On out of kilter algorithm

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Abstract

The out-of-kilter algorithm is one of the basic algorithms that solve the minimum cost flow problem. The aim of this paper is to shows that if the dual of out of kilter formulation has optimal solution then it has integer optimal solution too.

Keywords: Network flow, Minimum cost network flow

1. Introduction

Let $(\mathcal{N}, \mathcal{A})$ be an directed network defined by a set $\mathcal{N}$ of m nodes and a set $\mathcal{A}$ of n directed arcs. Each arc $(i, j) \in \mathcal{A}$ has a cost c_{ij}. We associate with each node $i \in \mathcal{N}$ a number b_i which indicates its supply or demand depending on whether $b_i > 0$ or $b_i < 0$. The minimum cost network flow problem is to determine the flow x_{ij} on each arc $(i, j) \in \mathcal{A}$ so as:

$$\min \quad \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{m} x_{ij} - \sum_{k=1}^{m} x_{ki} = b_i, \quad i = 1, \ldots, m \quad (1)$$

$$x_{ij} \geq 0 \qquad\qquad i, j = 1, \ldots, m$$

The following assumptions are hold for (1):

1. Total supply equal total demand i.e. $\sum_{i=1}^{m} b_i = 0$

2. All arc costs are nonnegative.

3. There is no any undirected arc (i, j) with $c_{ij} < 0$.

We associate a real number w_i with each node $i \in \mathcal{N}$. We refer to w_i as the potential of node i. In fact, w_i's are dual variables of the minimum cost network flow problem (1). There are several methods to solve the minimum cost network flow problem (1). One of them is the out-of-kilter algorithm which is similar to the primal-dual algorithm (see [1]). However it differs from the primal-dual algorithm in that the out-of-kilter algorithm does not always maintain the complementary slackness condition. In Section 2, we present the out-of-kilter algorithm.

2. The out-of-kilter algorithm

In this section we present a brief review of Fulkerson's out-of-kilter algorithm [1]. This review is only meant to refresh the reader's memory and to introduce our notation. (For a more complete presentation, the reader should refer to ([2], [4], [3]). For convenience of presentation, we onsider the minimum cost network flow problem as follows:

$$\min \quad \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{m} x_{ij} - \sum_{k=1}^{m} x_{ki} = 0, \quad i = 1, \ldots, m \quad (2)$$

$$l_{ij} \leq x_{ij} \leq u_{ij} \qquad\qquad i, j = 1, \ldots, m$$

We call a conserving flow any flow (choice of the x_{ij}) satisfying constraints (2). A conserving flow satisfies the remaining constraints $l_{ij} \le x_{ij} \le uij$ is a feasible flow (solution). We shall assume that , l_{ij} and u_{ij} are integers and $0 \le l_{ij} \le u_{ij} < \infty$. The dual of the out-of-kilter formulation for the minimum cost network flow problem is as follows:

$$\min \quad \sum_{i=1}^{m}\sum_{j=1}^{m} l_{ij}v_{ij} - \sum_{i=1}^{m}\sum_{j=1}^{m} u_{ij}h_{ij}$$
$$\text{s.t.} \quad w_i - w_j + v_{ij} - h_{ij} = c_{ij}, \quad i, j = 1, \ ..., \ m \quad (3)$$
$$v_{ij}, h_{ij} \ge 0 \qquad\qquad i, j = 1, \ ..., \ m$$
$$w_i \quad \text{unrestricted} \qquad i = 1, \ ..., \ m$$

in which dual variables w_i, h_{ij} and v_{ij} are associated with the conservation equation, $x_{ij} \le u_{ij}$, $x_{ij} \ge l_{ij}$ of (2), respectively. As stated in [4] the dual problem has very interesting structure. Suppose that we select any set of w_i's. Then the dual constraint for arc (i, j) becomes

$$v_{ij} - h_{ij} = c_{ij} - w_i + w_j$$

and is satisfied by:

$$v_{ij} = \max\{0, c_{ij} - w_i + w_j\}$$
$$h_{ij} = \min\{0, -(c_{ij} - w_i + w_j)\}$$

Thus the dual problem always possesses a feasible solution given any set of w_i. In fact, the choice of v_{ij} and h_{ij} yield the optimal values of v_{ij} and h_{ij} for a fixed set of w_i's. We use the following theorem in the sequel.

