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# **Generalized Fibonacci-Like Sequence and Some Identities**

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# Abstract

Sequences have been fascinating topic for mathematicians for centuries. The Fibonacci and Lucas sequences are examples of second order recursive sequences. Fibonacci sequence is defined by  $F_n = F_{n-1} + F_{n-2}$ ,  $n \ge 2$ , with  $F_0 = 0$ ,  $F_1 = 1$ . In recent years, few research scholars have been introduced Fibonacci-Like sequences which are similar to Fibonacci sequences in recurrence relation, but initial conditions are different. Due to this reason, these are known as Fibonacci-Like sequences. In this paper, we study a Generalized Fibonacci-Like sequence  $R_n = R_{n-1} + R_{n-2}$ ,  $n \ge 2$  with initial condition  $R_0=2b$  and  $R_1=a+b$ , where a and b are non-zero real numbers. Some identities are established by Binet's formula and generating function. Further, present connection formulae and some determinant identities.

**Keywords:** Fibonacci sequence; Lucas sequence; Fibonacci-Like sequence; Generalized Fibonacci-Like sequence.

#### 1. Introduction.

In modern science there is a huge interest in the theory and application of the Fibonacci numbers. The Fibonacci numbers  $F_n$  are the terms of the sequence 0, 1, 1, 2, 3, 5, ..., where the sequence of Fibonacci numbers  $F_n$  [1,2, 8] is define by

$$F_n = F_{n-1} + F_{n-2}, n \ge 2$$
 with the initial values  $F_0 = 0, F_1 = 1$  (1.1)

Generalized Fibonacci sequences have been intensively studied for many years and have become an interesting topic in Applied Mathematics. Fibonacci sequences and their related higher-order sequences are generally studied as sequence of integer. The sequence of Lucas numbers  $L_n$  [1, 2, 8] is defined by recurrence relation:

$$L_n = L_{n-1} + L_{n-2}, n \ge 2, \quad L_0 = 2, L_1 = 1$$
(1.2)

Few research scholars have been introduced Fibonacci-Like sequences which are similar to Fibonacci sequences in recurrence relation, but initial conditions are different. Due to this reason, these are known as Fibonacci-Like sequences. In 2010, Fibonacci-like sequence [3] is defined by recurrence relation:

$$S_n = S_{n-1} + S_{n-2}, n \ge 2$$
,  $S_0 = 2, S_1 = 2$ . (1.3)

The associated initial condition  $S_0$  and  $S_1$  are the sum of initial condition of Fibonacci and Lucas sequence respectively, i.e.  $S_0 = F_0 + L_0$  and  $S_1 = F_1 + L_1$ .

In 2013, Fibonacci-Like sequence [4] is defined by the recurrence relation

$$H_n = 2H_{n-1} + H_{n-2}$$
,  $n \ge 2$  with  $H_0 = 2$ ,  $H_1 = 1$ . (1.4)

The associated initial conditions  $H_0$  and  $H_1$  are the subtraction of initial conditions of Pell-Lucas and Pell sequence respectively.

In this paper, we study a Generalized Fibonacci-Like sequence and some identities. The Binet's formula and generating function are mainly use to derive identities of Generalized Fibonacci-Like sequence. Also we present Connection formulae and determinants identities which contain different terms of sequence.

#### 2. Generalized Fibonacci-Like Sequence.

Generalized Fibonacci-Like sequence is defined by the recurrence relation

$$R_n = R_{n-1} + R_{n-2}, n \ge 2$$
, with initial conditions  $R_0 = 2b, R_1 = a + b$  (2.1)  
where a and b are non zero real numbers.

Here the initial condition  $R_0$  and  $R_1$  are the sum of initial condition of Fibonacci sequence 'a' times and initial condition of Lucas sequence 'b' times respectively, i.e.  $R_0 = aF_0 + bL_0$  and

$$R_1 = aF_1 + bL_1$$

The few terms are as follows

 $R_0 = 2b,$   $R_1 = a + b,$   $R_2 = a + 3b,$   $R_3 = 2a + 4b,$   $R_4 = 3a + 7b,$  $R_5 = 5a + 11b, \dots$  and so on.

If a=b=1, then it becomes Conventional Fibonacci-Like sequence 2,2,4,6,10,16,...

If a=2, b=1, then it becomes Fibonacci sequence 2, 3, 5, 8, 13,... for  $n\geq 3$ .

