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Initial-boundary Value Problems to the One-dimensional Compressible Navier-Stokes-Poisson Equations with viscosity and heat conductivity coefficients

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Abstract

In this paper, the global, non-vacuum solutions with large amplitude to the initial-boundary value problem of the one-dimensional compressible Navier-Stokes-Poisson system with viscosity and heat conductivity coefficients are considered. The proof is based on the analysis on the positive lower and upper bounds on the specific volume and the absolute temperature.

Key words: compressible Navier-Stokes-Poisson system; global, non-vacuum solutions with large amplitude; viscosity and heat conductivity coefficients

1. Introduction

Magnetohydrodynamics, which combines the environmental fluid mechanics and electrodynamics theories to study the interaction discipline between the conduction fluids and electromagnetic, is the theory of the macroscopic, and it has spanned a very large range of applications[1]-{3}. The motion of compressible, viscous self-gravitating fluids can be expressed by Navier-Stokes-Poisson equations. In this paper, we consider the one-dimensional compressible Navier-Stokes-Poisson system with viscous coefficient and heat conductivity:

$$
\begin{cases}
v_t = u_x, \\
u_t - \sigma_x - \dfrac{\Phi_x}{v} = 0, \\
\left(e + \dfrac{u^2}{2}\right)_t - \left(\sigma u - q\right)_x = 0, \\
\left(\dfrac{\Phi_x}{v}\right)_x = 1 - v.
\end{cases}
\tag{1}
$$

Where v, u, σ, e, q and Φ denote the specific volume, velocity, stress, internal energy, heat flux, and the electrostatic potential function, respectively. For Newtonian fluid and Fourier's law

$$
\sigma(v,\theta,u_x) = -p(v,\theta) + \frac{\mu(v,\theta)}{v} u_x \ , \ q(v,\theta,u_x) = -\frac{\kappa(v,\theta)}{v}\theta_x \ . \tag{2}
$$

Here, p and θ are the pressure and the absolute temperature respectively. $\mu(v,\theta) > 0$ denotes the viscosity coefficient and $\kappa(v,\theta) > 0$ is heat conductivity coefficient.

Physically, the Navier–Stokes–Poisson equations describe the motion of compressible viscous isentropic gas flow under the self-gravitational force. Compressible Navier-Stokes type equation with density and temperature dependent transport coefficients arise in many applied sciences, such as certain class of solid-like materials [4], gases at very high temperatures [5], etc. Recently, there are many studies on the non-vacuum solutions to the one-dimensional compressible Navier-Stokes equations with density and temperature dependent transportation coefficients in various forms, cf. [6, 7, 8, 9, 10, 11, 12]. However, there is a gap between the physical models and the existence theory.

This manuscript is concerned with the construction of global, non-vacuum, large, smooth solutions to the one-dimensional compressible Navier-Stokes equation (1) in the domain $\{(x,t) \mid x \in I = [0,1], t \geq 0\}$ with prescribed initial condition

$$(v(x,0), v(x,0), \theta(x,0), \Phi_x(x,0)) = (v_0(x), u_0(x), \theta_0(x), \Phi_{0x}(x)), \quad x \in [0,1] \tag{3}$$

and boundary condition

$$\begin{cases} \sigma(0,t) = \sigma(1,t) = 0, \\ q(0,t) = q(1,t) = 0, \\ \Phi_x(0,t) = \Phi_x(1,t) = 0. \end{cases} \tag{4}$$

Throughout this manuscript, we will focus on the ideal, polytrophic gases which contain the case of gases for which kinetic theory provides constitutive relations, cf. [13, 14]

$$e = C_v\theta = \frac{R\theta}{\gamma-1}, p(v,\theta) = \frac{R\theta}{v} = Av^{-\gamma}\exp\left(-\frac{\gamma-1}{R}s\right) \tag{5}$$

where the specific gas constant R and the specific heat at constant volume C_v are positive constant and $\gamma > 1$ is the adiabatic constant. And our main interest concerns the case when the transport coefficients μ and κ may depend on the specific volume and/or the absolute temperature which are degenerate in the sense that μ and /or κ are not uniformly bounded from below or above some positive constants for all $v > 0$ and $\theta > 0$.

2. Results

To simplify the presentation, the result is concerned with the case

$$\mu(v) = v^{-a}, \kappa(v,\theta) = \theta^b, \tag{6}$$

Now we turn to state the main results obtained in this paper. The transport coefficients μ and κ are assumed to satisfy the following condition:

- μ is a positive constant and $\kappa(v,\theta)$ is a smooth function of v and θ satisfying $\kappa(v,\theta)>0$

for $v>0,\theta>0$ and there exist positive constants μ_0 and $\mathrm{K}(\tilde{v},\tilde{\theta})$ such that

$$\mu(v,\theta)=\mu_0>0,\ \min_{v\geq\tilde{v}>0,\theta\geq\tilde{\theta}>0}\kappa(v,\theta)=\mathrm{K}(\tilde{v},\tilde{\theta})>0 \tag{7}$$

hold true for each given positive constants $\tilde{v}>0$ and $\tilde{\theta}>0$,which is stated as follows.

Theorem 1 suppose

(i) $\left(v_0(x),u_0(x),\theta_0(x)\right)\in H^1(I)$;

(ii) $\inf\limits_{x\in I} v_0(x)>0,\inf\limits_{x\in I}\theta_0(x)>0$, the initial data $\left(v_0(x),u_0(x),\theta_0(x)\right)$ are compatible with the boundary condition (4);

(iii) μ and κ are assumed to satisfies one of the following two conditions

- μ is a positive constant and κ are assumed to satisfies $\kappa(v,\theta)>0$ for $v>0,\theta>0$ and

$$0\leq\kappa(v,\theta)\leq C(V)(1+\theta^c),\quad 0<V^{-1}\leq v\leq V \tag{8}$$

hold for some positive constant $C(V)>0$ and $\theta>0$ sufficiently large. Here $0\leq c<1$

is constant and $V>0$ is any given positive constant;

- μ and κ are given by (6) with a and b satisfying

$$0\leq a<\frac{1}{5},\ \ b\geq 2. \tag{9}$$

Then the initial-boundary value problem (1), (3), (4) exists a unique global solution $\left(v(x),u(x),\theta(x)\right)$ which satisfies

$$\big(v(x,t),u(x,t),\theta(x,t)\big)\in C^0(0,T;H^1(I)),$$

$$\big(u_x(x,t),\theta_x(x,t)\big)\in L^2(0,T;H^1(I)), \tag{10}$$

$$\underline{V}\le v\le \bar{V},\underline{\Theta}\le\theta\le\bar{\Theta},\forall(x,t)\in I\times[0,T].$$

Here $T>0$ is any given positive constant and $\underline{V},\bar{V},\underline{\Theta},\bar{\Theta}$ are some positive constants which may depend on T and the initial data $\big(v_0(x),u_0(x),\theta_0(x),\Phi_0(x)\big)$.

