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Technique for solving minimum of integral function by Legendre wavelet

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Abstract:

This presents a refined approach for computing integral and product operators by using the rich properties of Legendre wavelet.The most advantages of this refined calculations are that the operational matrices are lower dimensions, coarse and its elements are the same on each subinterval, respectively. The operational matrices are applied to solving the minimum of integral function from heat conduction problem. The essence of this technique is to transform the optimization of integral function into that of Legendre wavelet coefficients, which is solved by Lagrange-multiplier method. The good results demonstrate that this technique is valid and applicable.

Key words: Legendre wavelet; operational matrix of integration; operational matrix of product; refined calculation; minimum of integral function.

1. Introduction

Some applications of heat conduction problem are to minimize the integral function in essence as:

$$\min_{x\in[0,1]} J(y) = y(x) + \int_0^1 F[y(x), \dot{y}(x)]g(x)dx\,, \tag{1}$$

where $y(x)$ is an unknown function to be solved, $F(\cdot)$ is a differentiable function and $g(x) \in L_2[0,1]$ is a known function. Schechter [1] provided a physical interpretation for this type problem by noting an application in heat conduction problem. In general, orthogonal functions and polynomial series have received considerable attention in dealing with this type problem [1-9]. For example, Chen and Hsiao [2] considered an approximate solution using Walsh functions. Khellat and Yousefi [3] proposed the linear Legendre mother wavelets operational matrix of integration to solving this type minimization. Although Legendre wavelet method has many advantages because of the rich properties, such as compactly supported, polynomial, bounded and orthogonal on each subinterval. However, we note that there exists an aspect in computational complexity needed to be decreased, i.e., the dimensions of the integration and product operational matrices increase rapidly when the level of resolution is enlarged.

In order to decrease the complexity, we devise an alternative method called the refined approach to compute the integration and product operators by utilizing the rich properties of Legendre wavelet bases. More elaborately, the integration and product operational matrices are only calculated on the subinterval. The most interest is that each element of the matrices is the same on different subintervals, respectively. Compared with other methods, the essential advantages of this approach are briefly described as: The operational matrices are lower dimensions and can be easily inverted by hand and stored efficiently in computer-memory; The refined calculations of the integration and product operators on the subinterval can avoid the disadvantage of discontinuity of Legendre wavelet bases at nodes. The operational matrices included the above advantages are implemented to solve the minimum of the integral function from heat conduction problem.

The organization of the paper is as follows. In Section 2, descriptions of Legendre wavelet bases and its rich properties are elaborated. In Section 3, the refined calculation method is proposed to compute the operational matrices of the integration and product operators, respectively. Section 4 converts the minimization of the integral function into that of multivariable function with respect to Legendre wavelet by using the above operational matrices, which is solved by Lagrange-multiplier method. The good results of the numerical experiment show that our method is very effective. In section 5, conclusions of the proposed technique in this paper are given at the end.

2. Legendre wavelet

In this section, we first briefly introduce Legendre wavelet bases and our notations. Secondly, the approximation ability is elaborated. So that those who are familiar with this material can easily skip over this section.

Because Legendre wavelet is constructed by using Legendre polynomials [5-7], we introduce Legendre polynomials as:

$$L_0(x)=1, \qquad L_1(x)=x,$$

$$L_{k+2}(x)=\frac{2k+3}{k+2}xL_{k+1}(x)-\frac{k+1}{k+2}L_k(x), \tag{2}$$

where $L_k(x)$ is Legendre polynomials and $k=0,1,2,\cdots,p-1,\cdots$ is the order of Legendre polynomials. When the level of the resolution is $n=0$, Legendre wavelet bases defined on the interval $[0,1)$ has the form of

$$\phi_k(x)=\begin{cases}\sqrt{2k+1}L_k(2x-1), & x\in[0,\ 1),\\ 0, & x\notin[0,\ 1).\end{cases}$$

Then, when the level of the resolution is $n=1,2,\cdots$, Legendre wavelet bases in subspace $V_{p,nl}$ can be obtained by dilation and translation and represented as:

$$\phi_{k,nl}(x)=2^{n/2}\phi_{k,n}(2^n x-l), \quad x\in[l/2^n, (l+1)/2^n), \tag{4}$$

where $0\le k\le p-1,\ \ 0\le l\le 2^n-1$.

For a function $f\in L_2([0,1])$, the approximation of $f(x)$ at the decomposition level n is represented as:

$$f(x) := \sum_{l=0}^{2^n-1} \sum_{k=0}^{p-1} s_{k,nl}\phi_{k,nl}(x) = S^T\Phi(x), \tag{5}$$

where the wavelet coefficients are calculated by

$$s_{k,nl} = \int_{l/2^n}^{(l+1)/2^n} f(x)\phi_{k,nl}(x)dx, \tag{6}$$

where S, $\Phi(x)$ are $2^n p\times 1$ vectors:

$$S = [s_{0,n0}, s_{1,n0}, \cdots, s_{p-1,n0}, s_{0,n1}, s_{1,n1}, \cdots, s_{p-1,n1}, \cdots, s_{0,n(2^n-1)}, s_{1,n(2^n-1)}, \cdots, s_{p-1,n(2^n-1)}]^T,$$

$$\Phi(x) = [\phi_{0,n0}, \phi_{1,n0}, \cdots, \phi_{p-1,n0}, \phi_{0,n1}, \phi_{1,n1}, \cdots, \phi_{p-1,n1}, \cdots, \phi_{0,n(2^n-1)}, \phi_{1,n(2^n-1)}, \cdots, \phi_{p-1,n(2^n-1)}]^T.$$

