ON HYPERGEOMETRIC SERIES ASSOCIATED WITH THE GENERALIZED ZETA FUNCTIONS

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Abstract

In this work we will introduce and study a generalized zeta function of the general Hurwitz – Lerch zeta functions. Here, we aim to derive basic properties of generalized zeta function for the general Hurwitz – Lerch zeta functions include some integral representations for several general Hurwitz – Lerch zeta functions and fractional derivative. A number of known and new results which introduced are generalization of a known results introduced by Shy-Der and Srivastava [6].

Key words: Hurwitz-Lerch zeta functions; Mellin-Barnes integrals; hypergeometric functions; hypergeometric-type generating functions.
1. Introduction

For $\text{Re}(z) > 1$, the Riemann zeta function $\zeta(z)$ is defined as follows [1,p.88]

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \quad (1.1)$$

which can, except for a simple pole at $z = 1$ with its residue 1, be continued moromorphically to the whole complex $z$-plane. The Hurwitz (or generalized) zeta function denoted usually by $\zeta(z,a)$ and so for $\text{Re}(z) > 1$,

$$\zeta(z,a) = \sum_{n=0}^{\infty} \frac{1}{(a+n)^z}, \quad (1.2)$$

can just as $\zeta(z)$ be continued moromorphically to the whole complex $z$-plane except for a simple pole at $z = 1$ with its residue 1. When $a = 1$, (1.2) reduces to (1.1). In terms of Hurwitz-Lerch zeta function $\Phi(x,z,a)$ defined by [5,p.121] and [1]:

$$\Phi(x,z,a) = \sum_{n=0}^{\infty} \frac{x^n}{(a+n)^z}, |x| < 1, a \neq 0, -1, -2, .... \quad (1.3)$$

Equivalently, it has the integral expression

$$\Phi(x,z,a) = \frac{1}{\Gamma(z)} \int_{0}^{\infty} t^{z-1} e^{-at} (1-xe^{-t})^{-1} dt, \quad (1.4)$$

provided $\text{Re}(a) > 0$ and either $|x| \leq 1$ and $\text{Re}(z) > 0$ or $x = 1$ and $\text{Re}(z) > 1$.

So that obviously

$$\Phi(1,z,a) = \zeta(z,a). \quad (1.5)$$

The zeta function in (1.3) has since been extended by Goyal and Laddha [3,p.100(1.5)], in the form:

$$\Phi^\mu(x,z,a) = \sum_{n=0}^{\infty} \frac{\mu}{(a+n)^z} \frac{x^n}{n!}, \quad (1.6)$$

where $|x| < 1, \text{Re}(a) > 0, \mu \geq 1$.

Equivalently, it has the integral expression:
\[ \Phi^*(x, z, a) = \frac{1}{\Gamma(z)} \int_{0}^{\infty} t^{-1} e^{-at} (1-xe^{-t})^{-\mu} dt, \] 

(1.7)

provided that \( \mu \geq 1, \text{Re}(a) > 0, |x| \leq 1 \) and either \( \text{Re}(z) > 0 \) or \( \text{Re}(z) > \text{Re}(\mu) \) according to \( x \neq 1 \) or \( x = 1 \).

In [6] Shy-Der and Srivastava introduced the general family of the Hurwitz-Lerch zeta function in the form:

\[ \Phi^{(\rho, \sigma)}_{\mu, \nu}(x, z, a) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}} \frac{x^n}{(a+n)^{\mu}}, \]

(1.8)

where \( \mu \in \mathbb{C}, \nu \in \mathbb{C} \setminus \mathbb{Z}^{-}, \rho, \sigma \in \mathbb{R}^{+}, \rho > \sigma \) when \( x, z \in \mathbb{C}, \rho = \sigma, z \in \mathbb{C} \) \( |x| < 1, \rho = \sigma \)

and \( \text{Re}(z - \mu + \nu) > 1 \) when \( |x| = 1 \).

Equivalently, it has the following integral expressions:

\[ \Phi^{(\rho, \sigma)}_{\mu, \nu}(x, z, a) = \frac{1}{\Gamma(z)} \int_{0}^{\infty} t^{-1} e^{-at} \left[ x^t e^{-t} \right]^{(\mu, \rho), (1; 1)}_{(\nu, \sigma)} dt. \]

(1.9)

In this paper we will introduce and study a generalized zeta function of the general Hurwitz – Lerch zeta function \( \Phi^{(\rho, \sigma)}_{\mu, \nu}(x, z, a) \). This work aims to derive basic properties of generalized zeta function for the general Hurwitz – Lerch zeta functions include some integral representations for several general Hurwitz – Lerch zeta functions and fractional derivative. A number of known and new results which introduced are generalization of a known results introduced in [6].

