. Prove that

$(n+1)u_n(x)=u'_{n+1}(x)+u'_{n-1}(x),\ n\geq 1.$

Proof. By (3.1), we have

$u_{n+1}(x)-u_{n-1}(x)=xu_n(x),\ n\geq 1.$

Differentiating both sides with respect to x, we get

$$u'_{n+1}(x)-u'_{n-1}(x)=xu'_n(x)+u_n(x),$$

$$xu'_n(x)+u_n(x)=u'_{n+1}(x)-u'_{n-1}(x). \qquad (3.7)$$

Using (3.5) in (3.7), we get

$$nu_n(x)-2u'_{n-1}(x)+u_n(x)=u'_{n+1}(x)-u'_{n-1}(x).$$

$$nu_n(x)+u_n(x)=u'_{n+1}(x)+2u'_{n-1}(x)-u'_{n-1}(x),$$

$$(n+1)u_n(x)=u'_{n+1}(x)+u'_{n-1}(x). \qquad (3.8)$$

Theorem 3.5. Prove that

$$xu'_n(x)=2u'_{n+1}(x)-(n+2)u_n(x),\ n\geq 0.$$

Proof. Using (3.5) in (3.8), we get

$$(n+1)u_n(x)=u'_{n+1}(x)+\frac{1}{2}\left[nu_n(x)-xu'_n(x)\right],$$

$$2(n+1)u_n(x)=2u'_{n+1}(x)+\left[nu_n(x)-xu'_n(x)\right],$$

$$xu'_n(x)=2u'_{n+1}(x)+nu_n(x)-(2n+2)u_n(x),$$

$$xu'_n(x)=2u'_{n+1}(x)+(n-2n-2)u_n(x), \qquad (3.9)$$

Theorem 3.6. Prove that

$$(n+1)xu'_n(x)=nu'_{n+1}(x)-(n+2)u'_{n-1}(x),\ n\geq 1.$$

Proof. Using (3.8) in (3.2), we get

$$(n+1)\left\{u'_{n+1}(x)-xu'_n(x)-u'_{n-1}(x)\right\}=u'_{n+1}(x)+u'_{n-1}(x),$$

$$(n+1)u'_{n+1}(x)-(n+1)xu'_n(x)-(n+1)u'_{n-1}(x)$$
$$=u'_{n+1}(x)+u'_{n-1}(x),$$

$$(n+1)u'_{n+1}(x)-(n+1)u'_{n-1}(x)-u'_{n+1}(x)-u'_{n-1}(x)$$
$$=(n+1)xu'_n(x),$$

$$nu'_{n+1}(x)-(n+2)u'_{n-1}(x)=(n+1)xu'_n(x),$$

$$(n+1)xu'_n(x)=nu'_{n+1}(x)-(n+2)u'_{n-1}(x). \qquad (3.10)$$

Theorem 3.7. (Explicit Sum Formula) Prove that

$$u_n(x)=a\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-k}{k}x^{n-2k}. \qquad (3.11)$$

Proof. By generating function (2.5), we have

$$\sum_{n=0}^{\infty}u_n(x)t^n=\left[a+(2a+1-ax)t\right]\left(1-xt-t^2\right)^{-1}\sqrt{b^2-4ac}$$

$$=\left[a+(2a+1-ax)t\right]\left[1-(x+t)t\right]^{-1}$$

$$=\left[a+(2a+1-ax)t\right]\sum_{n=0}^{\infty}(x+t)^n t^n$$

$$=\left[a+(2a+1-ax)t\right]\sum_{n=0}^{\infty}t^n\sum_{k=0}^{n}\binom{n}{k}x^{n-k}t^k$$

$$\sum_{n=0}^{\infty}u_n(x)t^n=\left[a+(2a+1-ax)t\right]\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{n!}{k!n-k!}x^{n-k}t^{n+k}$$

$$=\left[a+(2a+1-ax)t\right]\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{n+k!}{k!n!}x^n t^{n+2k}$$

$$=\left[a+(2a+1-ax)t\right]\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n-k!}{k!n-2k!}x^{n-2k}t^n.$$

