Abstract

Sequences have been fascinating topic for mathematicians for centuries. The Fibonacci and Lucas sequences are examples of second order recursive sequences. Fibonacci sequence is defined by $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, with $F_0 = 0$, $F_1 = 1$. In recent years, few research scholars have been introduced Fibonacci-Like sequences which are similar to Fibonacci sequences in recurrence relation, but initial conditions are different. Due to this reason, these are known as Fibonacci-Like sequences. In this paper, we study a Generalized Fibonacci-Like sequence $R_n = R_{n-1} + R_{n-2}$, $n \geq 2$ with initial condition $R_0 = 2b$ and $R_1 = a+b$, where $a$ and $b$ are non-zero real numbers. Some identities are established by Binet’s formula and generating function. Further, present connection formulae and some determinant identities.

Keywords: Fibonacci sequence; Lucas sequence; Fibonacci-Like sequence; Generalized Fibonacci-Like sequence.
1. Introduction.

In modern science there is a huge interest in the theory and application of the Fibonacci numbers. The Fibonacci numbers \( F_n \) are the terms of the sequence 0, 1, 1, 2, 3, 5, \ldots, where the sequence of Fibonacci numbers \( F_n \) \([1, 2, 8]\) is define by

\[
F_n = F_{n-1} + F_{n-2}, \quad n \geq 2 \text{ with the initial values } F_0 = 0, F_1 = 1
\]  

(1.1)

Generalized Fibonacci sequences have been intensively studied for many years and have become an interesting topic in Applied Mathematics. Fibonacci sequences and their related higher-order sequences are generally studied as sequence of integer. The sequence of Lucas numbers \( L_n \) \([1, 2, 8]\) is defined by recurrence relation:

\[
L_n = L_{n-1} + L_{n-2}, \quad n \geq 2, \quad L_0 = 2, L_1 = 1
\]  

(1.2)

Few research scholars have been introduced Fibonacci-Like sequences which are similar to Fibonacci sequences in recurrence relation, but initial conditions are different. Due to this reason, these are known as Fibonacci-Like sequences. In 2010, Fibonacci-like sequence \([3]\) is defined by recurrence relation:

\[
S_n = S_{n-1} + S_{n-2}, \quad n \geq 2, \quad S_0 = 2, S_1 = 2.
\]  

(1.3)

The associated initial condition \( S_0 \) and \( S_1 \) are the sum of initial condition of Fibonacci and Lucas sequence respectively, i.e. \( S_0 = F_0 + L_0 \) and \( S_1 = F_1 + L_1 \).

In 2013, Fibonacci-Like sequence \([4]\) is defined by the recurrence relation

\[
H_n = 2H_{n-1} + H_{n-2}, \quad n \geq 2 \text{ with } H_0 = 2, H_1 = 1.
\]  

(1.4)

The associated initial conditions \( H_0 \) and \( H_1 \) are the subtraction of initial conditions of Pell-Lucas and Pell sequence respectively.

In this paper, we study a Generalized Fibonacci-Like sequence and some identities. The Binet’s formula and generating function are mainly use to derive identities of Generalized Fibonacci-Like sequence. Also we present Connection formulae and determinants identities which contain different terms of sequence.
2. Generalized Fibonacci-Like Sequence.

Generalized Fibonacci-Like sequence is defined by the recurrence relation

\[ R_n = R_{n-1} + R_{n-2}, \quad n \geq 2, \text{ with initial conditions } R_0 = 2b, R_1 = a + b \]  

(2.1)

where \( a \) and \( b \) are non zero real numbers.

Here the initial condition \( R_0 \) and \( R_1 \) are the sum of initial condition of Fibonacci sequence ‘\( a \)’ times and initial condition of Lucas sequence ‘\( b \)’ times respectively, i.e. \( R_0 = aF_0 + bL_0 \) and \( R_1 = aF_1 + bL_1 \).

The few terms are as follows

\[
\begin{align*}
R_0 &= 2b, \\
R_1 &= a + b, \\
R_2 &= a + 3b, \\
R_3 &= 2a + 4b, \\
R_4 &= 3a + 7b, \\
R_5 &= 5a + 11b, & \text{and so on.}
\end{align*}
\]

If \( a=b=1 \), then it becomes Conventional Fibonacci-Like sequence 2,2,4,6,10,16,...

If \( a=2 \), \( b=1 \), then it becomes Fibonacci sequence 2, 3, 5, 8, 13,... for \( n \geq 3 \).