The estimation of the approximation in (5) is obtained by the literature [5] as:

$$\left\| f - \sum_{l=0}^{2^n-1}\sum_{k=0}^{p-1} s_{k,nl}\phi_{k,nl}(x) \right\|_2 \sim O(2^{-np}), \tag{7}$$

which is exponential convergence with respect to the order p of Legendre wavelet and the level n of the resolution.

3. Operational matrices represented by Legendre wavelet

In this section, we first present the refine calculation for the integral operational matrix and address a general formulation by an algorithm for the above matrix. Then, we obtain the product operational matrix by using similar method.

3.1. Refine calculation of the integration operator

In fact, the approximation of the function by Legendre wavelet is realized through the piece wise approximation on the subinterval $[l/2^n, (l+1)/2^n)$ by using Legendre polynomials. Consequently, we can compute the integration and product operational matrices on each subinterval other than the whole interval $[0,1]$ as the literatures [3-7]. We define this computation as the refined calculation, which has advantage: The operational matrices are lower dimensions, which can be applied to devising fast algorithm.

We first let $p=3$ and the decomposition level be n, then we obtain Legendre wavelet base vector $\Phi_{p,nl}(x)=\left(\phi_{0,nl},\phi_{1,nl},\cdots,\phi_{p-1,nl}\right)^T$. For any $x\in[l/2^n,(l+1)/2^n)$, we obtain the calculation of the integration operational matrix as:

$$\int_{l/2^n}^{x}\Phi_{p,nl}(x)dx:=P_{3\times3}\Phi_{p,nl}(x)\,, \tag{8}$$

where

$$P_{3\times3}=(\Phi_{p,nl},\int_{l/2^n}^{x}\Phi_{p,nl}(x)dx)=\frac{1}{2^{n+1}}\begin{pmatrix}1 & \frac{\sqrt{3}}{3} & 0\\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{5}\times\sqrt{3}}{5\times3}\\ 0 & -\frac{\sqrt{5}\times\sqrt{3}}{5\times3} & 0\end{pmatrix}. \tag{9}$$

The integration operational matrix $P_{3\times3}$ only has a parameter, i.e., the n of the resolution. Furthermore, by simple variable substitute we can prove that each element of the matrix is the same for different $l=0,1,\cdots,2^{n-1}$, i.e., the matrix is the same on each subinterval.

Similarly, we obtain the integral operational matrix for any p as:

$$\int_{l/2^n}^{x}\Phi_{p,nl}(x)dx:=P_{p\times p}\Phi_{p,nl}(x)\,, \tag{10}$$

where

$$P_{p\times p}=\frac{1}{2^{n+1}}\begin{pmatrix}1 & \frac{\sqrt{3}}{3} & 0 & 0 & \cdots & 0\\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{5}\times\sqrt{3}}{5\times3} & 0 & \cdots & 0\\ 0 & -\frac{\sqrt{5}\times\sqrt{3}}{5\times3} & 0 & \frac{\sqrt{7}\times\sqrt{5}}{7\times5} & \cdots & 0\\ 0 & 0 & -\frac{\sqrt{7}\times\sqrt{5}}{7\times5} & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \frac{\sqrt{2p-1}\times\sqrt{2p-3}}{(2p-1)\times(2p-3)}\\ 0 & 0 & 0 & \cdots & -\frac{\sqrt{2p-1}\times\sqrt{2p-3}}{(2p-1)\times(2p-3)} & 0\end{pmatrix}.$$

Obviously, the matrix is sparse and lower dimensions $p\times p$ rather than $2^n p\times 2^n p$ in [3]. This advantage can be implemented to developing fast algorithm when the decomposition level n enlarged.

3.2. Refine calculation of the product operator

Let $S_{p,nl} = (s_{0,nl}, s_{1,nl}, \cdots, s_{p-1,nl})^T$, the product operational matrix $R_{p,nl}$ is calculated as:

$$\Phi_{p,nl}(x)\Phi^T_{p,nl}(x)S_{p,nl} := R_{p,nl}\Phi_{p,nl}(x), \quad (11)$$

where

$$R_{p,nl}(i+1, j+1) = \int_{l/2^n}^{(l+1)/2^n} \left(\sum_{k=0}^{p-1} s_{k,nl}\phi_{k,nl}(x) \cdot \phi_{j,nl}(x) \right) \cdot \phi_{i,nl}(x)dx,$$

which is symmetric matrix and $(i+1, j+1)$ denotes the $i+1\,th$ row and $j+1\,th$ column element of the product operational matrix and $i, j = 0, 1, \cdots, p-1$. By variable substitute, it is easy to prove that each element of the matrix $R_{p,nl}$ is the same for different l, i.e., on different subinterval $[l/2^n, (l+1)/2^n)$.

