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Generalized Fibonacci-Like Sequence and Some Identities

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Abstract

Sequences have been fascinating topic for mathematicians for centuries. The Fibonacci and Lucas sequences are examples of second order recursive sequences. Fibonacci sequence is defined by $F_n = F_{n-1} + F_{n-2},\ n \geq 2,$ with $F_0 = 0,\ F_1 = 1.$ In recent years, few research scholars have been introduced Fibonacci-Like sequences which are similar to Fibonacci sequences in recurrence relation, but initial conditions are different. Due to this reason, these are known as Fibonacci-Like sequences. In this paper, we study a Generalized Fibonacci-Like sequence $R_n = R_{n-1} + R_{n-2}, n \geq 2$ with initial condition R_0=2b and R_1= a+b, where a and b are non-zero real numbers. Some identities are established by Binet's formula and generating function. Further, present connection formulae and some determinant identities.

Keywords: Fibonacci sequence; Lucas sequence; Fibonacci-Like sequence; Generalized Fibonacci-Like sequence.

1. Introduction.

In modern science there is a huge interest in the theory and application of the Fibonacci numbers. The Fibonacci numbers F_n are the terms of the sequence 0, 1, 1, 2, 3, 5, . . . , where the sequence of Fibonacci numbers F_n [1,2, 8] is define by

$$F_n = F_{n-1} + F_{n-2}, n \geq 2 \text{ with the initial values } F_0 = 0, F_1 = 1 \qquad (1.1)$$

Generalized Fibonacci sequences have been intensively studied for many years and have become an interesting topic in Applied Mathematics. Fibonacci sequences and their related higher-order sequences are generally studied as sequence of integer. The sequence of Lucas numbers L_n [1, 2, 8] is defined by recurrence relation:

$$L_n = L_{n-1} + L_{n-2}, n \geq 2, \quad L_0 = 2, L_1 = 1 \qquad (1.2)$$

Few research scholars have been introduced Fibonacci-Like sequences which are similar to Fibonacci sequences in recurrence relation, but initial conditions are different. Due to this reason, these are known as Fibonacci-Like sequences. In 2010, Fibonacci-like sequence [3] is defined by recurrence relation:

$$S_n = S_{n-1} + S_{n-2}, n \geq 2 \ , \quad S_0 = 2, S_1 = 2. \qquad (1.3)$$

The associated initial condition S_0 and S_1 are the sum of initial condition of Fibonacci and Lucas sequence respectively, i.e. $S_0 = F_0 + L_0$ and $S_1 = F_1 + L_1$.

In 2013, Fibonacci-Like sequence [4] is defined by the recurrence relation

$$H_n = 2H_{n-1} + H_{n-2} \ , \ n \geq 2 \text{ with } H_0 = 2, \ H_1 = 1. \qquad (1.4)$$

The associated initial conditions H_0 and H_1 are the subtraction of initial conditions of Pell-Lucas and Pell sequence respectively.

In this paper, we study a Generalized Fibonacci-Like sequence and some identities. The Binet's formula and generating function are mainly use to derive identities of Generalized Fibonacci-Like sequence. Also we present Connection formulae and determinants identities which contain different terms of sequence.

2. Generalized Fibonacci-Like Sequence.

Generalized Fibonacci-Like sequence is defined by the recurrence relation

$R_n = R_{n-1} + R_{n-2}, n \geq 2$, with initial conditions $R_0 = 2b, R_1 = a+b$ (2.1)

where a and b are non zero real numbers.

Here the initial condition R_0 and R_1 are the sum of initial condition of Fibonacci sequence 'a' times and initial condition of Lucas sequence 'b' times respectively, i.e. $R_0 = aF_0 + bL_0$ and $R_1 = aF_1 + bL_1$.

The few terms are as follows

$R_0 = 2b,$
$R_1 = a+b,$
$R_2 = a+3b,$
$R_3 = 2a+4b,$
$R_4 = 3a+7b,$
$R_5 = 5a+11b, \ldots$ and so on.

If a=b=1 , then it becomes Conventional Fibonacci-Like sequence 2,2,4,6,10,16,...

If a=2 , b=1 , then it becomes Fibonacci sequence 2, 3, 5, 8, 13,... for n≥3.