The relation between Fibonacci sequence, Lucas sequence and Generalized Fibonacci-like sequence can be written as

$$R_n = aF_n + bL_n, n \ge 0.$$

The recurrence relation (2.1) has the characteristic equation

$$x^2 - x - 1 = 0 , (2.2)$$

which has two roots

$$\alpha = \frac{1+\sqrt{5}}{2}$$
 and  $\beta = \frac{1-\sqrt{5}}{2}$ , also  $\alpha\beta = -1, \alpha + \beta = 1, \alpha - \beta = \sqrt{5}, \alpha^2 + \beta^2 = 3.$ 

Generating function of generalized Fibonacci-like sequence is

$$\sum_{n=0}^{\infty} R_n x^n = r(x) = \frac{2b + (a-b)x}{(1-x-x^2)}$$
(2.3)

Binet's formula of generalized Fibonacci-like sequence is defined by

$$R_{n} = C_{1}\alpha^{n} + C_{2}\beta^{n} = C_{1}\left(\frac{1+\sqrt{5}}{2}\right)^{n} + C_{2}\left(\frac{1-\sqrt{5}}{2}\right)$$
(2.4)

Here, 
$$C_1 = \frac{a + b\sqrt{5}}{\sqrt{5}}$$
 and  $C_2 = \frac{b\sqrt{5} - a}{\sqrt{5}}$ , also  $C_1 + C_2 = R_0 = 2b$ ,  $C_1C_2 = \frac{5b^2 - a^2}{(\alpha - \beta)^2} = \frac{5b^2 - a^2}{5}$ 

## 3. Some Identities of Generalized Fibonacci-Like Sequence.

In this section, some identities of Generalized Fibonacci-Like sequence are presented and derived by Binet's formula and generating function.

**Theorem 3.1 (Explicit sum formula)** Let  $R_n$  be the  $n^{th}$  term of the generalized Fibonaccilike sequence. Then

$$R_{n} = 2b\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} {\binom{n-k}{k}} + (a-b)\sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} {\binom{n-k-1}{k}}$$
(3.1)

**Proof:-** By generating function (2.3), we have

$$\sum_{n=0}^{\infty} R_n x^n = R(x) = \frac{2b + (a-b)x}{1 - x - x^2} = \{2b + (a-b)x\}(1 - x - x^2)^{-1}$$
$$= \{2b + (a-b)x\}[1 - (x + x^2)]^{-1}$$
$$= \{2b + (a-b)x\}\sum_{n=0}^{\infty} (x + x^2)^n = \{2b + (a-b)x\}\sum_{n=0}^{\infty} x^n(1 + x^n)$$
$$= \{2b + (a-b)x\}\sum_{n=0}^{\infty} x^n \sum_{k=0}^n \binom{n}{k} x^k$$
$$= \{2b + (a-b)x\}\sum_{n=0}^{\infty} \sum_{k=0}^n \frac{|n|}{|k|n-k|} x^{n+k}$$

(Replacing n by n+k)

$$= \{2b + (a-b)x\} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{|n+k|}{|k|n|} x^{n+2k}$$

(Replacing n by n-2k)

$$= \{2b + (a-b)x\} \sum_{n=0}^{\infty} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{|n-k|}{|k|n-2k|} x^{n}$$
$$= 2b \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{|n-k|}{|k|n-2k|} \right\} x^{n} + (a-b) \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{|n-k|}{|k|n-2k|} \right\} x^{n+1}$$

Equating the coefficient of  $x^n$  we obtain

$$R_n = 2b\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} {\binom{n-k}{k}} + (a-b)\sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} {\binom{n-k-1}{k}}$$

For a=b=1 in above identity, explicit formula can be obtained for Fibonacci sequence.

Theorem 3.2 Sum of the first n terms of the generalized Fibonacci-like sequence is

$$\sum_{k=1}^{n} R_k = R_1 + R_2 + R_3 + \dots + R_n = R_{n+2} - (a+3b)$$
(3.2)

**Proof:-**By Binet's formula (2.4),we have

$$\sum_{k=1}^{n} R_{k} = \sum_{k=1}^{n} [C_{1}\alpha^{k} + C_{2}\beta^{k}]$$

$$= C_{1}\alpha \left(\frac{1-\alpha^{n}}{1-\alpha}\right) + C_{2}\beta \left(\frac{1-\beta^{n}}{1-\beta}\right)$$

$$= \left[\frac{(C_{1}\alpha + C_{2}\beta) - (C_{1} + C_{2})\alpha\beta - (C_{1}\alpha^{n+1} + C_{2}\beta^{n+1}) + (C_{1}\alpha^{n} + C_{2}\beta^{n})\alpha\beta}{1-(\alpha+\beta) + \alpha\beta}\right]$$

