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ON HYPERGEOMETRIC SERIES ASSOCIATED WITH THE GENERALIZED ZETA FUNCTIONS

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Abstract

In this work we will introduce and study a generalized zeta function of the general Hurwitz – Lerch zeta functions. Here, we aim to derive basic properties of generalized zeta function for the general Hurwitz – Lerch zeta functions include some integral representations for several general Hurwitz – Lerch zeta functions and fractional derivative. A number of known and new results which introduced are generalization of a known results introduced by Shy-Der and Srivastava [6].

Key words: Hurwitz-Lerch zeta functions; Mellin-Barnes integrals; hypergeometric functions; hypergeomteric-type generating functions.

1. Introduction

For $\mathrm{Re}(z) > 1$, the Riemann zeta function $\zeta(z)$ is defined as follows [1,p.88]

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \tag{1.1}$$

which can, except for a simple pole at $z = 1$ with its residue 1. be continued moromorphically to the whole complex z -plane. The Hurwitz (or generalized) zeta function denoted usually by $\zeta(z,a)$ and so for $\mathrm{Re}(z) > 1$,

$$\zeta(z,a) = \sum_{n=0}^{\infty} \frac{1}{(a+n)^z}, \tag{1.2}$$

can just as $\zeta(z)$. be continued moromorphically to the whole complex z-plane except for a simple pole at $z = 1$ with its residue 1. When $a = 1$, (1.2) reduces to (1.1). In terms of Hurwitz-Lerch zeta function $\Phi(x,z,a)$ defined by [5,p.121] and [1]:

$$\Phi(x,z,a) = \sum_{n=0}^{\infty} \frac{x^n}{(a+n)^z}, |x| < 1, a \neq 0, -1, -2, ... \tag{1.3}$$

Equivalently, it has the integral expression

$$\Phi(x,z,a) = \frac{1}{\Gamma(z)} \int_0^{\infty} t^{z-1} e^{-at} (1 - xe^{-t})^{-1} dt, \tag{1.4}$$

provided $\mathrm{Re}(a) > 0$ and either $|x| \leq 1$ and $\mathrm{Re}(z) > 0$ or $x = 1$ and $\mathrm{Re}(z) > 1$.

So that obviously

$$\Phi(1,z,a) = \zeta(z,a). \tag{1.5}$$

The zeta function in (1.3) has since been extended by Goyal and Laddha

[3,p.100(1.5)], in the form :

$$\Phi_{\mu}^{*}(x,z,a) = \sum_{n=0}^{\infty} \frac{(\mu)_n}{(a+n)^z} \frac{x^n}{n!}, \tag{1.6}$$

where $|x| < 1, \mathrm{Re}(a) > 0, \mu \geq 1$.

Equivalently , it has the integral expression :

$$\Phi^{*}_{\mu}(x,z,a)=\frac{1}{\Gamma(z)}\int_0^{\infty}t^{z-1}e^{-at}(1-xe^{-t})^{-\mu}dt, \qquad (1.7)$$

provided that $\mu\geq 1, \mathrm{Re}(a)>0, |x|\leq 1$ and either $\mathrm{Re}(z)>0$ or $\mathrm{Re}(z)>\mathrm{Re}(\mu)$ according to $x\neq 1$ or $x=1$.

In [6] Shy-Der and Srivastava introduced the general family of the Hurwitz-Lerch zeta function in the form:

$$\Phi^{(\rho,\sigma)}_{\mu,\nu}(x,z,a)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}}\frac{x^n}{(a+n)^z}, \qquad (1.8)$$

where $\mu\in\mathbb{C}, a,\nu\in\mathbb{C}\setminus\mathbb{Z}^{-}{}_{0}, \rho,\sigma\in\mathbb{R}^{+}, \rho>\sigma$ when $x,z\in\mathbb{C}, \rho=\sigma, z\in\mathbb{C}\ |x|<1, \rho=\sigma$

and $\mathrm{Re}(z-\mu+\nu)>1$ when $|x|=1$.