In section 2 we will introduce analogous definition of the general family of Hurwitz-Lerch zeta function denoted by \( \Phi^{(\rho, \sigma)}_{\mu, \nu, \lambda}(x, z, a) \) with a known special cases. Some integral representation for \( \Phi^{(\rho, \sigma)}_{\mu, \nu, \lambda}(x, z, a) \) are discussed in section 3 and fractional derivative in section 4.
2. The Function $\Phi_{\mu,\nu,\lambda}^{(\rho, \sigma)}(x, z, a)$

In this section we will give analogous definition of the function $\Phi_{\mu,\nu}^{(\rho, \sigma)}(x, z, a)$ which is defined by (1.3) in its form:

$$
\Phi_{\mu,\nu,\lambda}^{(\rho, \sigma)}(x, z, a) = (\lambda)^{-1/2} \Phi_{\mu,\nu}^{(\rho, \sigma)} \left( x, z, \frac{a}{\lambda} \right) = \sum_{n=0}^{\infty} \frac{(\mu)_n}{(\nu)_n} \frac{x^n}{(a + \lambda n)^{\mu}}.
$$

(2.1)

$$
\mu \in \mathbb{C}, \lambda, \nu \in \mathbb{C} \setminus \mathbb{Z}, \rho, \sigma \in \mathbb{R}^+, \rho > \sigma \text{ when } x, z \in \mathbb{C}, \rho = \sigma \text{ and when } |x| < 1, \rho = \sigma \text{ and } \text{Re}(z - \mu + \nu) > 1 \text{ when } |x| = 1.
$$

Clearly, we have

$$
\Phi_{\mu,\nu,1}^{(\rho, \sigma)}(x, z, a) = \Phi_{\mu,\nu,1}^{(0, 0)}(x, z, a) = \Phi(x, z, a),
$$

(2.2)

$$
\Phi_{\mu,1}^{(1,1)}(x, z, a) = \Phi(x, z, a).
$$

(2.3)

Obviously, when $\lambda = 1$, (2.1) reduce to (1.8). Further, on putting $x = a = 1$ in (2.2), it reduces to (1.1). Next, on putting $x = 1$ and $\mu = 1$ in (2.3), it reduces to (1.2).

3. Integral representations

By using the Fox-Wright extension of the generalized hypergeometric $\,_{p}F_{q}$ function with $p$ numerator and $q$ denominator parameters, defined by:

$$
\,_{p}\Psi_{q} \left[ \begin{array}{c}
(\alpha_1; A_1), (\alpha_2; A_2), ..., (\alpha_p; A_p); \\
(\beta_1; B_1), (\beta_2; B_2), ..., (\beta_q; B_q);
\end{array} \right] z^n = \sum_{n=0}^{\infty} \prod_{j=1}^{p} (\alpha_j)_{A_j} \frac{z^n}{n!},
$$

where the argument $z$, the complex parameters $\alpha_j (j = 1, ..., p)$ and $\beta_j (j = 1, ..., q)$, and the positive real parameters $A_j (j = 1, ..., p)$ and $B_j (j = 1, ..., q)$ are so constrained
that the series in (3.1) converges absolutely [1,p.183] and Eulerian integral formula of second kind [1]:

\[(n+a)^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty e^{-t-a} t^{z-1} e^{-t} e^{-\frac{a}{t}} dt, (\min \{\text{Re}(z), \text{Re}(n+a)\} > 0).\]

it is easy to derive the following integral representation.

**Theorem 1:** Let \(\text{Re}(a) > 0, \text{Re}(z) > 0\) when \(|x| < 1(x \neq 1)\) \(\text{Re}(z) > 1\) when

\[x = 1.\text{Then}
\]

\[\Phi_{\mu,\nu,\lambda}^{(\rho,\sigma)}(x, z, a) = (\lambda)^{-z} \int_0^\infty \frac{e^{-a}}{\Gamma(z)} t^{z-1} e^{-\frac{a}{t}} \left[ (\mu, \rho), (1;1); xe^{-t} \right] dt. \tag{3.1} \]

**Proof:** By starting from the left-hand side of formula (3.1), then in view of definition (2.1), it is easily seen that

\[\Phi_{\mu,\nu,\lambda}^{(\rho,\sigma)}(x, z, a) = (\lambda)^{-z} \Phi_{\mu,\nu}^{(\rho,\sigma)} \left\{ x, z, \frac{a}{\lambda} \right\}.\]

Upon using the integral formula (1.9), we are led finally to the right-hand side of formula (3.1).