Specially, let $p=3$, $n=1$, $l=0,1$, we obtain

$$R_{3,1l} = \sqrt{2}\begin{pmatrix} s_{0,1l} & s_{1,1l} & s_{2,1l} \\ s_{1,1l} & s_{0,1l} + \dfrac{2\sqrt{5}s_{2,1l}}{5} & \dfrac{2\sqrt{5}s_{1,1l}}{5} \\ s_{2,1l} & \dfrac{2\sqrt{5}s_{1,1l}}{5} & s_{0,1l} + \dfrac{2\sqrt{5}s_{2,1l}}{7} \end{pmatrix}.$$

Essentially, the elements of the product operational matrix $R_{p,nl}$ matrix are composed of linear combinations of the elements of the vector $S_{p,nl}$. The product operational matrix is mainly applied to the calculation of the nonlinear term.

Additionally, the integration and product operational matrices above computed can be used to solve problems such as identification, optimal control and analysis, etc. like that of the other orthogonal functions.

4. Solving the minimum of the integral function

In this section, we consider the specific problem as :

$$\min_{x\in[0,1]} J(y) = \int_0^1 [0.5\dot{y}^2 - yg(x)]dx\,, \tag{12}$$

where $y(x)$ is the function needed to solve and $g(x)$ has the form of

$$g(x) = \begin{cases} -1, & 0 \le x < \frac{1}{4},\ \ \frac{1}{2} \le x < 1, \\ 3, & \frac{1}{4} \le x < \frac{1}{2}. \end{cases} \tag{13}$$

The corresponding boundary conditions are determined as:

$$\dot{y}(0) = 0\,, \qquad \dot{y}(1) = 0\,. \tag{14}$$

Let $p = 4$, $n = 2$, then we obtain

$$\dot{y}(x) = S_{4,2l}^T \Phi_{4,2l}(x)\,,$$

where $l = 0,1,2,3$. According to (10), the function $y(x)$ defined on subinterval $[l/2^n, (l+1)/2^n)$ is approximated as:

$$y(x) = \int_{l/2^n}^{x} \dot{y}(x)dx + y(l/2^n) = S_{4,2l}^T (P_{4\times4}\Phi_{4,2l}(x) + \Phi_{4,2l}(l/2^n))\,. \tag{15}$$

Using (12), we obtain the approximation of the nonlinear term as:

$$\dot{y}^2(x) = S_{4,2l}^T \Phi_{4,2l}(x) \Phi_{4,2l}^T(x) S_{4,2l} = S_{4,2l}^T \overline{S_{4,2l}} \Phi_{4,2l}(x)\,. \tag{16}$$

Using (12), (15) and (16), we obtain

$$J(y) = \sum_{l=0}^{3} \int_{l/4}^{(l+1)/4} \left(0.5 S_{4,2l}^T \overline{S_{4,2l}} \Phi_{4,2l} - S_{4,2l}^T (P_{4\times4}\Phi_{4,2l} + \Phi_{4,2l}(l/4)) g \right) dx\,. \tag{17}$$

which is the goal function with respect to Legendre wavelet coefficients $S_{4,2l}$. Then, Lagrange function is constructed by using the boundary conditions (14) as:

$$L = J + \lambda_1 S_{4,2l}^T \Phi_{4,2l}(0) + \lambda_2 S_{4,2l}^T \Phi_{4,2l}(1)\,, \tag{18}$$

where λ_1, λ_2 are Lagrange-multiplier. The necessary condition of the minimization is demonstrated as:

$$\frac{\partial L}{\partial S_{4,2l}} = \frac{\partial J}{\partial S_{4,2l}} + \lambda_1 \Phi_{4,2l}(0) + \lambda_2 \Phi_{4,2l}(1) = 0\,, \tag{19a}$$

$$\frac{\partial L}{\partial \lambda_1} = S_{4,2l}^T \Phi_{4,2L}(0) = 0 , \quad (19b)$$

$$\frac{\partial L}{\partial \lambda_2} = S_{4,2l}^T \Phi_{4,2L}(1) = 0 . \quad (19c)$$

Solving Eqs.(19) for Legendre wavelet coefficients $S_{4,2l}$, we obtain the minimum of the intergral function (12) as:

$$y(x) = S_{4,2l}^T P_{4\times 4} \Phi_{4,2l}(x) = \begin{cases} \frac{1}{2}x^2, & 0 \le x < \frac{1}{4}, \\ -\frac{3}{2}x^2 + x - \frac{1}{8}, & \frac{1}{4} \le x < \frac{1}{2}, \\ \frac{1}{2}x^2 - x + \frac{3}{8}, & \frac{1}{2} \le x < 1. \end{cases} \quad (20)$$

5. Conclusion

In this article, the integration and product operational matrices obtained by the refined computations are applied to solving the minimum of the integral function. The results demonstrate that the refined calculations by Legendre wavelet are validity and applicability. Furthermore, the approximate accuracy can be improved by enlarging the p and n.

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