Equivalently , it has the following integral expressions :

$$\Phi^{(\rho,\sigma)}_{\mu,\nu}(x,z,a)=\frac{1}{\Gamma(z)}\int_0^{\infty}t^{z-1}e^{-at}\,{}_2\psi_1\left[\begin{matrix}(\mu;\rho),(1;1); & \\ & x^k e^{-t}\\ (\nu;\sigma); & \end{matrix}\right]dt. \qquad (1.9)$$

In this paper we will introduce and study a generalized zeta function of the general Hurwitz – Lerch zeta function $\Phi^{(\rho,\sigma)}_{\mu,\nu}(x,z,a)$. This work aims to derive basic properties of generalized zeta function for the general Hurwitz – Lerch zeta functions include some integral representations for several general Hurwitz – Lerch zeta functions and fractional derivative. A number of known and new results which introduced are generalization of a known results introduced in [6].

In section 2 we will introduce analogous definition of the general family of Hurwitz-Lerch zeta function denoted by $\Phi^{(\rho,\sigma)}_{\mu,\nu,\lambda}(x,z,a)$ with a known special cases. Some integral representation for $\Phi^{(\rho,\sigma)}_{\mu,\nu,\lambda}(x,z,a)$ are discussed in section 3 and fractional derivative in section 4.

2. **The Function** $\Phi^{(\rho,\sigma)}_{\mu,\nu,\lambda}(x,z,a)$

In this section we will give analogous definition of the function $\Phi^{(\rho,\sigma)}_{\mu,\nu}(x,z,a)$

which is defined by (1.3) in its form :

$$\Phi^{(\rho,\sigma)}_{\mu,\nu,\lambda}(x,z,a) = (\lambda)^{-z}\Phi^{(\rho,\sigma)}_{\mu,\nu}\left(x,z,\frac{a}{\lambda}\right) = \sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}}\frac{x^n}{(a+\lambda n)^z}, \tag{2.1}$$

$\mu \in \mathbb{C}, \lambda, a, \nu \in \mathbb{C}\setminus\mathbb{Z}^{-}{}_{0}, \rho, \sigma \in \mathbb{R}^{+}, \rho > \sigma \ when\ x, z \in \mathbb{C}, \rho = \sigma \ and$

$when\ |x| < 1, \rho = \sigma \ \ and\ \ \mathrm{Re}(z-\mu+\nu) > 1 \ \ when\ \ |x| = 1$.

Clearly, we have

$$\Phi^{(\sigma,\sigma)}_{\nu,\nu,1}(x,z,a) = \Phi^{(0,0)}_{\nu,\nu,1}(x,z,a) = \Phi(x,z,a), \tag{2.2}$$

$$\Phi^{(1,1)}_{\mu,1,1}(x,z,a) = \Phi^{*}_{\mu}(x,z,a). \tag{2.3}$$

Obviously, when $\lambda = 1$, (2.1) reduce to (1.8). Further, on putting $x = a = 1$ in (2.2), it reduces to (1.1). Next, on putting $x = 1$ and $\mu = 1$ in (2.3), it reduces to (1.2).

3. **Integral representations**

By using the Fox-Wright extension of the generalized hypergeometric ${}_pF_q$ function with p numerator and q denominator parameters, defined by:

$${}_p\Psi_q\left[\begin{matrix}(\alpha_1;A_1),(\alpha_2;A_2),...,(\alpha_p,A_p); \\ \\ (\beta_1;B_1),(\beta_2,B_2)...,(\beta_q,B_q);\end{matrix}\quad z\right] = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\alpha_j)_{A_{jn}}}{\prod_{j=1}^{q}(\beta_j)_{B_{jn}}}\frac{z^n}{n!},$$

where the argument z , the complex parameters $\alpha_j\,(j=1,...,p)$ and $\beta_j\,(j=1,...,q)$,

and the positive real parameters $A_j\,(j=1,...,p)$ and $B_j\,(j=1,...,q)$ are so constrained that the series in (3.1) converges absolutely [1,p.183] and Eulerian integral formula of second kind [1] :

$$(n+a)^{-z} = \frac{1}{\Gamma(z)}\int_0^{\infty} t^{z-1} e^{-(n+a)t}\,dt, (\min\{Re(z), Re(n+a)\} > 0).$$

it is easy to derive the following integral representation.