The relation between Fibonacci sequence, Lucas sequence and Generalized Fibonacci-like sequence can be written as

$R_n = aF_n + bL_n, n \geq 0$.

The recurrence relation (2.1) has the characteristic equation

$x^2 - x - 1 = 0$, (2.2)

which has two roots

$\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$,also $\alpha\beta = -1, \alpha+\beta = 1, \alpha-\beta = \sqrt{5}, \alpha^2+\beta^2 = 3.$

Generating function of generalized Fibonacci-like sequence is

$$\sum_{n=0}^{\infty} R_n x^n = r(x) = \frac{2b+(a-b)x}{(1-x-x^2)} \quad (2.3)$$

Binet's formula of generalized Fibonacci-like sequence is defined by

$$R_n = C_1\alpha^n + C_2\beta^n = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right) \tag{2.4}$$

Here, $C_1 = \dfrac{a+b\sqrt{5}}{\sqrt{5}}$ and $C_2 = \dfrac{b\sqrt{5}-a}{\sqrt{5}}$, also $C_1 + C_2 = R_0 = 2b$, $C_1C_2 = \dfrac{5b^2-a^2}{(\alpha-\beta)^2} = \dfrac{5b^2-a^2}{5}$

3. Some Identities of Generalized Fibonacci-Like Sequence.

In this section, some identities of Generalized Fibonacci-Like sequence are presented and derived by Binet's formula and generating function.

Theorem 3.1 (Explicit sum formula) Let R_n be the n^{th} term of the generalized Fibonacci-like sequence. Then

$$R_n = 2b\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-k}{k} + (a-b)\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n-k-1}{k} \tag{3.1}$$

Proof:- By generating function (2.3), we have

$$\sum_{n=0}^{\infty} R_n x^n = R(x) = \frac{2b+(a-b)x}{1-x-x^2} = \{2b+(a-b)x\}(1-x-x^2)^{-1}$$

$$= \{2b+(a-b)x\}[1-(x+x^2)]^{-1}$$

$$= \{2b+(a-b)x\}\sum_{n=0}^{\infty}(x+x^2)^n = \{2b+(a-b)x\}\sum_{n=0}^{\infty}x^n(1+x^n)$$

$$= \{2b+(a-b)x\}\sum_{n=0}^{\infty}x^n\sum_{k=0}^{n}\binom{n}{k}x^k$$

$$= \{2b+(a-b)x\}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\lfloor n}{\lfloor k\lfloor n-k}x^{n+k}$$

(Replacing n by n+k)

$$= \{2b+(a-b)x\}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\lfloor n+k}{\lfloor k\lfloor n}x^{n+2k}$$

(Replacing n by n-2k)

$$= \{2b + (a-b)x\} \sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{\lfloor n-k}{\lfloor k \lfloor n-2k} x^n$$

$$= 2b \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{\lfloor n-k}{\lfloor k \lfloor n-2k} \right\} x^n + (a-b) \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{\lfloor n-k}{\lfloor k \lfloor n-2k} \right\} x^{n+1}$$

Equating the coefficient of x^n we obtain

$$R_n = 2b \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k} + (a-b) \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-k-1}{k}$$

For a=b=1 in above identity, explicit formula can be obtained for Fibonacci sequence.

Theorem 3.2 Sum of the first n terms of the generalized Fibonacci-like sequence is

$$\sum_{k=1}^{n} R_k = R_1 + R_2 + R_3 + \ldots + R_n = R_{n+2} - (a+3b) \tag{3.2}$$

Proof:-By Binet's formula (2.4),we have

$$\sum_{k=1}^{n} R_k = \sum_{k=1}^{n} [C_1 \alpha^k + C_2 \beta^k]$$

$$= C_1 \alpha \left(\frac{1-\alpha^n}{1-\alpha} \right) + C_2 \beta \left(\frac{1-\beta^n}{1-\beta} \right)$$

$$= \left[\frac{(C_1\alpha + C_2\beta) - (C_1 + C_2)\alpha\beta - (C_1\alpha^{n+1} + C_2\beta^{n+1}) + (C_1\alpha^n + C_2\beta^n)\alpha\beta}{1 - (\alpha+\beta) + \alpha\beta} \right]$$