Remark The initial-boundary value problem (1), (3), (4) has also been studied in []. To deduced the desired lower and upper bound on the specific volume v, the viscosity coefficient $\mu(v)$ is assumed to satisfy

$$0<\mu_0\le\mu(v)\le\mu_1$$

and the entropy $s(v,\theta)$ and the internal energy $e(v,\theta)$ are assumed to satisfy

$$s(v,\theta)\le\left(\left|\int_0^v\frac{\mu(z)}{z}dz\right|^r+1\right)e(v,\theta) \tag{11}$$

in []. Here $\mu_0,\mu_1,$ and $r<2$ are positive constant. For the ideal polytrophic gas, if the transport coefficients μ and κ are assumed to satisfy (6), (11) holds only if $a=0$.

Notations: Throughout this manuscript, $C>1$ stands for a generic positive constant which may depend on $\inf_{x\in I}v_0(x),\inf_{x\in I}\theta_0(x),T,$ and $\left\|\big(v_0,u_0,\theta_0\big)\right\|_{H^1(I)}$. Note that all these constant may vary in different places. $H^s(I)$ represents the usual Sobolev spaces on I with the standard norm $\|\cdot\|_{H^s(I)}$ and for $1\le p\le+\infty, L^p(\mathrm{I})$ denotes the usual L^p spaces equipped with the usual norm $\|\cdot\|_{L^p(I)}$. For simplicity, we use $\|\cdot\|_\infty$ to denote the norm in $L^\infty(I\times[0,T])$.

3. Methods

Proof the Theorem 1

This section is devoted to the proof of Theorem 1 based on the continuation argument. Since the local solvability of the initial-boundary value problem is well-established [15], if we suppose that the local solution $\left(v(x,t),u(x,t),\theta(x,t)\right)$ to the initial-boundary value problem (1), (3), (4) has been extended to the time step $t=T>0$ for some $T>0$, then to extend such a solution $\left(v(x,t),u(x,t),\theta(x,t)\right)$ step by step to a global one.

Lemma 1 (Basic energy estimates) Let the conditions in Lemma 1 hold and suppose that the local solution $\left(v(x,t),u(x,t),\theta(x,t),\Phi(x,t)\right)$ constructed in Lemma 1 satisfies the a priori assumption (H), then we have for $0\le t\le T$ that

$$\int_0^1(\eta(v,u,\theta)+\frac{1}{2}\left(\frac{\Phi_x}{v}\right)^2)dx+\int_0^t\int_0^1(\frac{\mu(v)u_x^2}{v\theta}+\frac{\kappa(v,\theta)\theta_x^2}{v\theta^2})dxds$$

$$=\int_0^1(\eta(v_0,u_{0,}\theta_0)+\frac{1}{2}\left(\frac{\Phi_{0x}}{v_0}\right)^2)dx\,. \tag{12}$$

Lemma 2 (Estimate on the total energy) Let the conditions stated in Theorem 1 hold and suppose that $(v(x,t),u(x,t),\theta(x,t))$ is a solution to the initial-boundary value problem (1), (2), (4) defined on $I\times[0,T]$ for some $T>0$. If we assume further that $(v(x,t),u(x,t),\theta(x,t))$ satisfies the a priori assumption (H), then we have for $0\le t\le T$ that

$$\int_0^1\left(C_v\theta+\frac{u^2}{2}\right)dx=\int_0^1\left(C_v\theta_0+\frac{u_0^2}{2}\right)dx \tag{13}$$

First we consider the case when the transport coefficients μ and κ satisfy (6) and (7).

Lemma 3 Under the condition listed in Lemma 1, we have

$$\frac{1}{\theta(x,t)}\le C+C\left\|\frac{1}{\mu(v)v}\right\|_\infty,\forall(x,t)\in I\times[0,T]\,. \tag{14}$$

Proof: First of all, $(1)_3$ implies

$$C_v\left(\frac{1}{\theta}\right)_t=-\frac{\mu(v)u_x^2}{v\theta^2}+\frac{Ru_x}{v\theta}-\frac{1}{\theta^2}\left(\frac{\kappa(v,\theta)\theta_x}{v}\right)_x . \tag{15}$$

From (15), we can get for each $p>1$ that

$$\begin{aligned}&C_v\left[\left(\frac{1}{\theta}\right)^{2p}\right]_t+\frac{2p(2p+1)\kappa(v,\theta)\theta_x^2}{v\theta^{2p+2}}\\&=-2p\left(\frac{1}{\theta}\right)^{2p-1}\left[\frac{\mu(v)}{v}\left(\frac{u_x}{\theta}-\frac{R}{2\mu(v)}\right)-\frac{R^2}{2\mu(v)}\right]-\left(\frac{2p\kappa(v,\theta)\theta_x}{v\theta^{2p+1}}\right)_x .\end{aligned} \tag{16}$$

Integrating (16) with respect to *x* over *I*, we have

$$C_v\left[\left(\frac{1}{\theta}\right)^{2p}\right]_t\le 2p\int_0^1\frac{R^2}{4\mu(v)v}\left(\frac{1}{\theta}\right)^{2p-1}dx\le 2pC\left\|\frac{1}{\mu(v)v}\right\|_{L^{2p}}\left\|\frac{1}{\theta}\right\|_{L^{2p}}^{2p-1}, \tag{17}$$

which implies

$$\left\|\frac{1}{\theta}\right\|_{L^{2p}}\le C(\inf_{x\in I}\theta_0(x))^{-1}+C\int_0^t\left\|\frac{1}{\mu(v)v}\right\|_{L^{2p}}ds . \tag{18}$$

Letting $p\to+\infty$ in (18), we can deduce (14) immediately. This completes the proof of lemma 3.