Theorem 1: *Let* $\mathrm{Re}(a) > 0, \mathrm{Re}(z) > 0$ *when* $|x| < 1 (x \neq 1)$ $\mathrm{Re}(z) > 1$ *when*

$x = 1$. *Then*

$$\Phi^{(\rho,\sigma)}_{\mu,\nu,\lambda}(x,z,a) = \frac{(\lambda)^{-z}}{\Gamma(z)}\int_0^{\infty} t^{z-1} e^{-\frac{a}{\lambda}t}\, {}_2\psi_1\left[\begin{matrix}(\mu;\rho), & (1;1); & \\ & & xe^{-t} \\ & (\nu;\sigma); & \end{matrix}\right] dt \quad . \tag{3.1}$$

Proof: By starting from the left-hand side of formula (3.1), then in view of definition

(2.1), it is easily seen that

$$\Phi^{(\rho,\sigma)}_{\mu,\nu,\lambda}(x,z,a) = (\lambda)^{-z}\,\Phi^{(\rho,\sigma)}_{\mu,\nu}\left(x,z,\frac{a}{\lambda}\right).$$

Upon using the integral formula (1.9), we are led finally to the right-hand side of formula (3.1).

In terms of the Gauss hypergeometric function ${}_2F_1$ (see [7]):

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty}\frac{(a)_n (b)_n}{(c)_n}\frac{x^n}{n!}, \quad c \neq 0,-1,-2,\ldots,$$

a special cases of (3.1), occur when $\rho = \sigma = \lambda = 1$ as follows :

$$\Phi^{(1,1)}_{\mu,\nu,1}(x,z,a) = \frac{1}{\Gamma(z)}\int_0^{\infty} t^{z-1} e^{-at}\, {}_2F_1\left[\mu,1;\nu;xe^{-t}\right]dt. \tag{3.2}$$

By using the familiar Euler transformation formula [1,p.64]

$${}_2F_1(\alpha,\beta;\gamma;x) = (1-x)^{\gamma-\alpha-\beta}\,{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma;x),$$

$$(|\arg(1-x)| \leq \pi-\varepsilon, (0<\varepsilon<\pi), \gamma \notin \mathbb{Z}_0^-),$$

we get equivalently integral formula for (3.2) in form :

$$\Phi^{(1,1)}_{\mu,\nu,1}(x,z,a)=\frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}e^{-at}(1-xe^{-t})^{\nu-\mu-1}\,{}_2F_1\left[\nu-\mu,\nu-1;\nu;xe^{-t}\right]dt\ . \tag{3.3}$$

For $\nu=1$, (3.3) reduces to the well- known result of Goyal and Laddha [3]

$$\Phi^{(1,1)}_{\mu,1,1}(x,z,a)=\frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}e^{-at}(1-xe^{-t})^{-\mu}dt=\Phi^*_\mu(x,z,a). \tag{3.4}$$

Theorem 2: Let $\mathrm{Re}(\mu)>0, \rho\in\mathbb{R}^+$. Then

$$\Phi^{(\rho,\sigma)}_{\mu,\nu,\lambda}(x,z,a)=\frac{(\lambda)^{-z}}{\Gamma(\mu)}\int_0^\infty t^{\mu-1}e^{-t}\Phi^{(0,\sigma)}_{\mu,\nu}\left(xt^\rho,z,\frac{a}{\lambda}\right)dt, \tag{3.5}$$

and

$$\Phi^{(\rho,\sigma)}_{\mu,\nu,\lambda}(x,z,a)=\frac{(\lambda)^{-z}}{\Gamma(\mu)}\int_0^1 (Log\,\frac{1}{t})^{\mu-1}\Phi^{(0,\sigma)}_{\mu,\nu}\left(x\,(Log\,\frac{1}{t})^\rho,z,\frac{a}{\lambda}\right)dt\ . \tag{3.6}$$