Using subsequent result of Binet's formula, we get

$$\sum_{k=1}^{n} R_k = R_{n+1} + R_n - (a+3b) = R_{n+2} - (a+3b)$$

This identities becomes

$$R_1 + R_2 + R_3 + \ldots + R_{2n} = R_{2n+2} - (a+3b) \tag{3.3}$$

Theorem 3.3 Sum of the first n terms with odd indices is

$$\sum_{k=1}^{n} R_{2k-1} = R_1 + R_3 + R_5 + \ldots + R_{2n-1} = R_{2n} - 2b \tag{3.4}$$

Theorem 3.4 Sum of the first n terms with even indices is

$$\sum_{k=1}^{n} R_{2k} = R_2 + R_4 + R_6 + ... + R_{2n} = R_{2n+1} - (a+b) \quad (3.5)$$

If we subtract equation (3.5) term wise from equation (3.4) , we get alternative sum of first n numbers

$$R_1 - R_2 + R_3 - R_4 + R_5 - ... + R_{2n-1} - R_{2n} = R_{2n} - 2b - R_{2n+1} + (a+b)$$

$$= -R_{2n-1} + (a-b) \quad (3.6)$$

Adding R_{2n+1} to both side of equation (3.5)

$$R_1 - R_2 + R_3 - R_4 + R_5 - ... + R_{2n-1} - R_{2n} + R_{2n+1} = -R_{2n-1} + (a-b) + R_{2n+1}$$

$$= R_{2n} + (a-b) \quad (3.7)$$

Theorem 3.5 Sum of square of first n terms of the generalized Fibonacci-like sequence $\{R_n\}$ is

$$R_1^2 + R_2^2 + R_3^2 + ... + R_n^2 = R_n R_{n+1} - 2b(a+b) \quad (3.8)$$

Theorem 3.6 For every positive integer n

$$R_3 + R_6 + R_9 + ... + R_{3n} = \frac{1}{2}[R_{3n+2} - (a+3b)] \quad (3.9)$$

Theorem 3.7 For every positive integer n

$$R_5 + R_8 + R_{11} + ... + R_{3n+2} = \frac{1}{2}[R_{3n+4} - (3a+7b)] \quad (3.10)$$

Theorem 3.8 For every positive integer n, then

$$R_n^2 = R_n R_{n+1} - R_{n-1} R_n \quad (3.11)$$

Theorem 3.9 For every positive integer n, then

$$R_{2n} = \sum_{k=0}^{n} \binom{n}{k} R_{n-k} \quad (3.12)$$

Theorem 3.10 (Catalan's Identity) Let R_n be n^{th} term of the Generalized Fibonacci-like sequence, then

$$R_n^2 - R_{n+r}R_{n-r} = \frac{(-1)^{n-r}}{a^2-5b^2}\left[(a+b)R_r - 2bR_{r+1}\right]^2, n > r \geq 1. \quad (3.13)$$

Proof:-By Binet's formula (2.4), we have

$$\begin{aligned} R_n^2 - R_{n+r}R_{n-r} &= (C_1\alpha^n + C_2\beta^n)^2 - (C_1\alpha^{n+r} + C_2\beta^{n+r})(C_1\alpha^{n-r} + C_2\beta^{n-r}) \\ &= C_1C_2(\alpha\beta)^n(2 - \alpha^r\beta^{-r} - \alpha^{-r}\beta^r) \\ &= C_1C_2(\alpha\beta)^{n-r}(2\alpha^r\beta^r - \alpha^{2r} - \beta^{2r}) \\ &= -C_1C_2(\alpha\beta)^{n-r}(\alpha^r - \beta^r)^2 \end{aligned}$$

Using subsequent results of Binet's formula, we get

$$R_n^2 - R_{n+r}R_{n-r} = (a^2 - 5b^2)(-1)^{n-r}\left(\frac{\alpha^r - \beta^r}{\alpha - \beta}\right)$$

Since $\dfrac{\alpha^r - \beta^r}{\alpha - \beta} = \dfrac{(a+b)R_r - 2bR_{r+1}}{2aR_1 - (a+b)^2 - R_0^2} = \dfrac{(a+b)R_r - 2bR_{r+1}}{a^2 - 5b^2}$

We obtain

$$R_n^2 - R_{n+r}R_{n-r} = \frac{(-1)^{n-r}}{a^2-5b^2}\left[(a+b)R_r - 2bR_{r+1}\right]^2, n > r \geq 1.$$

Corrollary (3.10.1) (Cassini's Identity) Let R_n be n^{th} term of the Generalized Fibonacci-like sequence, then

$$R_n^2 - R_{n+1}R_{n-1} = (-1)^{n-r}(a^2 - 5b^2), n \geq 1. \quad (3.14)$$

Taking r=1 in the catalan's identity (3.13), the required identity is obtained.