Equating coefficients of t^n on both sides, we get

$$u_n(x)=a\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-k}{k}x^{n-2k}.$$

Theorem 3.8. For positive integer $n\geq 0$, prove that

$$u_n(x)=ax^n\ {}_2F_1\left(\frac{-n}{2},\ \frac{-n+1}{2};\ -n;\ \frac{-4}{x^2}\right). \qquad (3.12)$$

Proof. By explicit sum formula (3.11), it follows that

$$u_n(x)=ax^n\sum_{k=0}^{[n/2]}\frac{n-k!}{k!n-2k!}x^{-2k}$$

$$=ax^n\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k(1)_n(-n)_{2k}}{(-n)_k(-1)^{2k}(1)_n}\frac{x^{-2k}}{k!}$$

$$=ax^n\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k 2^{2k}\left(\frac{-n}{2}\right)_k\left(\frac{-n+1}{2}\right)_k}{(-n)_k(-1)^{2k}}\frac{x^{-2k}}{k!}$$

$$= ax^n \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{\left(\frac{-n}{2}\right)_k \left(\frac{-n+1}{2}\right)_k}{(-n)_k} \frac{\left(\frac{-4}{x^2}\right)^k}{k!}.$$

Hence, $u_n(x) = ax^n \, {}_2F_1\left(\frac{-n}{2}, \frac{-n+1}{2}; -n; \frac{-4}{x^2}\right)$.

Theorem 3.9. For positive integer $n \ge 0$, prove that

$$\sum_{n=0}^{\infty} (c)_n u_n(x) \frac{t^n}{n!}$$
$$= a(1-xt)^{-c} \, {}_3F_2\left(\frac{c}{2}, \frac{c+1}{2}, n+1; \frac{n+1}{2}; \frac{n+2}{2}; \frac{t^2}{(1-xt)^2}\right). \quad (3.13)$$

Proof. Multiplying both sides of the explicit sum formula by $(c)_n \frac{t^n}{n!}$ and summing between the limit $n=0$ to $n=\infty$, we obtain

$$\sum_{n=0}^{\infty} (c)_n u_n(x) \frac{t^n}{n!} = a \sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n-k!}{k!\,n-2k!} (c)_n x^{n-2k} \frac{t^n}{n!}$$

$$= a \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{n+k!}{k!\,n!\,n+2k!} (c)_{n+2k} x^n t^{n+2k}$$

$$= a \left\{ \sum_{n=0}^{\infty} (c+2k)_n \frac{(xt)^n}{n!} \right\} \sum_{k=0}^{\infty} \frac{n+k!}{k!\,n+2k!} (c)_{2k} t^{2k},$$

$$\sum_{n=0}^{\infty} (c)_n u_n(x) \frac{t^n}{n!}$$
$$= a \sum_{k=0}^{\infty} (1-xt)^{-(c+2k)} \frac{n+k!}{k!\,n+2k!} (c)_{2k} t^{2k}$$

$$= a(1-xt)^{-c} \sum_{k=0}^{\infty} \frac{n+k!}{k!\,n+2k!} (c)_{2k} \left[\frac{t^2}{(1-xt)^2}\right]^k$$

$$= a(1-xt)^{-c} \sum_{k=0}^{\infty} \frac{n+k!}{n+2k!} 2^{2k} \left(\frac{c}{2}\right)_k \left(\frac{c+1}{2}\right)_k \left[\frac{t^2}{(1-xt)^2}\right]^k \Big/ k!$$

$$= a(1-xt)^{-c} \sum_{k=0}^{\infty} \frac{(n+1)_k}{(n+1)_{2k}} 2^{2k} \left(\frac{c}{2}\right)_k \left(\frac{c+1}{2}\right)_k \left[\frac{t^2}{(1-xt)^2}\right]^k \Big/ k!,$$

$$= a(1-xt)^{-c} \sum_{k=0}^{\infty} \frac{\left(\frac{c}{2}\right)_k \left(\frac{c+1}{2}\right)_k (n+1)_k}{\left(\frac{n+1}{2}\right)_k \left(\frac{n+2}{2}\right)_k} \left[\frac{t^2}{(1-xt)^2}\right]^k \Big/ k!.$$