Lemma 4 Under the conditions listed in lemma 2 and assume that the transport coefficients μ and κ satisfy (6) and (7), there exist positive constants $\underline{V}_3,\bar{V}_3$, and $\underline{\Theta}_3$ depending only on T and the initial data $(v_0(x,t),u_0(x,t),\theta_0(x,t),\Phi_0(x,t))$ such that

$$\underline{V}_3\le v(x,t)\le\bar{V}_3,\qquad\forall(x,t)\in I\times[0,T], \tag{19}$$

and

$$\theta(x,t)\ge\underline{\Theta}_3,\qquad\forall(x,t)\in I\times[0,T]. \tag{20}$$

Proof: We first define

$$g(v):=\int_1^v\frac{\mu(\xi)}{\xi}d\xi . \tag{21}$$

Then we get

$$\left(\frac{\mu(v)u_x}{v}\right)_x = \left(\frac{\mu(v)\,\mathrm{v}_t}{v}\right)_x = [g(v)]_{xt} = u_t + p_x - \frac{\Phi_x}{v}. \tag{22}$$

Integrating (22) over $[y,x]\times[0,\mathrm{t}]$ yields

$$\begin{aligned}&-g(v(x,t))+\int_0^t p(x,s)ds\\ &=\int_y^x (u_0(z)-u(z,t)dz - g(v(y,t)) - g(v(x,0)) + g(v(\mathrm{y},0)) + \int_0^t p(\mathrm{y},s)ds + \int_0^t\int_y^x \frac{\Phi_z}{v}dzds.\end{aligned} \tag{23}$$

Set $y=0$ in (23), then involving the boundary condition (4), we have

$$-\mu_0 \log v(x,t) + \int_0^t p(x,s)ds = \int_0^x \big(u_0(z)-u(z,t)\big)dz - \mu_0 \log v_0(x) + \int_0^t\int_0^x \frac{\Phi_z}{v}dzds. \tag{24}$$

(24) together with the fact that $p(x,t)>0$ and the estimate (13), we can easily get the lower bound of $v(x,t)$ with (15). That is,

$$v(x,t)\ge \underline{V}_3\ , \qquad \theta(x,t)\ge \underline{\Theta}_3, \qquad \forall (x,t)\in I\times[0,T]. \tag{25}$$

The assumption (6) together with the estimates (25) imply that

$$\kappa(x,t)\ge K \tag{26}$$

hold for some positive constant K depending on $\underline{V}_3$ and $\underline{\Theta}_3$ for all v and θ under our consideration.

To deduce an upper bound on $v(x,t)$ by exploiting the argument used in Lemma 3, we only need to recover the dissipative estimates $\int_0^t\int_0^1\left(\frac{\mu u_x^2}{v\theta}+\frac{\kappa\theta_x^2}{v\theta^2}\right)dxds$. For this purpose, multiplying $(1)_3$ by θ^{-1} and integrating the resulting identity with respect to x and t over $I\times[0,t]$, one has

$$\begin{aligned}&\int_0^t\int_0^1 \frac{\mu_0 u_x^2}{v\theta}dxds + \int_0^t\int_0^1 \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}dxds\\ &= C_v\int_0^1 \log\theta dx - C_v\int_0^1 \log\theta_0 dx + R\int_0^1 \log v dx - R\int_0^1 \log v_0 dx\\ &\le C + R\int_0^1 \log v dx,\end{aligned} \tag{27}$$

where (13) and (25) are used.

As for the last term on the right hand side of (27), we have by integrating (24) with respect to x over [0, 1] that

$$\int_0^1 \mu_0 \log v dx \le C + \int_0^t \int_0^1 p(x,s) dxds \le C , \tag{28}$$

which together with (27) implies that

$$\int_0^t \int_0^1 \frac{\mu_0 u_x^2}{v\theta} dxds + \int_0^t \int_0^1 \frac{\kappa(v,\theta)}{v\theta^2} dxds \le C . \tag{29}$$

Having obtained (29), we can deduce the upper bound on $v(x,t)$ with the aid of the Gronwall inequality and (12). This completes the proof of Lemma 4.

Now we turn to deduce the upper bound on $\theta(x,t)$ for the case when the transport coefficients μ and κ satisfy (6) and (7).

Corollary 1 Under the conditions listed in Lemma 4, we have for $0 \le t \le T$ that

$$\int_0^t \|\theta(s)\|_{L^\infty(I)} ds \le C \tag{30}$$

and

$$\int_0^t \int_0^1 \theta^2(x,s) dxds \le C . \tag{31}$$

By (30), we can obtain

Lemma 5 Under the conditions listed in Lemma 4, we have for $0 \le t \le T$ that

$$\int_0^1 u^2 dx + \int_0^t \int_0^1 u_x^2 dxds \le C . \tag{32}$$

Proof: Multiplying $(1)_2$ by u and integrating the resulting equation with respect to x and t over $I \times [0,t]$, one has

$$\int_0^t \int_0^1 (uu_t + up_x) dxdt = \int_0^t \int_0^1 \left[u\left(\frac{\mu(v)u_x}{v} \right)_x + u\frac{\Phi_x}{v} \right] dxdt \tag{33}$$

and

$$\int_0^1 \frac{u^2}{2}dx + \int_0^t\int_0^1 \frac{\mu u_x^2}{v}dxds \le C\|u_0\|_{L^2}^2 + C\int_0^t\int_0^1 \frac{\theta^2}{\mu v}dxds\,, \tag{34}$$

where $$\int_0^t\int_0^1 \frac{u_x\theta}{v}dxds \le \frac{1}{2}\int_0^t\int_0^1 \frac{\mu u_x^2}{v}dxds + \frac{1}{2}\int_0^t\int_0^1 \frac{\theta^2}{\mu v}dxds\,,$$

$$\begin{aligned}\int_0^t\int_0^1 \frac{u\Phi_x}{v}dxds &\le \frac{1}{2}\int_0^t\int_0^1 u^2 dxds + \frac{1}{2}\int_0^t\int_0^1 \left(\frac{\Phi_x}{v}\right)^2 dxds \\ &\le \frac{1}{2}T\int_0^1 u^2 dx + \frac{1}{2}\int_0^t\int_0^1 \left(\frac{\Phi_x}{v}\right)^2 dxds.\end{aligned}$$

Thus applying (22) and (31), we get (32). This proves Lemma 5.

Lemma 6 Under the conditions listed in Lemma 4, we have for $0 \le t \le T$ that

$$\|\theta\|_{L^\infty(I)} \le C + C\int_0^1 (\|u_x(s)\|_{L^\infty(I)}^2 + \|\theta(s)\|_{L^\infty(I)}^2)ds\,. \tag{35}$$

Proof: From $(1)_3$, we can easily deduce for each $p>1$ that

$$C_v(\theta^{2p})_t + 2p(2p-1)\theta^{2P-2}\frac{\kappa\theta_x^2}{v} = \left(2p\theta^{2p-1}\frac{\kappa\theta_x}{v}\right)_x + 2p\theta^{2p-1}\frac{\mu u_x^2}{v} - 2p\theta^{2p-1}\frac{R\theta u_x}{v}\,. \tag{36}$$

Integrating (36) with respect to *x* over *I*, one has

$$C_v(\|\theta(t)\|_{L^2}^{2p})_t \le 2p\int_0^1 \theta^{2p-1}\frac{\mu u_x^2}{v}dx + 2p\int_0^1 \theta^{2p-1}\frac{R\theta u_x}{v}dx\,. \tag{37}$$

By exploiting the Holder inequality and letting $p \to +\infty$, we get from (37) that

$$\|\theta(t)\|_{L^\infty(I)} \le C\|\theta_0\|_{L^\infty(I)} + C\int_0^t \left(\left\|\frac{\mu u_x^2}{v}\right\|_{L^\infty(I)} + \left\|\frac{\theta u_x}{v}\right\|_{L^\infty(I)}\right)ds\,. \tag{38}$$

Then with the help of Cauchy's inequality, we can deduce (35) and the proof of Lemma 6 is complete.