Theorem 3.11 (D'Ocagne's Identity) Let R_n be n^{th} term of the generalized Fibonacci-like sequence, then

$$R_mR_{n+1} - R_{m+1}R_n = (-1)^n(a+b)R_{m-n} - 2bR_{m-n+1}, m > n \geq 0. \quad (3.15)$$

Theorem 3.12 (Generalized Identity) Let R_n be the n^{th} term of the generalized Fibonacci-like sequence. Then

$$R_m R_n - R_{m-r} R_{n+r} = \frac{(-1)^{m-r}}{a^2 - 5b^2} \{(a+b)R_r - 2bR_{r+1}\} \{(a+b)R_{n-m+r} - 2bR_{n-m+r+1}\}, n > m \geq r \geq 1$$

(3.16)

4. Connection Formulae.

In this section, connection formulae of Generalized Fibonacci-Like sequence, Fibonacci and Lucas sequences are presented.

Theorem 4.1 Let n be a positive integer, then

$$R_{n+1} + R_{n-1} = (a+b)L_n + 2bL_{n-1} \tag{4.1}$$

Proof:- We shall prove this identity by induction over n,

For n=1 , we have

$$R_2 + R_0 = (a+b)L_1 + 2bL_2$$
$$a + 3b + 2b = (a+b)1 + 2b * 2$$
$$a + 5b = a + 5b$$

Suppose that the identity holds for n=k-2 and n=k-1 , then

$$R_{k-1} + R_{k-3} = (a+b)L_{k-2} + 2bL_{k-3}$$
$$R_k + R_{k-2} = (a+b)L_{k-1} + 2bL_{k-2}$$

Adding both equations , we get

$$(R_{k-1} + R_k) + (R_{k-3} + R_{k-2}) = (a+b)(L_{k-2} + L_{k-1}) + 2b(L_{k-3} + L_{k-2})$$
$$R_{k+1} + R_{k-1} = (a+b)L_k + 2bL_{k-1}$$

Which is precisely our identity when n=k.

Hence, $R_{n+1} + R_{n-1} = (a+b)L_n + 2bL_{n-1}$

Theorem 4.2 Let n be a positive integer, then

$$R_{n+1} - R_{n-1} = (a+b)F_n + 2bF_{n-1} \qquad n \geq 1 \tag{4.2}$$

Theorem 4.3 For every integer n≥0,

$$R_{n+1} = aF_{n+1} + bL_{n+1} \tag{4.3}$$

Theorem 4.4 For every integer n≥0,

$$R_{2n} = aF_{2n} + bL_{2n} \tag{4.4}$$

Theorem 4.5 for every integer n≥2,

$$R_m R_{n-1} + R_{m+1} + R_n = aF_{n-1}R_{m+2} + bL_{n-1}R_{m+2} + R_{n-2}R_{m+1} \tag{4.5}$$

5. Determinant Identities.

Problems on determinants of Fibonacci sequence and Lucas sequence are appeared in various issues of Fibonacci Quarterly. In this section some determinant identities of Generalized Fibonacci-Like sequence are derived. Entries of determinants are satisfying the recurrence relation of Generalized Fibonacci-Like sequence, Fibonacci and Lucas sequences.