The relation between Fibonacci sequence, Lucas sequence and Generalized Fibonacci-like sequence can be written as

\[ R_n = aF_n + bL_n, \quad n \geq 0. \]

The recurrence relation (2.1) has the characteristic equation

\[ x^2 - x - 1 = 0, \]  

(2.2)

which has two roots

\[
\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}, \quad \text{also} \quad \alpha \beta = -1, \alpha + \beta = 1, \alpha - \beta = \sqrt{5}, \alpha^2 + \beta^2 = 3.
\]

Generating function of generalized Fibonacci-like sequence is

\[
\sum_{n=0}^{\infty} R_n x^n = r(x) = \frac{2b + (a-b)x}{(1-x-x^2)} \]  

(2.3)
Binet’s formula of generalized Fibonacci-like sequence is defined by

\[ R_n = C_1\alpha^n + C_2\beta^n = C_1\left(\frac{1 + \sqrt{5}}{2}\right)^n + C_2\left(\frac{1 - \sqrt{5}}{2}\right)^n \]  

(2.4)

Here, \( C_1 = \frac{a + b\sqrt{5}}{\sqrt{5}} \) and \( C_2 = \frac{b\sqrt{5} - a}{\sqrt{5}} \), also \( C_1 + C_2 = R_0 = 2b, C_1C_2 = \frac{5b^2 - a^2}{(\alpha - \beta)^2} = \frac{5b^2 - a^2}{5} \)

### 3. Some Identities of Generalized Fibonacci-Like Sequence.

In this section, some identities of Generalized Fibonacci-Like sequence are presented and derived by Binet’s formula and generating function.

**Theorem 3.1 (Explicit sum formula)** Let \( R_n \) be the \( n \)th term of the generalized Fibonacci-like sequence. Then

\[ R_n = 2b \sum_{k=0}^{n-2} \binom{n-k-1}{k} + (a-b) \sum_{k=0}^{n-1} \binom{n-k-1}{k} \]

(3.1)

**Proof:** By generating function (2.3), we have

\[ \sum_{n=0}^{\infty} R_n x^n = R(x) = \frac{2b + (a-b)x}{1-x-x^2} = \frac{[2b + (a-b)x](1-x-x^2)^{-1}}{1-x-x^2} \]

\[ = \{2b + (a-b)x\}[1-(x+x^2)]^{-1} \]

\[ = \{2b + (a-b)x\} \sum_{n=0}^{\infty} (x+x^2)^n = \{2b + (a-b)x\} \sum_{n=0}^{\infty} x^n(1+x^n) \]

\[ = \{2b + (a-b)x\} \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \binom{n}{k} x^k \]

\[ = \{2b + (a-b)x\} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{n}{k|n-k|} x^{n+k} \]

(Replacing \( n \) by \( n+k \))

\[ = \{2b + (a-b)x\} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{n+k}{k|n-k|} x^{n+2k} \]

(Replacing \( n \) by \( n-2k \))
\[ = 2b + (a - b)x \sum_{n=0}^{\infty} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n-k}{k|n-2k|} x^n \]

\[ = 2b \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n-k}{k|n-2k|} \right) x^n + (a - b) \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n-k}{k|n-2k|} \right) x^{n+1} \]

Equating the coefficient of \( x^n \) we obtain

\[ R_n = 2b \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n-k}{k} + (a - b) \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{n-k-1}{k} \]

For \( a=b=1 \) in above identity, explicit formula can be obtained for Fibonacci sequence.

**Theorem 3.2** Sum of the first \( n \) terms of the generalized Fibonacci-like sequence is

\[ \sum_{k=1}^{n} R_k = R_1 + R_2 + R_3 + \ldots + R_n = R_{n+2} - (a + 3b) \]  \((3.2)\)

**Proof:** By Binet’s formula (2.4), we have

\[ \sum_{k=1}^{n} R_k = \sum_{k=1}^{n} [C_1 \alpha^k + C_2 \beta^k] \]

\[ = C_1 \alpha \left( \frac{1 - \alpha^n}{1 - \alpha} \right) + C_2 \beta \left( \frac{1 - \beta^n}{1 - \beta} \right) \]

\[ = \left[ (C_1 \alpha + C_2 \beta) - (C_1 + C_2) \alpha \beta - (C_1 \alpha^{n+1} + C_2 \beta^{n+1}) + (C_1 \alpha^n + C_2 \beta^n) \alpha \beta \right] \]