Proof: Denote, for convenience, the right-hand side of formula (3.5) by *I*. Then it is easily seen that

$$I=(\lambda)^{-z}\,\frac{1}{\Gamma(\mu)}\int_0^\infty t^{\mu-1}e^{-t}\Phi^{(0,\sigma)}_{\mu,\nu}\left(xt^\rho,z,\frac{a}{\lambda}\right)dt\ .$$

$$=\sum_{n=0}^\infty\frac{(\mu)_{0n}x^n}{\Gamma(\mu)(\nu)_{\sigma n}(a+\lambda n)^z}\int_0^\infty t^{\mu+\rho n-1}e^{-t}dt\ .$$

Upon using the Hankel's formula [6, p.13(2)]

$$\frac{1}{\Gamma(z)}=\frac{1}{2\pi i}\int_{-\infty}^{(0+)}e^t\quad t^{-z}\quad dt\,,\qquad |\arg(t)|\le\pi, z\in\mathbb{C}. \tag{3.7}$$

and definition (2.1) , we are led finally to the left-hand side of formula (3.5).

In the same manner one can derive the formula (3.6).

In their special cases, when $\rho=\sigma=\nu=\lambda=1$,the integral representations

(3.5) and (3.6) reduce to the following forms:

$$\Phi^*_\mu(x,z,a)=\frac{1}{\Gamma(\mu)}\int_0^\infty t^{\mu-1}e^{-t}\Phi^{(0,1)}_{\mu,1}(xt,z,a)dt, \tag{3.8}$$

and

$$\Phi_{\mu}^{*}(x,z,a)=\frac{1}{\Gamma(\mu)}\int_{0}^{1}(Log\,\frac{1}{t})^{\mu-1}\Phi_{\mu,1}^{(0,1)}\left(x\,(Log\,\frac{1}{t}),z,a\right)dt\ , \tag{3.9}$$

respectively.

Theorem 3: Let $0<\mathrm{Re}(\mu)<\mathrm{Re}(\nu)$, $\sigma\in\mathbb{R}^{+}$. Then

$$\Phi_{\mu,\nu,\lambda}^{(\sigma,\sigma)}(x,z,a)=\frac{(\lambda)^{-z}\,\Gamma(\nu)}{\Gamma(\nu-\mu)\Gamma(\mu)}\int_{0}^{\infty}\frac{t^{\mu-1}}{(1+t)^{\nu}}\Phi\left(\frac{xt^{\sigma}}{(1+t)^{\sigma}},z,\frac{a}{\lambda}\right)dt\ . \tag{3.10}$$

Proof: Denote, for convenience, the right-hand side of formula (3.10) by *I*. Then it is easily seen that

$$I=(\lambda)^{-z}\,\frac{\Gamma(\nu)}{\Gamma(\nu-\mu)\Gamma(\mu)}\int_{0}^{\infty}\frac{t^{\mu-1}}{(1+t)^{\nu}}\Phi\left(\frac{xt^{\sigma}}{(1+t)^{\sigma}},z,\frac{a}{\lambda}\right)dt\ .$$

Now, in view of integral formula

$$\Phi_{\mu,\nu}^{(\rho,\sigma)}(x,z,a)=\frac{\Gamma(\nu)}{\Gamma(\nu-\mu)\Gamma(\mu)}\int_{0}^{\infty}\frac{t^{\mu-1}}{(1+t)^{\nu}}\Phi(\frac{xt^{\sigma}}{(1+t)^{\sigma}},z,a)dt\ ,$$

$$(0<\mathrm{Re}(\mu)<\mathrm{Re}(\nu),\sigma\in\mathbb{R}^{+}),$$

and the definition (2.1), we obtain the left-hand side of formula (3.10).