Hence,

$$\sum_{n=0}^{\infty} (c)_n u_n(x) \frac{t^n}{n!}$$
$$= a(1-xt)^{-c} \, {}_3F_2\left(\frac{c}{2}, \frac{c+1}{2}, n+1; \frac{n+1}{2}; \frac{n+2}{2}; \frac{t^2}{(1-xt)^2}\right).$$

Theorem 3.10. (Catalan's Identity) Let $u_n(x)$ be the n^th^ term of generalized Fibonacci-Like polynomials, then

$$u_n^2(x) - u_{n+r}(x)u_{n-r}(x)$$
$$= \frac{(-1)^{n-r}}{a^2+3a+1}\left[(2a+1)u_r(x) - au_{r+1}(x)\right],\ n > r \ge 1. \quad (3.14)$$

Proof. Using Binet's formula (2.5), we have

$$u_n^2(x) - u_{n+r}(x)u_{n-r}(x)$$
$$= (A\alpha^n + B\beta^n)^2 - (A\alpha^{n+r} + B\beta^{n+r})(A\alpha^{n-r} + B\beta^{n-r})$$
$$= AB(\alpha\beta)^n \left(2 - \alpha^r\beta^{-r} - \alpha^{-r}\beta^r\right)$$
$$= -AB(-1)^{n-r}\left(\alpha^r - \beta^r\right)^2$$
$$= \frac{(a^2+3a+1)}{(\alpha-\beta)^2}(-1)^{n-r}\left(\alpha^r - \beta^r\right)^2$$
$$= (a^2+3a+1)(-1)^{n-r}\frac{\left(\alpha^r - \beta^r\right)^2}{(\alpha-\beta)^2}.$$

Since

$$\frac{\alpha^r - \beta^r}{\alpha - \beta} = \frac{(2a+1)u_r(x) - au_{r+1}(x)}{(2a+1)^2 - a(2a+1) - a^2} = \frac{(2a+1)u_r(x) - au_{r+1}(x)}{(a^2+3a+1)}.$$

$$u_n^2(x) - u_{n+r}(x)u_{n-r}(x)$$
$$= \frac{(-1)^{n-r}}{(a^2+3a+1)}\left[(2a+1)u_r(x) - au_{r+1}(x)\right]^2,\ n > r \ge 1.$$

Theorem 3.11. (Cassini's Identity) Let $u_n(x)$ be the n[th] term of generalized Fibonacci polynomials, then

$$u_n^2(x)-u_{n+1}(x)u_{n-1}(x)=(-1)^{n-1}(a^2+3a+1),\ n\geq 1. \tag{3.15}$$

Proof. If r = 1 in the Catalan's Identity, then obtained required result.

Theorem 3.12. (d'Ocagne's Identity) Let $u_n(x)$ be the n[th] term of generalized Fibonacci polynomials, then

$$u_m(x)u_{n+1}(x)-u_{m+1}(x)u_n(x)$$
$$=(-1)^n\left[\frac{(2a+1)u_{m-n}(x)-au_{m-n+1}(x)}{a^2+3a+1}\right],\ m\geq 1, n\geq 0, m>n. \tag{3.16}$$

Proof. Using Binet's formula (2.5), we have

$$u_m(x)u_{n+1}(x)-u_{m+1}(x)u_n(x)$$
$$=(A\alpha^n+B\beta^m)(A\alpha^{n+1}+B\beta^{n+1})-(A\alpha^{m+1}+B\beta^{m+1})(A\alpha^n+B\beta^n)$$
$$=AB\left(\alpha^m\beta^{n+1}+\alpha^{n+1}\beta^m-\alpha^n\beta^{m+1}-\alpha^{m+1}\beta^n\right)$$
$$=AB(\alpha\beta)^n\left[\beta\left(\alpha^{m-n}-\beta^{m-n}\right)\alpha\left(\alpha^{m-n}-\beta^{m-n}\right)\right]$$
$$=AB(-1)^n\left(\beta-\alpha\right)\left(\alpha^{m-n}-\beta^{m-n}\right)$$
$$=\frac{(a^2+3a+1)}{\left(\alpha-\beta\right)^2}(-1)^n\left(\alpha-\beta\right)\left(\alpha^{m-n}-\beta^{m-n}\right)$$
$$=(a^2+3a+1)(-1)^n\frac{\left(\alpha^{m-n}-\beta^{m-n}\right)}{\left(\alpha-\beta\right)}.$$