Theorem 1: *Let x be any conserving flow, and let w= (w_1, ..., w_m) any vector. Then x and w are respectively primal and dual optimal solutions to problem (2) if and only if for all (i, j)*

- $z_{ij} - c_{ij} < 0$ *implies* $x_{ij} = l_{ij}$

- $z_{ij} - c_{ij} > 0$ *implies* $x_{ij} = u_{ij}$

- $z_{ij} - c_{ij} = 0$ *implies* $l_{ij} \le x_{ij} \le u_{ij}$

where $z_{ij} - c_{ij} = c_{ij} - w_i + w_j$ for all (i, j).

Proof. See [4].

Any arc (i, j) satisfies the conditions of Theorem 1 is in-kilter; otherwise it is out-of-kilter. The kilter number K_{ij} of an arc (i, j) is defined here to be the minimal change of flow on the arc that is needed to bring it into kilter i.e.:

$$K_{ij} = \begin{cases} \left| x_{ij} - l_{ij} \right| & z_{ij} - c_{ij} < 0 \\ \left| x_{ij} - u_{ij} \right| & z_{ij} - c_{ij} > 0 \\ \left| x_{ij} - l_{ij} \right| & z_{ij} - c_{ij} = 0, \ x_{ij} < l_{ij} \\ \left| x_{ij} - u_{ij} \right| & z_{ij} - c_{ij} = 0, \ x_{ij} > u_{ij} \\ 0 & z_{ij} - c_{ij} = 0, \ u_{ij} \leq x_{ij} \leq l_{ij} \end{cases}$$

Also notice that if the arc is in kilter, the associated kilter number is zero, and if the arc is out-of-kilter, the associated kilter number is strictly positive. The out-of-kilter algorithm starts with a conserving flow $x = 0$ and a set of node potentials $w = 0$. The algorithm maintains all of in-kilter arcs as in-kilter arcs and successively transforms the out-of-kilter arcs into in-kilter arcs by changing node potentials and by augmenting flow on appropriate directed cycles. The algorithm terminates when all arcs in become in-kilter, i.e. when the current flow is a minimum cost flow. The following theorem is the main results of the work.

Theorem 2: *If the dual of the out-of-kilter formulation (i.e. model (3)) has optimal solution then it has integer optimal solution too.*

Proof. Suppose $\mathbf{w^*} = (w^*_1, ..., w^*_m)$ is optimal solution of (3). We show that $\mathbf{w^*} = ([w^*_1], ...,[w^*_m])$ is optimal solution of (3) in which $[w_i]$ is integral part of w_I. Let $w_i = [w_i] + p_i$ in which $0 < p_i \leq 1$.

$[w^*_i] - [w^*] - c_{ij} < 0 \Rightarrow [w^*_i] - [w^*_i] - c_{ij} + p^*_i - p^*_j < 0 \Rightarrow w^*_i - w^*_j - c_{ij} < 0$ so by Theorem1 we have $x_{ij} = l_{ij}$.

$[w^*_i] - [w^*] - c_{ij} > 0 \Rightarrow [w^*_i] - [w^*_i] - c_{ij} + p^*_i - p^*_j > 0 \Rightarrow w^*_i - w^*_j - c_{ij} > 0$ so by Theorem1 we have $x_{ij} = u_{ij}$.

Now we consider the case that $[w^*] - [w^*_j] - c_{ij} = 0$. Since $-1 < p^*_i - p^*_j < 1$; three cases can be occurred:

- Case 1: $[w^*_i] - [w^*_i] - c_{ij} + p^*_i - p^*_j < 0 \Rightarrow w^*_i - w^*_j - c_{ij} < 0 \Rightarrow x_{ij} = l_{ij}$

- Case 1: $[w^*_i] - [w^*_i] - c_{ij} + p^*_i - p^*_j = 0 \Rightarrow w^*_i - w^*_j - c_{ij} = 0 \Rightarrow l_{ij} \leq x_{ij} \leq u_{ij}$.

- Case 1: $[w^*_i] - [w^*_i] - c_{ij} + p^*_i - p^*_j > 0 \Rightarrow w^*_i - w^*_j - c_{ij} < 0 \Rightarrow x_{ij} = u_{ij.}$

Therefore $[w^*] - [w^*_j] - c_{ij} = 0$ implies $l_{ij} \leq x_{ij} \leq u_{ij}$.

Hence the proof is complete.

3. Conclusion

In this paper we show that if the dual of the out-of-kilter formulation (i.e. model (3)) has optimal solution then it

has integer optimal solution too. So, by the proof of Theorem 2 one can obtain any integer optimal solution of the model (3) using any non-integer optimal solution of it.

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