In terms of the Gauss hypergeometric function \(_2F_1\) (see [7]):

\[_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, c \neq 0, -1, -2, \ldots,\]

a special cases of (3.1), occur when \(\rho = \sigma = \lambda = 1\) as follows:

\[\Phi_{\mu,\nu,1}^{(1,1)}(x, z, a) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{e^{-a}}{\Gamma(z)} t^{z-1} e^{-\frac{a}{t}} \left[ \mu, 1; xe^{-t} \right] dt. \tag{3.2} \]

By using the familiar Euler transformation formula [1,p.64]

\[_2F_1(\alpha, \beta; \gamma; x) = (1-x)^{\gamma-\alpha-\beta} _2F_1(\gamma - \alpha, \gamma - \beta; \gamma; x), \]

\(|\arg(1-x)| \leq \pi - \varepsilon, (0 < \varepsilon < \pi), \gamma \notin \mathbb{Z}^+),\]

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we get equivalently integral formula for (3.2) in form:

\[
\Phi_{\mu,\nu,\lambda}(x, z, a) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1}e^{-at}(1-axe^{-t})^{\nu-1}F_1\left[\nu-\mu,\nu-1;axe^{-t}\right]dt.
\] (3.3)

For \(\nu = 1\), (3.3) reduces to the well-known result of Goyal and Laddha [3]

\[
\Phi_{\mu,\nu,\lambda}(x, z, a) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1}e^{-at}(1-axe^{-t})^{-\mu}dt = \Phi_{\mu}^*(x, z, a).
\] (3.4)

**Theorem 2**: Let \(\text{Re}(\mu) > 0, \rho \in \mathbb{R}^+\). Then

\[
\Phi_{\rho,\sigma,\lambda}(x, z, a) = \frac{(\lambda)^{\rho}}{\Gamma(\mu)} \int_0^\infty t^{\rho-1}e^{-t} \Phi_{\mu,\nu}^{(0,\sigma)}\left(x t^{\rho}, z, \frac{a}{\lambda}\right)dt,
\] (3.5)

and

\[
\Phi_{\mu,\nu,\lambda}(x, z, a) = \frac{(\lambda)^{-z}}{\Gamma(\mu)} \int_0^\infty (\log t)^{\mu-1} \Phi_{\mu,\nu}^{(0,\sigma)}\left(x (\log t)^\rho, z, \frac{a}{\lambda}\right)dt.
\] (3.6)

**Proof**: Denote, for convenience, the right-hand side of formula (3.5) by \(I\). Then it is easily seen that

\[
I = (\lambda)^{z} \frac{1}{\Gamma(\mu)} \int_0^\infty t^{\mu+1}e^{-t} \Phi_{\mu,\nu}^{(0,\sigma)}\left(x t^{\rho}, z, \frac{a}{\lambda}\right)dt.
\]

\[
= \sum_{n=0}^\infty \frac{(\mu_{\mu,\nu})x^n}{\Gamma(\mu)(\nu)_n(a+\lambda n)} \int_0^\infty t^{\mu+\rho-1}e^{-t}dt.
\]

Upon using the Hankel's formula [6, p.13(2)]

\[
\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{C} e^{zt} \, dt,
\] (3.7)

and definition (2.1), we are led finally to the left-hand side of formula (3.5).

In the same manner one can derive the formula (3.6).

In their special cases, when \(\rho = \sigma = \nu = \lambda = 1\), the integral representations (3.5) and (3.6) reduce to the following forms:

\[
\Phi_{\mu}^*(x, z, a) = \frac{1}{\Gamma(\mu)} \int_0^\infty t^{\mu-1}e^{-t} \Phi_{\mu,1}^{(0,3)}(xt, z, a)dt,
\] (3.8)
and
\[ \Phi^*_\mu(x, z, a) = \frac{1}{\Gamma(\mu)} \int_0^1 (\log \frac{1}{t})^{\mu-1} \Phi_{\mu,1}^{(0,1)} \left( x \left( \log \frac{1}{t} \right), z, a \right) dt , \]  
(3.9)
respectively.