Lemma 7 Under the conditions listed in Lemma 4, we have for $0 \le t \le T$ that

$$\int_0^t\int_0^1 \frac{\kappa(v,\theta)\theta_x^2}{\theta^{1-r}}dxds \le C + C\|\theta\|_\infty^r\,,\ \forall r \in (0,1)\,. \tag{39}$$

Proof: Multiplying $(1)_3$ by θ^r and integrating the resulting equation with respect to x over I yield

$$
\begin{aligned}
&C_v\int_0^1 \theta^{1+r}dx+\int_0^t\int_0^1 \frac{r\kappa(v,\theta)\theta_x^2}{v\theta^{1-r}}dxds \\
&=C_v\int_0^1 \theta_0^{1+r}dx+\int_0^t\int_0^1 \frac{\mu_0\theta^r u_x^2}{v}dxds-R\int_0^t\int_0^1 \frac{\theta^{1+r}u_x}{v}dxds \\
&\le C\|\theta_0\|_\infty^{1+r}+C\|\theta\|_\infty^r\left(\int_0^t\int_0^1\theta^2dxds+\int_0^t\int_0^1 u_x^2dxds\right) \\
&\le C+C\|\theta\|_\infty^r,
\end{aligned}
\tag{40}
$$

where (30) and (31) are used. This is (39) and completes the proof of Lemma 7.

A direct consequence of (37) is

Lemma 8 Under the conditions listed in Lemma 4, we have for $0\le t\le T$ that

$$
\int_0^t \|\theta(s)\|_{L^\infty(I)}^2 ds\le C+C\|\theta\|_\infty^{\frac{1}{2}}. \tag{41}
$$

Proof: Observe that (31) imply

$$
\begin{aligned}
\theta^2(x,t)&=\theta^2(b(t),t)+\int_{b(t)}^x 2\theta(y,t)\theta_y(y,t)dy \\
&\le C+C\|\theta(t)\|_{L^\infty(I)}^{1-\frac{r}{2}}\left(\int_0^1\theta(x,t)dx\right)^{\frac{1}{2}}\left(\int_0^1\left(\frac{\theta_x^2}{\theta^{1-r}}\right)(x,t)dx\right)^{\frac{1}{2}} \\
&\le C+C\|\theta(t)\|_{L^\infty(I)}^{1-\frac{r}{2}}\left(\int_0^1\left(\frac{\theta_x^2}{\theta^{1-r}}\right)(x,t)dx\right)^{\frac{1}{2}}.
\end{aligned}
\tag{42}
$$

From the above inequality together with the estimates (39) and (40), we can get that

$$\int_0^t \|\theta(s)\|^2_{L^\infty(I)} \le C + C\int_0^t \left(\|\theta(s)\|^{1-\frac{r}{2}}_{L^\infty(I)} \left(\int_0^1 \left(\frac{\theta_x^2}{\theta^{1-r}} \right)(x,s)dx \right)^{\frac{1}{2}} \right) ds$$

$$\le C + C\left(\int_0^t \|\theta(s)\|^{2-r}_{L^\infty(I)} \right)^{\frac{1}{2}} \left(\int_0^t \int_0^1 \left(\frac{\theta_x^2}{\theta^{1-r}} \right)(x,s)dx \right)^{\frac{1}{2}} \quad (43)$$

$$\le C + C\|\theta\|_\infty^{\frac{1-r}{2}} \left(\int_0^t \|\theta(s)\|_{L^\infty(I)} \right)^{\frac{1}{2}} \left(\int_0^t \int_0^1 \left(\frac{\theta_x^2}{\theta^{1-r}} \right)(x,s)dx \right)^{\frac{1}{2}}$$

$$\le C + C\|\theta\|_\infty^{\frac{1}{2}}.$$

This is exactly (41) and the proof of Lemma 8 is complete.

Lemma 9 Under the conditions listed in Lemma 4, we have for $0 \le t \le T$ that

$$\int_0^1 v_x^2 dx + \int_0^t \int_0^1 \theta v_x^2 dxds \le C + C\|\theta\|_\infty^r, \ \forall r \in (0,1). \quad (44)$$

Proof: As in (38), we can rewrite $(1)_2$ as

$$u_t + \left(\frac{R\theta}{v} \right)_x = \left(\frac{\mu v_x}{v} \right)_t + \frac{\Phi_x}{v}. \quad (45)$$

Multiplying the identity (45) by $\frac{\mu v_x}{v}$, we get that

$$\left(\frac{\mu^2 v_x^2}{2v^2} \right)_t = \left(\frac{\mu u_x}{v} \right)_t - (u\sigma)_x + \frac{\mu u_x^2}{v} - (up)_x + \frac{\mu p_x v_x}{v} + \frac{\mu v_x \Phi_x}{v^2}. \quad (46)$$

Integrating (46) with respect to x and t over $I \times [0,t]$, with the help of (12), Cauchy's inequality, and the fact $\sigma(0,t) = \sigma(1,t) = 0$, we have

$$\int_0^1 v_x^2 dx + \int_0^t \int_0^1 \theta v_x^2 dxds \le C + \int_0^t \int_0^1 \left(u_x^2 + u^2\theta + \frac{\theta_x^2}{\theta} + \theta^2 \right) dxds$$
$$\le C + C\int_0^t \int_0^1 \frac{\theta_x^2}{\theta^{1-r}} dxds, \quad (47)$$

where (32)-(33) are used. Then by (39), we can easily get (43).

By employing the arguments used in [6, 9, 16], we can control $\int_0^t\int_0^1 u_x^4 dxds$ as in the following lemma

Lemma 10 Under the conditions listed in Lemma 4, we have for $0 \le t \le T$ that

$$\int_0^t\int_0^1 u_x^4 dxds \le C\left(1+\|\theta\|_\infty^2\right). \tag{48}$$

Proof: Set
$$U(x,t)=\int_0^x u(y,t)dy. \tag{49}$$

Under the boundary condition

$$\sigma(0,t)=\sigma(1,t)=0, \tag{50}$$

We can get by integrating $(1)_2$ over (0, x) and by using (50) that

$$\begin{cases} U_t-\dfrac{\mu}{v}U_{xx}=-p(x,t)+\displaystyle\int_0^t \dfrac{\Phi_y}{v}dy, \\ U(x,0)=\displaystyle\int_0^x u_0(y)dy, \\ U(0,t)=0, \\ U(1,t)=\displaystyle\int_0^1 u_0(x)dx. \end{cases} \tag{51}$$

Hence the standard L^p -estimates for solutions to the linear problem (51), cf. [16], yields

$$\int_0^t\int_0^1 U_{xx}^4 dxds \le C+C\left(\|u_0\|_{L^2(I)}\right)+C\int_0^t\int_0^1 p^4 dxds \le C+C\int_0^t\int_0^1 \theta^4 dxds. \tag{52}$$

Thus by (31), we get (48) and the proof of Lemma 10 is complete.