Using subsequent result of Binet's formula, we get

$$\sum_{k=1}^{n} R_k = R_{n+1} + R_n - (a+3b) = R_{n+2} - (a+3b)$$
  
This identities becomes

$$R_1 + R_2 + R_3 + \dots + R_{2n} = R_{2n+2} - (a+3b)$$
(3.3)

Theorem 3.3 Sum of the first n terms with odd indices is

$$\sum_{k=1}^{n} R_{2k-1} = R_1 + R_3 + R_5 + \dots + R_{2n-1} = R_{2n} - 2b$$
(3.4)

Theorem 3.4 Sum of the first n terms with even indices is

$$\sum_{k=1}^{n} R_{2k} = R_2 + R_4 + R_6 + \dots + R_{2n} = R_{2n+1} - (a+b)$$
(3.5)

If we subtract equation (3.5) term wise from equation (3.4), we get alternative sum of first n numbers

$$R_{1} - R_{2} + R_{3} - R_{4} + R_{5} - \dots + R_{2n-1} - R_{2n} = R_{2n} - 2b - R_{2n+1} + (a+b)$$
$$= -R_{2n-1} + (a-b)$$
(3.6)

Adding  $R_{2n+1}$  to both side of equation (3.5)

$$R_{1} - R_{2} + R_{3} - R_{4} + R_{5} - \dots + R_{2n-1} - R_{2n} + R_{2n+1} = -R_{2n-1} + (a-b) + R_{2n+1}$$
$$= R_{2n} + (a-b)$$
(3.7)

**Theorem 3.5** Sum of square of first n terms of the generalized Fibonacci-like sequence  $\{R_n\}$  is

$$R_1^2 + R_2^2 + R_3^2 + \dots + R_n^2 = R_n R_{n+1} - 2b(a+b)$$
(3.8)

Theorem 3.6 For every positive integer n

$$R_3 + R_6 + R_9 + \dots + R_{3n} = \frac{1}{2} [R_{3n+2} - (a+3b)]$$
(3.9)

Theorem 3.7 For every positive integer n

$$R_5 + R_8 + R_{11} + \dots + R_{3n+2} = \frac{1}{2} [R_{3n+4} - (3a+7b)]$$
(3.10)

Theorem 3.8 For every positive integer n, then

$$R_n^2 = R_n R_{n+1} - R_{n-1} R_n \tag{3.11}$$

Theorem 3.9 For every positive integer n, then

$$R_{2n} = \sum_{k=0}^{n} \binom{n}{k} R_{n-k}$$
(3.12)

**Theorem 3.10 (Catalan's Identity)** Let  $R_n$  be  $n^{th}$  term of the Generalized Fibonacci-like sequence, then

$$R_n^2 - R_{n+r}R_{n-r} = \frac{(-1)^{n-r}}{a^2 - 5b^2} \left[ (a+b)R_r - 2bR_{r+1} \right]^2, n > r \ge 1.$$
(3.13)

**Proof:-**By Binet's formula (2.4), we have

$$\begin{aligned} R_n^2 - R_{n+r} R_{n-r} &= (C_1 \alpha^n + C_2 \beta^n)^2 - (C_1 \alpha^{n+r} + C_2 \beta^{n+r})(C_1 \alpha^{n-r} + C_2 \beta^{n-r}) \\ &= C_1 C_2 (\alpha \beta)^n (2 - \alpha^r \beta^{-r} - \alpha^{-r} \beta^r) \\ &= C_1 C_2 (\alpha \beta)^{n-r} (2\alpha^r \beta^r - \alpha^{2r} - \beta^{2r}) \\ &= -C_1 C_2 (\alpha \beta)^{n-r} (\alpha^r - \beta^r)^2 \end{aligned}$$

Using subsequent results of Binet's formula, we get

$$R_{n}^{2} - R_{n+r}R_{n-r} = (a^{2} - 5b^{2})(-1)^{n-r} \left(\frac{\alpha^{r} - \beta^{r}}{\alpha - \beta}\right)$$

Since 
$$\frac{\alpha^r - \beta^r}{\alpha - \beta} = \frac{(a+b)R_r - 2bR_{r+1}}{2aR_1 - (a+b)^2 - R_0^2} = \frac{(a+b)R_r - 2bR_{r+1}}{a^2 - 5b^2}$$