**Theorem 3:** Let \( 0 < \text{Re}(\mu) < \text{Re}(\nu) , \sigma \in \mathbb{R}^+ \). Then
\[ \Phi_{\mu,\nu,\lambda}^{(\sigma,\sigma)}(x, z, a) = \frac{(\lambda)^{-\gamma} \Gamma(\nu)}{\Gamma(\nu - \mu) \Gamma(\mu)} \int_0^\infty t^{\mu-1} \Phi \left( \frac{x t^{\sigma}, a}{\lambda}, z \right) dt . \]  
(3.10)

**Proof:** Denote, for convenience, the right-hand side of formula (3.10) by \( I \). Then it is easily seen that
\[ I = (\lambda)^{-\gamma} \frac{\Gamma(\nu)}{\Gamma(\nu - \mu) \Gamma(\mu)} \int_0^\infty t^{\mu-1} \Phi \left( \frac{x t^{\sigma}}{(1+t)^\sigma}, \frac{a}{\lambda} \right) dt . \]

Now, in view of integral formula
\[ \Phi_{\mu,\nu}^{(\rho,\sigma)}(x, z, a) = \frac{\Gamma(\nu)}{\Gamma(\nu - \mu) \Gamma(\mu)} \int_0^\infty t^{\mu-1} \Phi \left( \frac{x t^{\rho}}{(1+t)^\rho}, \frac{z, a}{\lambda} \right) dt , \]

(0 < \text{Re}(\mu) < \text{Re}(\nu), \sigma \in \mathbb{R}^+) ,

and the definition (2.1), we obtain the left-hand side of formula (3.10).

Obviously, for \( \sigma = \nu = \lambda = 1 \), (3.10) reduces to the following result :
\[ \Phi^*_\mu(x, z, a) = \frac{\sin(\pi \mu)}{\pi} \int_0^\infty t^{\mu-1} \Phi \left( \frac{x t}{1+t}, z, a \right) dt . \]  
(3.11)

Furthermore we derive the following result :

**Theorem 4:** Let \( \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-, \sigma \in \mathbb{R}^+ , |\text{arg}(w)| < \pi \). Then
\[ \Phi_{\mu,\nu}^{(\rho,\sigma)}(x, z, a) = \frac{(\lambda)^{-\gamma} \Gamma(\nu)}{2\pi i} \int_{-\infty}^{+\infty} w^{-\gamma} e^{\pi i} \Phi_{\mu,\nu}^{(\rho,0)} \left( \frac{x w^{-\sigma}, a}{\lambda} \right) dw . \]  
(3.12)

**Proof:** Denote, for convenience, the right-hand side of formula (3.12) by \( I \). Then in
view of (1.8), it is easily seen that

\[ I = \sum_{n=0}^{\infty} \frac{(\mu)_n}{(a + \lambda n)^{2n}} \frac{\Gamma(\nu)}{2\pi i} \int_{-\infty}^{0+} w^{-(\nu + \sigma n)} e^{w} dw . \]

Upon using the formula (3.7) and definition (2.1), we are led finally to left-hand side of formula (3.12).

Obviously, for \( \rho = \sigma = \nu = \lambda = 1 \), (3.12) reduces to the following result:

\[ \Phi^*_{\mu}(x, z, a) = \frac{1}{2\pi i} \int_{-\infty}^{0+} w^{-1} e^{w} \Phi^{(1,0)}_{\mu,1}(xw^{-1}, z, a) dw , \]  

\(|\text{arg}(w)| < \pi| \).  

Furthermore, in its limit case when \( \rho \to 0 \), (3.12) would simplify immediately to the following formula:

\[ \Phi^{(\rho,\sigma)}_{\mu,\nu,\lambda}(x, z, a) = \frac{(\lambda)^{1-\sigma} \Gamma(\nu)}{2\pi i} \int_{-\infty}^{0+} w^{-\nu} e^{w} \Phi^{(\rho,0)}_{\mu,\nu,\lambda}(xw^{-1}, z, a) dw , \]  

\(|\text{arg}(w)| \leq \pi| \).

If we replace the Hankel’s loop in (3.12) to (3.14) by a suitable Mellin – Barnes contour integral we get:

**Theorem 5**: Let \( \text{Re}(\nu) > 0, \sigma, \tau \in \mathbb{R}^+ \). Then

\[ \Phi^{(\rho,\sigma)}_{\mu,\nu,\lambda}(x, z, a) = \frac{(\lambda)^{1-\sigma} \Gamma(\nu)}{2\pi i} \int_{-\tau-i\infty}^{\tau+i\infty} w^{-\nu} e^{w} \Phi^{(\rho,0)}_{\mu,\nu,\lambda}(xw^{-1}, z, a) dw , \]  

\[ \Phi^*_{\mu}(x, z, a) = \frac{1}{2\pi i} \int_{-\tau-i\infty}^{\tau+i\infty} w^{-1} e^{w} \Phi^{(1,0)}_{\mu,\nu,\lambda}(xw^{-1}, z, a) dw , \]  

and

\[ \Phi^{(\rho,\sigma)}_{\mu,\nu,\lambda}(x, z, a) = \frac{(\lambda)^{1-\sigma} \Gamma(\nu)}{2\pi i} \int_{-\tau-i\infty}^{\tau+i\infty} w^{-\nu} e^{w} \Phi^{(\rho,0)}_{\mu,\nu,\lambda}(xw^{-1}, z, a) dw . \]