On the other hand, noticing that

$$u_x^2(y,t) \le \int_0^1 u_x^2(x,t)dx+2\int_0^1 |u_x(x,t)|\|u_{xx}(x,t)\|dx, \tag{53}$$

We have from (31) and Holder's inequality that

Lemma 11 Under the conditions listed in Lemma 4, we have for $0 \le t \le T$ that

$$\int_0^t \|u_x(s)\|_{L^\infty(I)}^2 ds \le C+C\left(\int_0^t\int_0^1 u_{xx}(x,s)^2 dxds\right)^{\frac{1}{2}}. \tag{54}$$

Lemma 12 Under the conditions listed in Lemma 4, we have for $0 \le t \le T$ that

$$\int_0^t\int_0^1 u_{xx}^2 dxds \le C + C\|\theta\|_\infty^{\max\{2r,1,c+1\}}. \tag{55}$$

Proof: By differentiating $(1)_2$ with respect to x and multiplying the resulting equation by $u_x - \frac{R\theta}{\mu_0}$, we have

$$\left(\frac{u_x^2}{2} - \frac{R\theta u_x}{\mu_0}\right)_t = -u_x\left(\frac{R\theta}{\mu_0}\right)_t + \left(\frac{v\sigma\sigma_x}{\mu_0}\right)_x - \sigma_x\left(\frac{v\sigma}{\mu_0}\right)_x + \left(\frac{\Phi_x}{v}\right)_x \cdot \left(u_x - \frac{R\theta}{\mu_0}\right). \tag{56}$$

Integrating (56) with respect to x and t over $I\times[0,t]$, one has

$$\int_0^1\left(\frac{u_x^2}{2} - \frac{R\theta u_x}{\mu_0}\right)dx \le C - \frac{R}{\mu_0}\int_0^t\int_0^1 u_x\theta_t dxds - \int_0^t\int_0^1 \sigma_x\left(\frac{v\sigma}{\mu_0}\right)_x dxds + \int_0^t\int_0^1(1-v)\left(u_x - \frac{R\theta}{\mu_0}\right)dxds. \tag{57}$$

Since by (8), (32), (39), (41), (44), (54) and (55), we have

$$\begin{aligned}
&-\int_0^t\int_0^1 \sigma_x\left(\frac{v\sigma}{\mu_0}\right)_x dxds \\
&\le -\frac{\underline{V}_3}{\mu_0}\int_0^t\int_0^1 \sigma_x^2 dxds + C\int_0^t\int_0^1 \sigma\sigma_x v_x dxds \\
&\le -\frac{\underline{V}_3}{2\mu_0}\int_0^t\int_0^1 \sigma_x^2 dxds + C\int_0^t\int_0^1 \sigma_x^2 v_x^2 dxds \\
&\le -\frac{\mu_0}{4\bar{V}_3^2}\int_0^t\int_0^1 u_{xx}^2 dxds + C\int_0^t\int_0^1 (u_x^2+\theta^2)v_x^2 dxds + C\int_0^t\int_0^1 \theta_x^2 dxds \\
&\le -\frac{\mu_0\underline{V}_3}{4\bar{V}_3^2}\int_0^t\int_0^1 u_{xx}^2 dxds + C\int_0^t\left(\|u_x(s)\|_{L^\infty(I)}^2 + \|\theta(s)\|_{L^\infty(I)}^2\right)\|v_x(s)\|_{L^2(I)}^2 + C\|\theta\|_\infty^{1-r}\int_0^t\int_0^1 \frac{\theta_x^2}{\theta^{1-r}} dxds \\
&\le -\frac{\mu_0\underline{V}_3}{8\bar{V}_3^2}\int_0^t\int_0^1 u_{xx}^2 dxds + C\|\theta\|_\infty^{\max\{2r,r+\frac{1}{2},1\}},
\end{aligned} \tag{58}$$

and

$$
\begin{aligned}
&-\frac{R}{\mu_0}\int_0^t\int_0^1 u_x\theta_t dxds\\
&\le -\frac{R}{\mu_0 C_v}\int_0^t u_x\left[\left(\frac{\kappa\theta_x}{v}\right)_x+\frac{\mu_0 u_x^2}{v}-\frac{R\theta u_x}{v}\right]dxds\\
&\le \frac{\mu_0 \underline{V}_3}{16\overline{V}_3^2}\int_0^t\int_0^1 u_{xx}^2 dxds + C\int_0^t\int_0^1\left(\kappa^2(v,\theta)\theta_x^2+u_x^3+\theta u_x^2\right)dxds\\
&\le \frac{\mu_0 \underline{V}_3}{16\overline{V}_3^2}\int_0^t\int_0^1 u_{xx}^2 dxds + C\left(\int_0^t\int_0^1 u_x^4 dxds\right)^{\frac{1}{2}}\left(\int_0^t\int_0^1\left(u_x^2+\theta^2\right)dxds\right)^{\frac{1}{2}} + C\|\theta\|_\infty^{c+1-r}\int_0^t\int_0^1\frac{\kappa(v,\theta)\theta_x^2}{\theta^{1-r}}dxds\\
&\le \frac{\mu_0 \underline{V}_3}{16\overline{V}_3^2}\int_0^t\int_0^1 u_{xx}^2 dxds + C\|\theta\|_\infty^{\max\{1,c+1\}}.
\end{aligned}
\tag{59}
$$

The above two estimates together with (13), (57) and Cauchy's inequality, we get (55). This completes the proof of the lemma 12.

Having obtained (35), (41), (54), and (55), we can obtain the upper bound on $\theta(x,t)$ if the parameter c is chosen such that $c<1$. Here we have used the fact that $r>0$ can be chosen as small as wanted.