Using subsequent result of Binet’s formula, we get

\[ \sum_{k=1}^{n} R_k = R_{n+1} + R_n - (a + 3b) = R_{n+2} - (a + 3b) \]

This identities becomes

\[ R_1 + R_2 + R_3 + \ldots + R_{2n} = R_{2n+2} - (a + 3b) \]  \((3.3)\)

**Theorem 3.3** Sum of the first \( n \) terms with odd indices is

\[ \sum_{k=1}^{n} R_{2k-1} = R_1 + R_3 + \ldots + R_{2n-1} = R_{2n} - 2b \]  \((3.4)\)

**Theorem 3.4** Sum of the first \( n \) terms with even indices is
\[
\sum_{k=1}^{n} R_{2k} = R_2 + R_4 + R_6 + \ldots + R_{2n} = R_{2n+1} - (a + b) \tag{3.5}
\]

If we subtract equation (3.5) term wise from equation (3.4), we get alternative sum of first \( n \) numbers

\[
R_1 - R_2 + R_3 - R_4 + R_5 - \ldots + R_{2n-1} - R_{2n} = R_{2n} - 2b - R_{2n+1} + (a + b)
\]

\[= -R_{2n-1} + (a - b) \tag{3.6}\]

Adding \( R_{2n+1} \) to both side of equation (3.5)

\[
R_1 - R_2 + R_3 - R_4 + R_5 - \ldots + R_{2n-1} - R_{2n} + R_{2n+1} = -R_{2n-1} + (a - b) + R_{2n+1}
\]

\[= R_{2n} + (a - b) \tag{3.7}\]

**Theorem 3.5** Sum of square of first \( n \) terms of the generalized Fibonacci-like sequence \( \{R_n\} \) is

\[
R_1^2 + R_2^2 + R_3^2 + \ldots + R_n^2 = R_n R_{n+1} - 2b(a + b) \tag{3.8}
\]

**Theorem 3.6** For every positive integer \( n \)

\[
R_3 + R_6 + R_9 + \ldots + R_{3n} = \frac{1}{2}[R_{3n+2} - (a + 3b)] \tag{3.9}
\]

**Theorem 3.7** For every positive integer \( n \)

\[
R_5 + R_8 + R_{11} + \ldots + R_{3n+2} = \frac{1}{2}[R_{3n+4} - (3a + 7b)] \tag{3.10}
\]

**Theorem 3.8** For every positive integer \( n \), then

\[
R_n^2 = R_n R_{n+1} - R_{n-1} R_n \tag{3.11}
\]

**Theorem 3.9** For every positive integer \( n \), then

\[
R_{2n} = \sum_{k=0}^{n} \binom{n}{k} R_{n-k} \tag{3.12}
\]

**Theorem 3.10 (Catalan’s Identity)** Let \( R_n \) be \( n^{th} \) term of the Generalized Fibonacci-like sequence, then
\[ R_n^2 - R_{n+r}^2 = \frac{(-1)^{n-r}}{a^2 - 5b^2} \left[ (a+b)R_r - 2bR_{r+1} \right]^2, \quad n > r \geq 1. \]  

(3.13)

**Proof:** By Binet’s formula (2.4), we have

\[
R_n^2 - R_{n+r}^2 = (C_1\alpha^n + C_2\beta^n)^2 - (C_1\alpha^{n+r} + C_2\beta^{n+r})(C_1\alpha^{n-r} + C_2\beta^{n-r}) \\
= C_1C_2(\alpha\beta)^r(2 - \alpha'\beta' - \alpha''\beta'') \\
= C_1C_2(\alpha\beta)^r(2\alpha'\beta' - \alpha''^2 - \beta''^2) \\
= -C_1C_2(\alpha\beta)^r(\alpha' - \beta')^2
\]

Using subsequent results of Binet’s formula, we get

\[
R_n^2 - R_{n+r}^2 = (a^2 - 5b^2)(-1)^{n-r}\left(\frac{\alpha' - \beta'}{\alpha - \beta}\right)
\]

Since \[ \frac{\alpha' - \beta'}{\alpha - \beta} = \frac{(a+b)R_r - 2bR_{r+1}}{2aR_1 - (a+b)^2 - R_0^2} = \frac{(a+b)R_r - 2bR_{r+1}}{a^2 - 5b^2} \]

We obtain

\[ R_n^2 - R_{n+r}^2 = \frac{(-1)^{n-r}}{a^2 - 5b^2} \left[ (a+b)R_r - 2bR_{r+1} \right]^2, \quad n > r \geq 1. \]

**Corollary (3.10.1) (Cassini’s Identity)** Let \( R_n \) be \( n^{th} \) term of the Generalized Fibonacci-like sequence, then

\[ R_n^2 - R_{n+1}R_{n-1} = (-1)^{n-r}(a^2 - 5b^2), \quad n \geq 1. \]  

(3.14)

Taking \( r=1 \) in the Catalan’s identity (3.13), the required identity is obtained.