**Proof**: Denote, for convenience, the right-hand side of formula (3.15) by \( I \). Then it is easily seen that
\[ I = \frac{(\lambda)\Gamma(\nu)}{2\pi i} \int_{t+i\infty}^{t+i\infty} w^{-\nu} e^{\nu} \Phi^{(\rho,0)}_{\mu,\nu}(xw^{-\sigma},z,\frac{a}{\lambda}) dw . \]

Now, in view of integral formula [23]

\[ \Phi^{(\rho,\sigma)}_{\mu,\nu}(x,z,a) = \frac{\Gamma(\nu)}{2\pi i} \int_{t+i\infty}^{t+i\infty} w^{-\nu} e^{\nu} \Phi^{(\rho,0)}_{\mu,\nu}(xw^{-\sigma},z,a) dw , \]

\[ (\text{Re}(\nu) > 0, \sigma, \tau \in \mathbb{R}^+) , \]

and definition (2.1), we are led finally to the left-hand side of formula (3.15).

In the same manner one can derive formulas (3.16) and (3.17).

4. Fractional Derivative

We recall the Riemann - Liouville fractional derivative operator \( D^\mu_x \) [2,p.181] and [4]

\[
D^\mu_x f(x) = \begin{cases} 
\frac{1}{\Gamma(-\mu)} \int_0^x (x-t)^{-\mu-1} f(t) dt, & (\text{Re}(\mu) < 0) \\
\frac{d^m}{dx^m} D^\mu_x f(x), & m - 1 \leq \text{Re}(\mu) < m (m \in \mathbb{N})
\end{cases}
\]

Clearly, we have:

\[
D^\mu_x \{x^\alpha\} = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - \mu + 1)} x^{\alpha-\mu}, \quad (\text{Re}(\alpha) > -1, \alpha - \mu \geq 0). \quad (4.2)
\]

It is easy to derive the following results.

**Theorem 5:** Let \( \text{Re}(\mu) > 0, \ \sigma \in \mathbb{R}^+ \). Then

\[
D^\mu_{x^\nu} \left\{ x^{\nu-1}(\lambda)^{-\nu} \Phi \left(x^\alpha, z, \frac{a}{\lambda}\right) \right\} = \frac{\Gamma(\mu)}{\Gamma(\nu)} x^{\nu-1} \Phi^{(\sigma,\alpha)}_{\mu,\nu,\lambda}(x^\alpha, z, a). \quad (4.3)
\]

**Proof:** Denote, for convenience, the left-hand side of formula (4.3) by \( F^\nu \). Then in view of (1.3), it is easily seen that
\[ F^l = D_x^{\mu - \nu} \left\{ (\lambda)^{-\nu} \sum_{n=0}^{\infty} \frac{x^{\mu + \alpha n - 1}}{(a + n)^{\nu}} \right\}. \]

Upon using the relation (4.2), we are led finally to right-hand side of formula (4.3).

In particular, when \( \lambda = 1 \), (4.3) would reduce to the following result:

\[ D_x^{\mu - \nu} x^{\mu - 1} \Phi(x^\sigma, z, a) = \frac{\Gamma(\mu)}{\Gamma(\nu)} x^{\mu - 1} \Phi_{\mu, \nu}(x^\sigma, z, a). \quad (4.4) \]

On putting \( \sigma = \nu = \lambda = 1 \) in (4.3), formula (4.3) reduces to the interesting result

\[ \Phi^\mu(x, z, a) = \frac{1}{\Gamma(\mu)} D_x^{\mu - 1} x^{\mu - 1} \Phi(x, z, a). \quad (4.5) \]

When \( \nu = \lambda = 1 \) in (4.3), we arrive to the following interesting result

\[ \Phi_{\mu, 1, 1}^{(\sigma, \alpha)}(x^\sigma, z, a) = \frac{1}{\Gamma(\mu)} D_x^{\mu - 1} x^{\mu - 1} \Phi(x^\sigma, z, a). \quad (4.6) \]

**REFERENCES**