Now we consider the case when the transport coefficients μ and κ satisfy (5) with $0\le a<\frac{1}{5}$ and $b\ge 2$. For such a case, (24) should be replaced by

$$
-g(v(x,t))+\int_0^t p(x,t)ds=\int_0^x (u_0(z)-u(z,t))dz+g(v_0(x))+\int_0^t\int_0^1\frac{\Phi_z}{v}dzds \tag{60}
$$

with

$$
g(v)=\begin{cases}\dfrac{1-v^{-a}}{a}, & a>0,\\ \ln v, & a=0.\end{cases}
$$

With (58) in hand, we can deduce by repeating the argument used in the proof of Lemma 4, especially the way to deduce (23)-(25), that there exist some positive constants $\underline{V}_3>0$ and $\underline{\Theta}_3>0$ such that

$$
v(x,t)\ge \underline{V}_3,\qquad \theta(x,t)\ge\underline{\Theta}_3
$$

hold for all $(x,t)\in I\times[0,T]$. But since the boundary condition (4) does not yield any L^p -estimates on v, we can deduce from the fact $|\ln v|<\|v\|_\infty^\varepsilon$ for any $\varepsilon>0$ that

$$\int_0^t\int_0^1\frac{\theta^{b-2}\theta_x^2}{v}dxds\le C+C\|v\|_\infty^\varepsilon. \tag{61}$$

To deduce as upper bound on $v(x,t)$, we try to recover the L^1-estimates on $v(x,t)$, which plays an important role in deriving the upper bound on $v(x,t)$ for the case when the transport coefficients μ and κ satisfy (7). To do so, integrating $(1)_1$ with respect to x and t over $I\times[0,t]$, we get

$$\begin{aligned}\int_0^1 vdx&\le\int_0^1 v_0dx+\int_0^t\int_0^1 u_xdxds\\&\le C+C\|v\|_\infty^{\frac{a}{2}}\left(\int_0^t\int_0^1\frac{u_x^2}{v^{1+a}}dxds\right)^{\frac12}\left(\int_0^t\int_0^1 vdxds\right)^{\frac12}\\&\le C+C\|v\|_\infty^{a}\int_0^t\int_0^1\frac{u_x^2}{v^{1+a}}dxds+\int_0^t\int_0^1 vdxds.\end{aligned}\tag{62}$$

Then by the Gronwall inequality, we can easily deduce that

$$\int_0^1 vdx\le C+C\|v\|_\infty^{a}\int_0^t\int_0^1\frac{u_x^2}{v^{1+a}}dxds\,. \tag{64}$$

Since $b\ge 2$, we have

$$\begin{aligned}\int_0^t\|\theta(s)\|_{L^\infty(\mathrm{I})}ds&\le C+C\int_0^t\|\theta(s)\|_{L^\infty(I)}^{\frac{b}{2}}ds\\&\le C+C\int_0^t\int_0^1\theta^{\frac{b}{2}-1}|\theta_x|dxds\\&\le C+C\left(\int_0^t\int_0^1 vdxds\right)^{\frac12}\left(\int_0^t\int_0^1\frac{\theta^{b-2}\theta_x^2}{v}dxds\right)^{\frac12}\\&\le C+C\|v\|_\infty^{\frac{\varepsilon+a}{2}}\left(\int_0^t\frac{u_x^2}{v^{1+a}}dxds\right)^{\frac12},\end{aligned}\tag{64}$$

which implies that

$$\int_0^t\int_0^1 \theta^2 dxds \le C + C\|v\|_\infty^{\frac{\varepsilon+a}{2}} \left(\int_0^t\int_0^1 \frac{u_x^2}{v^{1+a}} dxds\right)^{\frac{1}{2}}. \tag{65}$$

Thus with the help of (33), we have

$$\int_0^t\int_0^1 \frac{u_x^2}{v^{1+a}} dxds \le C + C\|v\|_\infty^{\frac{\varepsilon+a}{2}} \left(\int_0^t\int_0^1 \frac{u_x^2}{v^{1+a}} dxds\right)^{\frac{1}{2}}, \tag{66}$$

Then by Cauchy's inequality, we can easily obtain the following results

Lemma 13 Under the conditions listed in Lemma 3.4, we have for $0 \le t \le T$ that

$$\int_0^t\int_0^1 \frac{u_x^2}{v^{1+a}} dxds \le C + C\|v\|_\infty^{\varepsilon+a}, \tag{67}$$

$$\int_0^1 v dx \le C + C\|v\|_\infty^{\varepsilon+2a}, \tag{68}$$

$$\int_0^t \|\theta(s)\|_{L^\infty(I)} ds \le C + C\|v\|_\infty^{\varepsilon+a}, \tag{69}$$

and

$$\int_0^t\int_0^1 \theta^2 dxds \le C + C\|v\|_\infty^{\varepsilon+a}. \tag{70}$$

To estimate $\|v_x(t)\|_{L^2(I)}$, we have by integrating (46) with respect to x and t over $I\times[0,t]$ and with the help of (13) and Cauchy's inequality that

$$\begin{aligned}
&\int_0^1 \frac{v_x^2}{v^{2(1+a)}} dx + \int_0^t\int_0^1 \frac{\theta v_x^2}{v^{3+a}} dxds \\
&\le C + C\int_0^t\int_0^1 \left(\frac{u_x^2}{v^{1+a}} + \frac{u^2\theta}{v^{1-a}} + \frac{\theta_x^2}{v^{1+a}\theta} + \frac{\theta^2}{v^{1-a}}\right) dxds + C\int_0^t\int_0^1 \left(\frac{\Phi_x}{v}\right)^2 dxds \\
&\le C + C\|v\|_\infty^{\varepsilon+a} + \int_0^t\int_0^1 \frac{\theta_x^2}{v\theta} dxds.
\end{aligned} \tag{71}$$

To control $\int_0^t\int_0^1 \frac{\theta_x^2}{v\theta} dxds$, we have by multiplying $(1)_3$ by θ^{-b}, and integrating the resulting identity over $I\times[0,t]$ that

$$\int_0^t\int_0^1 \frac{u_x^2}{v^{1+a}\theta^b}dxds+\int_0^t\int_0^1\frac{\theta_x^2}{v\theta}dxds \le C+C\int_0^t\int_0^1\frac{|u_x|}{v}dxds \le C+C\|\theta\|_\infty^{\frac{\varepsilon+a}{2}}, \tag{72}$$

and the above estimate together with (71) imply

$$\int_0^t\int_0^1\frac{v_x^2}{v^{2(1+a)}}dxds \le C+C\|v\|_\infty^{\varepsilon+a}. \tag{73}$$

Since

$$\begin{aligned} v(y,t) &\le \int_0^1 v(x,t)dx+\int_0^1|v_x|dx \\ &\le C+C\|v\|_\infty^{\varepsilon+a}+C\|v\|_\infty^{\frac{1}{2}+a}\left(\int_0^1 vdx\right)^{\frac{1}{2}}\left(\int_0^1\frac{v_x^2}{v^{2(1+a)}}\right)^{\frac{1}{2}} \\ &\le C+C\|v\|_\infty^{\varepsilon+\frac{1}{2}+\frac{5a}{2}}, \end{aligned} \tag{74}$$

for which and the assumption $0\le a<\dfrac{1}{5}$, we can deduce that

$$v(x,t)\le \bar{V}_3, \quad \forall (x,t)\in I\times[0,T] \tag{75}$$

holds for some positive constant $\bar{V}_3$ which depends only on T and the initial data $(v_0(x,t),u_0(x,t),\theta_0(x,t),\Phi_0(x,t))$. As a by-produce of the estimate (75), we can deduce that the terms on the right-hand side of the inequalities in Lemma 13 and (73) can all be bounded by some constant C depending only on T and the initial data $(v_0(x,t),u_0(x,t),\theta_0(x,t),\Phi_0(x,t))$.