Obviously, for $\sigma=\nu=\lambda=1$, (3.10) reduces to the following result :

$$\Phi_{\mu}^{*}(x,z,a)=\frac{\sin(\pi\mu)}{\pi}\int_{0}^{\infty}\frac{t^{\mu-1}}{(1+t)^{\nu}}\Phi\left(\frac{xt}{(1+t)},z,a\right)dt\ . \tag{3.11}$$

Furthermore we derive the following result .

Theorem 4: Let $\nu\in\mathbb{C}/\mathbb{Z}_{0}^{-},\sigma\in\mathbb{R}^{+}$, $\left|\arg(w)\right|<\pi$ .Then

$$\Phi_{\mu,\nu,\lambda}^{(\rho,\sigma)}(x,z,a)=\frac{(\lambda)^{-z}\,\Gamma(\nu)}{2\pi i}\int_{-\infty}^{(0+)}w^{-\nu}e^{w}\,\Phi_{\mu,\nu}^{(\rho,0)}\left(xw^{-\sigma},z,\frac{a}{\lambda}\right)dw\ . \tag{3.12}$$

Proof: Denote, for convenience, the right-hand side of formula (3.12) by I . Then in view of (1.8),it is easily seen that

$$I = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n} x^n}{(a+\lambda n)^z} \frac{\Gamma(\nu)}{2\pi i} \int_{-\infty}^{(0+)} w^{-(\nu+\sigma n)} e^w dw \ .$$

Upon using the formula (3.7) and definition (2.1) , we are led finally to left-hand side of formula (3.12).

Obviously, for $\rho = \sigma = \nu = \lambda = 1$, (3.12) reduces to the following result :

$$\Phi^*_\mu(x, z, a) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} w^{-1} e^w \Phi^{(1,0)}_{\mu,1}(xw^{-1}, z, a) dw \ , \tag{3.13}$$

$$(|\arg(w)| < \pi) .$$

Furthermore , in its limit case when ρ→0, (3.12) would simplify immediately to the following formula :

$$\Phi^{(0,\sigma)}_{\mu,\upsilon,\lambda}(x,z,a) = \frac{(\lambda)^{-z}\Gamma(\nu)}{2\pi i} \int_{-\infty}^{(0+)} w^{-\nu} e^w \Phi\left(xw^{-\sigma}, z, \frac{a}{\lambda}\right) dw \ , \tag{3.14}$$

$$(\nu \in \mathbb{C}/\mathbb{Z}_0^-, \sigma \in \mathbb{R}^+, |\arg(w)| \le \pi) .$$

If we replace the Hankel's loop in (3.12) to (3.14) by a suitable Mellin – Barnes contour integral we get :

Theorem 5 : Let $\mathrm{Re}(\nu) > 0, \sigma, \tau \in \mathbb{R}^+$.Then

$$\Phi^{(\rho,\sigma)}_{\mu,\nu,\lambda}(x,z,a) = \frac{(\lambda)^{-z}\Gamma(\nu)}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} w^{-\nu} e^w \Phi^{(\rho,0)}_{\mu,\nu}\left(xw^{-\sigma}, z, \frac{a}{\lambda}\right) dw \ , \tag{3.15}$$

$$\Phi^*_\mu(x, z, a) = \frac{1}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} w^{-1} e^w \Phi^{(1,0)}_{\mu,1}(xw^{-1}, z, a) dw \ , \tag{3.16}$$

and

$$\Phi^{(0,\sigma)}_{\mu,\nu,\lambda}(x,z,a) = \frac{(\lambda)^{-z}\Gamma(\nu)}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} w^{-\nu} e^w \Phi\left(xw^{-\sigma}, z, \frac{a}{\lambda}\right) dw \ . \tag{3.17}$$