Since,

$$\frac{\alpha^{m-n}-\beta^{m-n}}{\alpha-\beta}$$
$$=\frac{(2a+1)u_{m-n}(x)-au_{m-n+1}(x)}{\left(2a+1\right)^2-a(2a+1)-a^2}=\frac{(2a+1)u_{m-n}(x)-au_{m-n+1}(x)}{(a^2+3a+1)},$$

we obtain

$$u_m(x)u_{n+1}(x)-u_{m+1}(x)u_n(x)$$
$$=(-1)^n\left[\frac{(2a+1)u_{m-n}(x)-au_{m-n+1}(x)}{a^2+3a+1}\right],\ m\geq 1, n\geq 0, m>n.$$

Theorem 3.13. (Generalized Identity) Let $u_n(x)$ be n[th] term of generalized Fibonacci polynomials, then

$$u_m(x)u_n(x)-u_{m-r}(x)u_{n+r}(x)$$
$$=(a^2+3a+1)\left(-1\right)^{m-r}\times$$
$$\left[(2a+1)u_r(x)-au_{r+1}(x)\right]\left[(2a+1)u_{n-m+r}(x)-au_{n-m+r+1}(x)\right],$$
$$n>m\geq r\geq 1 \tag{3.17}$$

$$u_m(x)u_n(x)-u_{m-r}(x)u_{n+r}(x)$$
$$=(A\alpha^m+B\beta^m)(A\alpha^n+B\beta^n)-(A\alpha^{m-r}+B\beta^{m-r})(A\alpha^{n+r}+B\beta^{n+r})$$
$$=AB(\alpha^r-\beta^r)\left[\frac{\alpha^m\beta^n}{\alpha^r}-\frac{\alpha^n\beta^m}{\beta^r}\right]$$
$$=AB\left(-1\right)^{-r}(\alpha^r-\beta^r)(\alpha^m\beta^{n+r}-\alpha^{n+r}\beta^m)$$
$$=AB\left(-1\right)^{-r}(\alpha^m\beta^m)(\alpha^r-\beta^r)(\beta^{n-p+r}-\alpha^{n-p+r})$$
$$=-AB\left(-1\right)^{-r}(\alpha^m\beta^m)(\alpha^r-\beta^r)(\alpha^{n-p+r}-\beta^{n-p+r})$$
$$=\frac{(a^2+3a+1)}{(\alpha-\beta)^2}\left(-1\right)^{-r}(\alpha^m\beta^m)(\alpha^r-\beta^r)(\alpha^{n-p+r}-\beta^{n-p+r}).$$

Proof. Using Binet's formula (2.5), we have

Using subsequent results of Binet's formula, we get

Since, $\dfrac{\alpha^r-\beta^r}{\alpha-\beta}=\dfrac{(2a+1)u_r(x)-au_{r+1}(x)}{(a^2+3a+1)}$ and

$$\frac{\alpha^{n-m+r}-\beta^{n-m+r}}{\alpha-\beta}=\frac{(2a+1)u_{n-m+r}(x)-au_{n-m+r+1}(x)}{(a^2+3a+1)}.$$

$$u_m(x)u_n(x)-u_{m-r}(x)u_{n+r}(x)$$
$$=(a^2+3a+1)\left(-1\right)^{m-r}\times$$
$$\left[(2a+1)u_r(x)-au_{r+1}(x)\right]\left[(2a+1)u_{n-m+r}(x)-au_{n-m+r+1}(x)\right],$$
$$n>m\geq r\geq 1$$

The identity (3.13) provides Catalan's identity, Cassini's and d'Ocagne and other identities.

4. Conclusion

In this paper, we have introduced generalized Fibonacci polynomials by variation of initial conditions and presented some basic results. Finally explain some fundamental properties with derivations by standard methods.

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