Theorem 5.1 Let n be a positive integer , then

$$\begin{vmatrix} R_n & F_n & 1 \\ R_{n+1} & F_{n+1} & 1 \\ R_{n+2} & F_{n+2} & 1 \end{vmatrix} = [F_n R_{n+1} - F_{n+1} R_n]$$

Proof: Let $\Delta = \begin{vmatrix} R_n & F_n & 1 \\ R_{n+1} & F_{n+1} & 1 \\ R_{n+2} & F_{n+2} & 1 \end{vmatrix}$ (5.1)

Let $R_n = a,\ R_{n+1} = b,\ R_{n+2} = a+b,\ F_n = p,\ F_{n+1} = q,\ F_{n+2} = p+q$ (5.2)

Substituting the values of equation (5.2) in equation (5.1), we get

$$\Delta = \begin{vmatrix} a & p & 1 \\ b & q & 1 \\ a+b & p+q & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$

$$\Delta = \begin{vmatrix} a-b & p-q & 0 \\ b & q & 1 \\ a+b & p+q & 1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_3$

$$\Delta = \begin{vmatrix} a-b & p-q & 0 \\ -a & -p & 0 \\ a+b & p+q & 1 \end{vmatrix} = [pb-aq]. \quad (5.3)$$

Substituting the values of the equation (5.2) in equation (5.3), we get

$$\begin{vmatrix} R_n & F_n & 1 \\ R_{n+1} & F_{n+1} & 1 \\ R_{n+2} & F_{n+2} & 1 \end{vmatrix} = [F_n R_{n+1} - F_{n+1} R_n].$$

Proofs of following determinant identities can be given same as theorem 5.1.

Theorem 5.2. For every positive integer n,

$$\begin{vmatrix} R_n & L_n & 1 \\ R_{n+1} & L_{n+1} & 1 \\ R_{n+2} & L_{n+2} & 1 \end{vmatrix} = [L_n R_{n+1} - L_{n+1} R_n] \quad (5.4)$$

Theorem 5.3. For every positive integer n ,

$$\begin{vmatrix} R_{n+1} & R_{n+2} & R_{n+3} \\ R_{n+4} & R_{n+5} & R_{n+6} \\ R_{n+7} & R_{n+8} & R_{n+9} \end{vmatrix} = 0 \quad (5.5)$$

Theorem 5.4. For every positive integer n,

$$\begin{vmatrix} 1+R_n & R_{n+1} & R_{n+2} \\ R_n & 1+R_{n+1} & R_{n+2} \\ R_n & R_{n+1} & 1+R_{n+2} \end{vmatrix} = 1 + R_n + R_{n+1} + R_{n+2} \quad (5.6)$$

Theorem 5.5. For every positive integer n,

$$\begin{vmatrix} R_n + R_{n+1} & R_{n+1} + R_{n+2} & R_{n+2} + R_n \\ R_{n+2} & R_n & R_{n+1} \\ 1 & 1 & 1 \end{vmatrix} = 0 \quad (5.7)$$

Theorem 5.6. For every positive integer n,

$$\begin{vmatrix} R_n - R_{n+1} & R_{n+1} - R_{n+2} & R_{n+2} - R_n \\ R_{n+1} - R_{n+2} & R_{n+2} - R_n & R_n - R_{n+1} \\ R_{n+2} - R_n & R_n - R_{n+1} & R_{n+1} - R_{n+2} \end{vmatrix} = 0 \quad (5.8)$$

Theorem 5.7. For every positive integer n,

$$\begin{vmatrix} R_n & R_n + R_{n+1} & R_n + R_{n+1} + R_{n+2} \\ 2R_n & 2R_n + 3R_{n+1} & 2R_n + 3R_{n+1} + 4R_{n+2} \\ 3R_n & 3R_n + 6R_{n+1} & 3R_n + 6R_{n+1} + 12R_{n+2} \end{vmatrix} = 3R_n R_{n+1} R_{n+2} \tag{5.9}$$

Theorem 5.8. For every positive integer n,

$$\begin{vmatrix} 0 & R_n R_{n+1}^2 & R_n R_{n+2}^2 \\ R_n^2 R_{n+1} & 0 & R_{n+1} R_{n+2}^2 \\ R_n^2 R_{n+2} & R_{n+2} R_{n+1}^2 & 0 \end{vmatrix} = 2R_n^3 R_{n+1}^3 R_{n+2}^3 \tag{5.10}$$

6. Conclusion.

In this paper, we have described fundamental identities, connection formulae of Generalized Fibonacci-Like sequence. Determinant identities are established. We may find out new identities by Binet's formula and generating function.

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