Now we turn to derive the upper bound on $\theta(x,t)$. For this purpose, we have by multiplying $(1)_3$ by $\theta^{-\gamma}$ for some $\gamma\in(0,1)$ and integrating the resulting identity over $I\times[0,t]$ that

$$\int_0^t\int_0^1\frac{u_x^2}{v^{1+a}\theta^\gamma}dxds+\int_0^t\int_0^1\frac{\theta^{b-1-\gamma}\theta_x^2}{v}dxds\le C. \tag{76}$$

Then by (52), we have

$$
\begin{aligned}
\int_0^t\int_0^1 u_x^4 dxds &\le C + C\int_0^t\int_0^1 \theta^4 dxds \\
&\le C + C\int_0^t \|\theta(s)\|_{L^\infty(I)}^3 ds \\
&\le C + C\int_0^t \left(\int_0^1 \sqrt{\theta}\,|\theta_x|\,dx\right)^2 ds \\
&\le C + C\|\theta\|_\infty^{\max\{2+\gamma-b,0\}} \int_0^t\int_0^1 \theta^{b-1-\gamma}\theta_x^2 dxds \\
&\le C + C\|\theta\|_\infty^{\max\{2+\gamma-b,0\}} .
\end{aligned} \tag{77}
$$

Now we set

$$
X := \int_0^t\int_0^1 \theta^b \theta_t^2 dxds \ , \qquad Y := \max_t \int_0^1 \theta^{2b}\theta_x^2 dx\,, \qquad Z := \max_t \int_0^1 u_{xx}^2 dx\,. \tag{78}
$$

Observe that

$$
\begin{aligned}
\theta^{2b+2} &\le C + C\int_0^1 \theta^{2b+1}|\theta_x|\,dx \\
&\le C + C\|\theta\|_{L^\infty(I)}^{b+\frac{1}{2}} \left(\int_0^1 \theta dx\right)^{\frac{1}{2}} \left(\int_0^1 \theta^{2b}\theta_x^2 dx\right)^{\frac{1}{2}} \\
&\le C + C\|\theta\|_{L^\infty(I)}^{b+\frac{1}{2}}\, Y^{\frac{1}{2}},
\end{aligned} \tag{79}
$$

which implies

$$
\|\theta\|_{L^\infty(I)} \le C + CY^{\frac{1}{2b+3}} . \tag{80}
$$

Combining (53), the inequality

$$
\int_0^1 u_x^2 dx \le C\int_0^1 u^2 dx + C\left(\int_0^1 u^2 dx\right)^{\frac{1}{2}} \left(\int_0^1 u_{xx}^2 dx\right)^{\frac{1}{2}}, \tag{81}
$$

and by (13), we have

$$
\max_t \int_0^1 u_x^2 dx \le C + CZ^{\frac{1}{2}}, \tag{82}
$$

and

$$
\|u_x\|_{L^\infty(I)} \le C + CZ^{\frac{3}{8}} . \tag{83}
$$

Our next result is to show that *X* and *Y* can be controlled by *Z*.

Lemma 14 Under the conditions listed in Lemma 3.4, we have for $0 \le t \le T$ that

$$X + Y \le C + CZ^{\frac{3}{4}} . \tag{84}$$

Proof: Multiplying $(1)_3$ by $\theta^b \theta_t$, and integrating the resulting identity over $I \times [0,t]$, one has

$$X + Y \le C + C\int_0^t \int_0^1 \left(\theta^{b+1} |u_x||\theta_t| + \theta^b u_x^2 |\theta_t| + \theta^{2b} |u_x| \theta_x^2 \right) dxds . \tag{85}$$

Since by Cauchy's inequality and (75), (76), (77), we can get from (80) and (83) that

$$\int_0^t \int_0^1 \theta^{b+1} |u_x||\theta_t| dxds \le \frac{X}{4} + C\|\theta\|_\infty^{b+2} \int_0^t \int_0^1 u_x^2 dxds \le \frac{X}{4} + CY^{\frac{b+2}{2b+3}}, \tag{86}$$

$$\int_0^t \int_0^1 \theta^b u_x^2 |\theta_t| dxds \le \frac{X}{4} + C\|\theta\|_\infty^{b} \int_0^t \int_0^1 u_x^4 dxds \le \frac{X}{4} + CY^{\frac{\max\{b,2+\gamma\}}{2b+3}} , \tag{87}$$

and

$$\int_0^t \int_0^1 \theta^{2b} |u_x| \theta_x^2 dxds \le \|u_x\|_\infty \|\theta\|_\infty^{b+1+\gamma} \int_0^t \int_0^1 \theta^{b-1-\gamma} \theta_x^2 dxds \le CY^{\frac{b+1+\gamma}{2b+3}} \left(1 + Z^{\frac{3}{8}} \right). \tag{88}$$

Based on the above three estimates and (85) and by employing the Cauchy inequality, we can get (84) immediately if we choose $\gamma \in (0, \frac{1}{2})$. This completes the proof of Lemma 14.

Our last result in this section is to show that *Z* can be bounded by *X* and *Y*.

Lemma 15 Under the conditions listed in Lemma 4, we have for $0 \le t \le T$ that

$$Z \le C + CY^{\frac{2+\gamma}{2b+3}} + CX + CZ^{\frac{3}{4}} . \tag{89}$$

Proof: Using $(1)_2$, we can easily get the following identity

$$u_{xx} = v^{1+a} \left(u_t + p_x + \frac{(1+a) v_x u_x}{v^{2+a}} \right) - \Phi_x v^a . \tag{90}$$

Integrating (90) with respect to x and t over $I \times [0,t]$ yields

$$\int_0^t\int_0^1 u_{xx}^2 \le C\int_0^t\int_0^1\left(u_t^2+\theta_x^2+\theta^2 v_x^2+v_x^2 u_x^2\right)dxds$$
$$\le C\int_0^t\int_0^1 u_t^2 dxds + C\int_0^t\int_0^1 \frac{\theta^{b-1-\gamma}\theta_x^2}{v}dxds$$
$$+C\left(\|\theta\|_\infty^2+\|u_x\|_\infty^2\right)\int_0^t\int_0^1 v_x^2 dxds \tag{91}$$
$$\le C\int_0^t\int_0^1 u_t^2 dxds + CY^{\frac{2}{2b+3}}+CZ^{\frac{3}{4}}.$$