**Theorem 3.11 (D’Ocagne’s Identity)** Let \( R_n \) be \( n^{th} \) term of the generalised Fibonacci-like sequence, then

\[ R_mR_{m+1} - R_{m+1}R_m = (-1)^{n-r}(a+b)R_{m-n} - 2bR_{m-n+1}, \quad m > n \geq 0. \]  

(3.15)

**Theorem 3.12 (Generalized Identity)** Let \( R_n \) be the \( n^{th} \) term of the generalized Fibonacci-like sequence. Then
\[ R_m R_n - R_{m-r} R_{n+r} = \frac{(-1)^{m-r}}{a^2 - 5b^2} \{(a+b)R_r - 2bR_{r+1}\} \{(a+b)R_{n-m+r} - 2bR_{n+m+r+1}\}, \ n > m \geq r \geq 1 \]  

(3.16)

4. Connection Formulae.

In this section, connection formulae of Generalized Fibonacci-Like sequence, Fibonacci and Lucas sequences are presented.

**Theorem 4.1** Let \( n \) be a positive integer, then

\[ R_{n+1} + R_{n-1} = (a+b)L_n + 2bL_{n-1} \]  

(4.1)

**Proof:** We shall prove this identity by induction over \( n \).

For \( n=1 \), we have

\[ R_1 + R_0 = (a+b)L_1 + 2bL_0 \]

\[ a + 3b + 2b = (a+b)1 + 2b \times 2 \]

\[ a + 5b = a + 5b \]

Suppose that the identity holds for \( n=k-2 \) and \( n=k-1 \), then

\[ R_{k-1} + R_{k-3} = (a+b)L_{k-2} + 2bL_{k-3} \]

\[ R_k + R_{k-2} = (a+b)L_{k-1} + 2bL_{k-2} \]

Adding both equations, we get

\[(R_{k-1} + R_k) + (R_{k-3} + R_{k-2}) = (a+b)(L_{k-2} + L_{k-1}) + 2b(L_{k-3} + L_{k-2}) \]

\[ R_{k+1} + R_{k-1} = (a+b)L_k + 2bL_{k-1} \]

Which is precisely our identity when \( n=k \).

Hence, \( R_{n+1} + R_{n-1} = (a+b)L_n + 2bL_{n-1} \)

**Theorem 4.2** Let \( n \) be a positive integer, then

\[ R_{n+1} - R_{n-1} = (a+b)F_n + 2bF_{n-1} \quad n \geq 1 \]  

(4.2)

**Theorem 4.3** For every integer \( n \geq 0 \),

\[ R_{n+1} = aF_{n+1} + bL_{n+1} \]  

(4.3)

**Theorem 4.4** For every integer \( n \geq 0 \),

\[ a_{n+1} = aF_{n+1} + bL_{n+1} \]
\[ R_{2n} = aF_{2n} + bL_{2n} \] (4.4)

**Theorem 4.5** for every integer \( n \geq 2 \),

\[ R_m R_{m-1} + R_{m+1} + R_n = aF_{n-1} R_{m+2} + bL_{n-1} R_{m+2} + R_{n-2} R_{m+1} \] (4.5)

### 5. Determinant Identities.

Problems on determinants of Fibonacci sequence and Lucas sequence are appeared in various issues of Fibonacci Quarterly. In this section some determinant identities of Generalized Fibonacci-Like sequence are derived. Entries of determinants are satisfying the recurrence relation of Generalized Fibonacci-Like sequence, Fibonacci and Lucas sequences.