We obtain

$$R_n^2 - R_{n+r}R_{n-r} = \frac{(-1)^{n-r}}{a^2 - 5b^2} [(a+b)R_r - 2bR_{r+1}]^2, n > r \ge 1.$$

**Corrollary (3.10.1) (Cassini's Identity)** Let  $R_n$  be  $n^{th}$  term of the Generalized Fibonacci-like sequence, then

$$R_n^2 - R_{n+1}R_{n-1} = (-1)^{n-r}(a^2 - 5b^2), n \ge 1.$$
(3.14)

Taking r=1 in the catalan's identity (3.13), the required identity is obtained.

**Theorem 3.11 (D'Ocagne's Identity)** Let  $R_n$  be  $n^{th}$  term of the generalized Fibonacci-like sequence, then

$$R_m R_{n+1} - R_{m+1} R_n = (-1)^n (a+b) R_{m-n} - 2b R_{m-n+1}, m > n \ge 0.$$
(3.15)

**Theorem 3.12 (Generalized Identity)** Let  $R_n$  be the  $n^{th}$  term of the generalized Fibonaccilike sequence. Then

$$R_{m}R_{n} - R_{m-r}R_{n+r} = \frac{(-1)^{m-r}}{a^{2} - 5b^{2}} \{(a+b)R_{r} - 2bR_{r+1}\} \{(a+b)R_{n-m+r} - 2bR_{n-m+r+1}\}, n > m \ge r \ge 1$$

(3.16)

## 4. Connection Formulae.

In this section, connection formulae of Generalized Fibonacci-Like sequence, Fibonacci and Lucas sequences are presented.

**Theorem 4.1** Let n be a positive integer, then

$$R_{n+1} + R_{n-1} = (a+b)L_n + 2bL_{n-1}$$
(4.1)

**Proof:-** We shall prove this identity by induction over n,

For n=1, we have

$$R_{2} + R_{0} = (a+b)L_{1} + 2bL_{2}$$
  

$$a + 3b + 2b = (a+b)1 + 2b * 2$$
  

$$a + 5b = a + 5b$$

Suppose that the identity holds for n=k-2 and n=k-1, then

$$R_{k-1} + R_{k-3} = (a+b)L_{k-2} + 2bL_{k-3}$$
$$R_k + R_{k-2} = (a+b)L_{k-1} + 2bL_{k-2}$$

Adding both equations, we get

$$(R_{k-1} + R_k) + (R_{k-3} + R_{k-2}) = (a+b)(L_{k-2} + L_{k-1}) + 2b(L_{k-3} + L_{k-2})$$
$$R_{k+1} + R_{k-1} = (a+b)L_k + 2bL_{k-1}$$

Which is precisely our identity when n=k.

Hence,  $R_{n+1} + R_{n-1} = (a+b)L_n + 2bL_{n-1}$ 

**Theorem 4.2** Let n be a positive integer, then

$$R_{n+1} - R_{n-1} = (a+b)F_n + 2bF_{n-1} \qquad n \ge 1$$
(4.2)

**Theorem 4.3** For every integer  $n \ge 0$ ,

$$R_{n+1} = aF_{n+1} + bL_{n+1} \tag{4.3}$$

**Theorem 4.4** For every integer  $n \ge 0$ ,

$$R_{2n} = aF_{2n} + bL_{2n} \tag{4.4}$$

**Theorem 4.5** for every integer  $n \ge 2$ ,

$$R_m R_{n-1} + R_{m+1} + R_n = a F_{n-1} R_{m+2} + b L_{n-1} R_{m+2} + R_{n-2} R_{m+1}$$
(4.5)

## 5. Determinant Identities.

Problems on determinants of Fibonacci sequence and Lucas sequence are appeared in various issues of Fibonacci Quarterly. In this section some determinant identities of Generalized Fibonacci-Like sequence are derived. Entries of determinants are satisfying the recurrence relation of Generalized Fibonacci-Like sequence, Fibonacci and Lucas sequences.