Proof : Denote, for convenience, the right-hand side of formula (3.15) by I . Then it is easily seen that

$$I=\frac{(\lambda)^{-z}\,\Gamma(\nu)}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} w^{-\nu}e^{w}\,\Phi_{\mu,\nu}^{(\rho,0)}\left(xw^{-\sigma},z,\frac{a}{\lambda}\right)dw\ .$$

Now, in view of integral formula[23]

$$\Phi_{\mu,\nu}^{(\rho,\sigma)}(x,z,a)=\frac{\Gamma(\nu)}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} w^{-\nu}e^{w}\,\Phi_{\mu,\nu}^{(\rho,0)}(xw^{-\sigma},z,a)dw\ ,$$

$$(\mathrm{Re}(\nu)>0,\sigma,\tau\in\mathbb{R}^{+}),$$

and definition (2.1), we are led finally to the left-hand side of formula (3.15).

In the same manner one can derive formulas (3.16) and (3.17).

4. Fractional Derivative

We recall the Riemann - Liouville fractional derivative operator D_x^{μ} [2,p.181] and [4]

$$D_x^{\mu}\ f(x)\ =\begin{cases}\dfrac{1}{\Gamma(-\mu)}\displaystyle\int_0^x (x-t)^{-\mu-1}f(t)dt,(\mathrm{Re}(\mu)<0)\\[2ex] \dfrac{d^m}{dx^m}D_x^{\mu-m}\ f(x)\quad,\quad m-1\le \mathrm{Re}(\mu)<m(m\in N)\end{cases} \tag{4. 1}$$

Clearly , we have :

$$D_x^{\mu}\{x^{\alpha}\}=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\mu+1)}x^{\alpha-\mu},\qquad (\mathrm{Re}(\alpha)>-1,\alpha-\mu\ge 0)\,. \tag{4.2}$$

It is easy to derive the following results.

Theorem 5: $Let\ \mathrm{Re}(\mu)>0,\ \sigma\in\mathbb{R}^{+}.Then$

$$D_x^{\mu-\nu}\left\{x^{\mu-1}(\lambda)^{-z}\Phi\left(x^{\sigma},z,\frac{a}{\lambda}\right)\right\}=\frac{\Gamma(\mu)}{\Gamma(\nu)}\ \ x^{\nu-1}\Phi_{\mu,\nu,\lambda}^{(\sigma,\sigma)}(x^{\sigma},z,a)\ \ . \tag{4.3}$$

Proof: Denote, for convenience, the left-hand side of formula (4.3) by F^{l} . Then in view of (1.3),it is easily seen that

$$F^l = D_x^{\mu-\nu}\left\{(\lambda)^{-z}\sum_{n=0}^{\infty}\frac{x^{\mu+\sigma n-1}}{(a+n)^z}\right\}.$$

Upon using the relation (4.2), we are led finally to right -hand side of formula (4.3).

In particular , when $\lambda = 1$, (4.3) would reduce to the following result :

$$D_x^{\mu-\nu} \quad x^{\mu-1}\Phi(x^\sigma, z, a) \;= \frac{\Gamma(\mu)}{\Gamma(\nu)} \;\; x^{\nu-1}\Phi_{\mu,\nu}^{(\sigma,\sigma)}(x^\sigma, z, a) \;\; . \tag{4.4}$$

On putting $\sigma = \nu = \lambda = 1$ in (4.3) , formula (4.3) reduces to the

interesting result

$$\Phi_\mu^*(x\,, z\,, a) \;= \frac{1}{\Gamma(\mu)} D_x^{\mu-1} \quad x^{\mu-1}\Phi(x, z, a) \;\; . \tag{4.5}$$

When $\nu = \lambda = 1$ in (4.3) , we arrive to the following interesting result

$$\Phi_{\mu,1,1}^{(\sigma,\sigma)}(x^\sigma, z, a) \;= \frac{1}{\Gamma(\mu)} D_x^{\mu-1} \quad x^{\mu-1}\Phi(x^\sigma, z, a) \;\; . \tag{4.6}$$

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