Next we need to estimate $\int_0^t\int_0^1 u_t^2 dxds$ to complete the proof of this lemma. To this end, we have by differentiating $(1)_2$ with respect to t and multiplying the resulting identity by u_t that

$$\left(\frac{u_t^2}{2}\right)_t + \frac{u_{xt}^2}{v^{1+a}} = \left(\sigma_t u_t\right)_x + \frac{(1+a)u_x^2 u_{xt}}{v^{2+a}} + \frac{R\theta_t u_{xt}}{v} - \frac{R\theta u_x u_{xt}}{v^2} + \frac{\Phi_{xt}u_t}{v} - \frac{\Phi_x v_t u_t}{v^2}. \tag{92}$$

Integrating (92) with respect to x and t over $I\times[0,t]$ and with the help of Cauchy's inequality, one has

$$\int_0^1 u_t^2 dx + \int_0^t\int_0^1 u_{xt}^2 dxds \le C + C\int_0^t\int_0^1\left(u_x^4+\theta_t^2+\theta^2 u_x^2+\left(\frac{\Phi_x}{v}\right)^2\right)dxds$$
$$\le C + CY^{\frac{2+\gamma}{2b+3}} + CX. \tag{93}$$

(93)together with (91) implies (89) and the proof of Lemma 15 .

Combining (84) and (89), we can obtain $Y \le C$, then we derive the upper bounds on $\theta(x,t)$ from (80).

In summary, we have obtained the desired lower and upper bounds on v and θ provide that the transport coefficients μ and κ satisfy the conditions listed in Theorem 1 and then Theorem 1 can be proved by employing the continuation argument.

4. Discussion

Compressible Navier-Stokes type equation with density and temperature dependent transport coefficients arise in many applied sciences, such as certain class of solid-like materials, gases at very high temperatures, etc. Such a dependence of μ and κ on v and θ will obviously influence the solutions of the field equations as well as the mathematical analysis and to establish the corresponding well-posedness theory has been the subject of many recent researches. These studies indicate that temperature dependence of the viscosity in μ is especially challenging but one can incorporate various forms of density dependence in μ and also temperature dependence in κ. We note, however, that in all the previous studies although the viscosity coefficient μ may depend on v and the heat conductivity κ may depend on both v and θ, they ask that at least one of μ and κ is non-degenerate. What we are interested in this paper focuses on the case when μ is a function of v and κ depend on v and/or θ and both μ and κ are degenerate. We hope the study here can shed some light on the construction of global classical solutions to the fluid model derived from the Vlasov-Possion-Boltzmann system.

5. Conclusions

In this paper, we concern with the construction of global solutions with large amplitude to the initial-boundary value problem of the one-dimensional compressible NSP system with degenerate transport coefficients. The results can not only provide the global existence of solutions, but also reduce the gap between the physical models and the satisfactory existence theory. It implies that the viscous fluid under the influence of the self-induced electric force will doesn't affect construction of global, non-vacuum solutions.

For the case when $\mu(v)$ is a smooth function of v satisfying $\mu(v) > 0$ for $v > 0$ and $\kappa(\theta) = \theta^b$, if the specific volume v is bounded both from below and from above and the absolute temperature θ is bounded from below, i.e., there exist some positive constants $\underline{V}_3 > 0, \overline{V}_3 > 0,$ and $\underline{\Theta}_3 > 0$ such that

$$\underline{V}_3 \le v(x,t) \le \overline{V}_3, \quad \theta(x,t) \ge \underline{\Theta}_3 > 0$$

hold for $(x,t) \in I \times [0,T]$, then the argument used above can be employed to derive the upper hound on $\theta(x,t)$ provide that $b \geq 0$.

Reference

[1] Gerebeau, J. F. and Bris, C. L. and Lelievre,T. Mathematical methods for the magneto-hydrodynamics of liquid metals. Oxford University Press, Oxford, (2006)

[2] Tan, Z., Yang, T., Zhao H. J. and Zhou, Q. Y. Global solutions to the one-dimensional compressible Navier-Stokes-Poisson equations with large data. Society for Industrial and Applied Mathematics, 45(2), 547-571(2013)

[3] Okada M. and Kawashima, S. On the equations of one-dimensional motion of compressible viscous fluids. J. Math. Kyoto Univ., 1, 55–7123 (1983)

[4] Grad, H. Asymptotic Theory of the Boltzmann Equation II. Rarefied Gas Dynamics. J. A. Laurmann, ed., Vol. 1, Academic Press, New York, 26-59 (1963)

[5] Zeldovich, Y. B. and Raizer, Y. P. Physics of Shock Waves and High-Temperature Hydrodynamic
Phenomena. Academic Press, New York, (1967)

[6] Dafermos, C. M. and Hsiao, C. M. Global smooth thermomechanical processes in one-dimensional nonlinear thermoviscoelasticity. Nonlinear Anal., 6, 435-454(1982)

[7] Jenssen, H. K. and Karper, T. K. One-dimensional compressible flow with temperature dependent transport coefficients. SIAM J. Math. Anal., 42, 904-930(2010)

[8] Jiang, S. and Racke, R. Evolution Equations in Thermoelasticity. Monographs and surveys in pure and applied mathematics, volume 112, Chapman & Hall/CRC Press, Boca Raton, (2000)

[9] Kawohl, B. Global existence of large solutions to initial-boundary value problems for a viscous, heat-conducting, one-dimensional real gas. J. Differential Equations, 58, 76-103(1985)

[10] Kawashima, S. and Nishida, T. Global solutions to the initial value problem for the equations of one-dimensional motion of viscous polytrophic gases. J. Math. Kyoto Univ., 21(4), 825-837(1981)

[11] Kazhikhow, A. V. and Shelukhin, V. V. Unique global solution with respect to time of initial-boundary calue problem for one-dimensional equations of a viscous gas. J. Appl. Math. Mech., 41(2), 273-282(1977)

[12] Kanel, Y. On a model system of equations of one-dimensional gas motion. Differencial'nya Uravnenija, 4, 374-380(1968)

[13] Chapman, S. and Colwing, T. G. The Mathematical Theory of Nonuniform Gases. Cambrige Math. Lib., 3rd ed., Cambridge University Press, Cambridge, (1990)

[14] Vincenti, W. G. and Kruger, C. H. Introduction to physical Gas Dynamics. Cambridge Math. Lib., Cambridge University Press, Cambridge, (1975)

[15] Tani, A. On the free boundary value problem for compressible viscous fluid motion. J. Math. Kyoto Univ., 21(4), 839–859) (1981)

[16] Ladyzhenskaya, O. A., Solonnikov, V. A. and Ural'ceva, N. N. Linear and quasi-equations of parabolic type. Amer. Math. Sot., Providence, R. I., (1968)