**Theorem 5.1** Let \( n \) be a positive integer , then

\[
\begin{vmatrix}
R_n & F_n & 1 \\
R_{n+1} & F_{n+1} & 1 \\
R_{n+2} & F_{n+2} & 1
\end{vmatrix}
= [F_n R_{n+1} - F_{n+1} R_n]
\]

Proof: Let \( \Delta = \begin{vmatrix} R_n & F_n & 1 \\ R_{n+1} & F_{n+1} & 1 \\ R_{n+2} & F_{n+2} & 1 \end{vmatrix} \) (5.1)

Let \( R_n = a, \ R_{n+1} = b, \ R_{n+2} = a + b, F_n = p, F_{n+1} = q, F_{n+2} = p + q \) (5.2)

Substituting the values of equation (5.2) in equation (5.1), we get

\[
\Delta = \begin{vmatrix} a & p & 1 \\ b & q & 1 \\ a+b & p+q & 1 \end{vmatrix}
\]

Applying \( R_1 \rightarrow R_1-R_2 \)

\[
\Delta = \begin{vmatrix} a-b & p-q & 0 \\ b & q & 1 \\ a+b & p+q & 1 \end{vmatrix}
\]

Applying \( R_2 \rightarrow R_2-R_3 \)
\[
\Delta = \begin{vmatrix}
-a-b & p-q & 0 \\
-a & -p & 0 \\
a+b & p+q & 1
\end{vmatrix} = [pb - aq].
\] (5.3)

Substituting the values of the equation (5.2) in equation (5.3), we get

\[
\begin{vmatrix}
R_n & F_n & 1 \\
R_{n+1} & F_{n+1} & 1 \\
R_{n+2} & F_{n+2} & 1
\end{vmatrix} = [F_n R_{n+1} - F_{n+1} R_n].
\]

Proofs of following determinant identities can be given same as theorem 5.1.

**Theorem 5.2.** For every positive integer \( n \),

\[
\begin{vmatrix}
R_n & L_n & 1 \\
R_{n+1} & L_{n+1} & 1 \\
R_{n+2} & L_{n+2} & 1
\end{vmatrix} = [L_n R_{n+1} - L_{n+1} R_n].
\] (5.4)

**Theorem 5.3.** For every positive integer \( n \),

\[
\begin{vmatrix}
R_{n+1} & R_{n+2} & R_{n+3} \\
R_{n+4} & R_{n+5} & R_{n+6} \\
R_{n+7} & R_{n+8} & R_{n+9}
\end{vmatrix} = 0
\]

(5.5)

**Theorem 5.4.** For every positive integer \( n \),

\[
\begin{vmatrix}
1 + R_n & R_{n+1} & R_{n+2} \\
R_n & 1 + R_{n+1} & R_{n+2} \\
R_n & R_{n+1} & 1 + R_{n+2}
\end{vmatrix} = 1 + R_n + R_{n+1} + R_{n+2}
\]

(5.6)

**Theorem 5.5.** For every positive integer \( n \),

\[
\begin{vmatrix}
R_n + R_{n+1} & R_{n+1} + R_{n+2} & R_{n+2} + R_n \\
R_{n+2} & R_n & R_{n+1} \\
1 & 1 & 1
\end{vmatrix} = 0
\]

(5.7)

**Theorem 5.6.** For every positive integer \( n \),

\[
\begin{vmatrix}
R_n - R_{n+1} & R_{n+1} - R_{n+2} & R_{n+2} - R_n \\
R_{n+1} - R_{n+2} & R_{n+2} - R_n & R_n - R_{n+1} \\
R_{n+2} - R_n & R_n - R_{n+1} & R_{n+1} - R_{n+2}
\end{vmatrix} = 0
\]

(5.8)
Theorem 5.7. For every positive integer $n$,

\[
\begin{vmatrix}
R_n & R_n + R_{n+1} & R_n + R_{n+1} + R_{n+2} \\
2R_n & 2R_n + 3R_{n+1} & 2R_n + 3R_{n+1} + 4R_{n+2} \\
3R_n & 3R_n + 6R_{n+1} & 3R_n + 6R_{n+1} + 12R_{n+2}
\end{vmatrix}
= 3R_n R_{n+1} R_{n+2}
\]  

(5.9)

Theorem 5.8. For every positive integer $n$,

\[
\begin{vmatrix}
0 & R_n R_{n+1}^2 & R_n R_{n+2}^2 \\
R_n R_{n+1}^2 & 0 & R_{n+1} R_{n+2}^2 \\
R_n^2 R_{n+2} & R_{n+1}^2 R_{n+2} & 0
\end{vmatrix}
= 2R_n^3 R_{n+1} R_{n+2}
\]  

(5.10)

6. Conclusion.

In this paper, we have described fundamental identities, connection formulae of Generalized Fibonacci-Like sequence. Determinant identities are established. We may find out new identities by Binet’s formula and generating function.

References