Theorem 5.1 Let n be a positive integer, then

$$\begin{vmatrix} R_{n} & F_{n} & 1 \\ R_{n+1} & F_{n+1} & 1 \\ R_{n+2} & F_{n+2} & 1 \end{vmatrix} = [F_{n}R_{n+1} - F_{n+1}R_{n}]$$
Proof: Let  $\Delta = \begin{vmatrix} R_{n} & F_{n} & 1 \\ R_{n+1} & F_{n+1} & 1 \\ R_{n+2} & F_{n+2} & 1 \end{vmatrix}$ 
(5.1)

Let 
$$R_n = a$$
,  $R_{n+1} = b$ ,  $R_{n+2} = a + b$ ,  $F_n = p$ ,  $F_{n+1} = q$ ,  $F_{n+2} = p + q$  (5.2)

Substituting the values of equation (5.2) in equation (5.1), we get

$$\Delta = \begin{vmatrix} a & p & 1 \\ b & q & 1 \\ a+b & p+q & 1 \end{vmatrix}$$

Applying  $R_1 \rightarrow R_1 - R_2$ 

$$\Delta = \begin{vmatrix} a-b & p-q & 0 \\ b & q & 1 \\ a+b & p+q & 1 \end{vmatrix}$$

Applying  $R_2 \rightarrow R_2 - R_3$ 

$$\Delta = \begin{vmatrix} a-b & p-q & 0 \\ -a & -p & 0 \\ a+b & p+q & 1 \end{vmatrix} = [pb-aq].$$
(5.3)

Substituting the values of the equation (5.2) in equation (5.3), we get

$$\begin{vmatrix} R_n & F_n & 1 \\ R_{n+1} & F_{n+1} & 1 \\ R_{n+2} & F_{n+2} & 1 \end{vmatrix} = [F_n R_{n+1} - F_{n+1} R_n].$$

Proofs of following determinant identities can be given same as theorem 5.1.

**Theorem 5.2.** For every positive integer n,

$$\begin{vmatrix} R_n & L_n & 1 \\ R_{n+1} & L_{n+1} & 1 \\ R_{n+2} & L_{n+2} & 1 \end{vmatrix} = [L_n R_{n+1} - L_{n+1} R_n]$$
(5.4)

Theorem 5.3. For every positive integer n,

$$\begin{vmatrix} R_{n+1} & R_{n+2} & R_{n+3} \\ R_{n+4} & R_{n+5} & R_{n+6} \\ R_{n+7} & R_{n+8} & R_{n+9} \end{vmatrix} = 0$$
(5.5)

Theorem 5.4. For every positive integer n,

$$\begin{vmatrix} 1+R_n & R_{n+1} & R_{n+2} \\ R_n & 1+R_{n+1} & R_{n+2} \\ R_n & R_{n+1} & 1+R_{n+2} \end{vmatrix} = 1+R_n+R_{n+1}+R_{n+2}$$
(5.6)

**Theorem 5.5.** For every positive integer n,

$$\begin{vmatrix} R_n + R_{n+1} & R_{n+1} + R_{n+2} & R_{n+2} + R_n \\ R_{n+2} & R_n & R_{n+1} \\ 1 & 1 & 1 \end{vmatrix} = 0$$
(5.7)

Theorem 5.6. For every positive integer n,

$$\begin{vmatrix} R_n - R_{n+1} & R_{n+1} - R_{n+2} & R_{n+2} - R_n \\ R_{n+1} - R_{n+2} & R_{n+2} - R_n & R_n - R_{n+1} \\ R_{n+2} - R_n & R_n - R_{n+1} & R_{n+1} - R_{n+2} \end{vmatrix} = 0$$
(5.8)

**Theorem 5.7.** For every positive integer n,

$$\begin{vmatrix} R_n & R_n + R_{n+1} & R_n + R_{n+1} + R_{n+2} \\ 2R_n & 2R_n + 3R_{n+1} & 2R_n + 3R_{n+1} + 4R_{n+2} \\ 3R_n & 3R_n + 6R_{n+1} & 3R_n + 6R_{n+1} + 12R_{n+2} \end{vmatrix} = 3R_n R_{n+1} R_{n+2}$$
(5.9)

Theorem 5.8. For every positive integer n,

$$\begin{vmatrix} 0 & R_n R_{n+1}^2 & R_n R_{n+2}^2 \\ R_n^2 R_{n+1} & 0 & R_{n+1} R_{n+2}^2 \\ R_n^2 R_{n+2} & R_{n+2} R_{n+1}^2 & 0 \end{vmatrix} = 2R_n^3 R_{n+1}^3 R_{n+2}^3$$
(5.10)

## 6. Conclusion.

In this paper, we have described fundamental identities, connection formulae of Generalized Fibonacci-Like sequence. Determinant identities are established. We may find out new identities by Binet's formula and generating function.

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