SCIREA Journal of Mathematics

http://www.scirea.org/journal/Mathematics

November 19, 2016

Volume 1, Issue1, October 2016

Pile-up flow-solutions to Navier-Stokes equations in the Millennium Prize Problem-version with isochoric condition and regularity requirements

Lena J-T Strömberg

Previously Dep of Solid Mechanics, Royal Inst of Technology, KTH

e-mail: lena_str@hotmail.com

Abstract

Solutions to the Navier-Stokes equations for the Millenium Prize Problem are provided. These consist of a transient Pile-Up flow. A proof is given to show that the flow functions satisfy the Boundary Conditions at infinity. The proof for the spatial derivatives of velocity, u, and force, f, relies on decomposition of an exponential function, Cauchy-Schwarz and induction.

Keywords: Navier-Stokes, Millennium Prize, Pile Up flow, Lena Pile-Up, Theorem, Proof, decomposition, Cauchy-Schwarz, induction, exponential function, velocity, coordinates

Introduction

In the present context, solutions to Navier-Stokes equation as formulated in the Millennium Prize will be considered and notations from Clay Mathematics Institute's prize problem, Fefferman (2000)[1] are used. In general, the equations have several solutions describing physical properties of fluids and flows.

For the static case, there will be a pressure p when outer force due to gravity exists. This is linear in depth, and results in buoyancy, for objects in the fluid.

For stationary flow, at a streamline, the property p + $u^2/2$ remain constant, which is known as the Bernoulli's principle, BP. Assuming this, in conjunction with other solutions, BP linearizes the equation. In industrial applications, pumps are provided with characteristics in terms of flow pressure curves Strömberg (2007). Result is BP, with losses due to friction at walls, and damping.

For example a whirlwind, characterised by its pressure, pressure gradient, and wind velocity, is a flow-formation subjected to different boundary conditions, when traveling in varying surrounding flows. At sea, energy is collected from the hot surface water. Entering land from sea, BC changes, such that pressure decreases and wind velocity increases.

In the present context we will focus on flow, consisting of so-called transient pile-ups, such that all the requirements in [1] are met.

Solutions

Pile-up flow

To meet requirements of incompressibility and the requirements (A) [1], we consider a flow such that u is transient and decreases with spatial coordinates as $\exp(-br^2)$, where $r^2=x*x$.

Theorem1. A solution fulfilling requirements (A) reads $u_1=x_2x_3\exp(-br^2)\exp(-at)$, $u_2=x_3x_1\exp(-br^2)\exp(-at)$, $u_3=-2x_1x_2\exp(-br^2)\exp(-at)$,

This will be denoted Lena Pile-Up flow.

Proof. Since there are no restrictions on the functions f and p, except the continuity of derives and that Navier-Stokes shall be fulfilled, there are several possibilities. For example, let f be such that $f=f_l+f_{nl}+f_p$ where f_l balance the linear terms in velocity, f_{nl} balance the nonlinear terms in velocity and f_p balance the gradient of pressure. Another possibility is that p balances the nonlinear terms such that BP is fulfilled/satisfied i.e. p + $u^2/2$ is constant, and f balance the linear terms in velocity (due to viscosity, and inertia).

Detailed proof of required regularity

By regularity, we will mean 'how continuous, the functions are', e.g. C^2, means that two derives are continuous and limited. It should be showed that the solutions are C^{inf}, meaning that the functions and all its' derives should be limited, i.e. smaller than the function of a sphere given in [1].

For the derive, it is sufficient to consider the 'largest' term, i.e. when the exponential function in the product is differentiated. For simplicity, we will omit factor b, in the proof.

Theorem 1. The Lena Pile-Up, together with the above preliminaries for f and p, fulfils the requirements in [1].
Proof. Regularity, such that all derives and functions are limited in R^3:
The proof relies on decomposition, Cauchy-Schwarz inequality and induction.

Let w denote the j:th derive multiplied with exp(r).

Then we may write the (j+1): th derive as $\exp(-r)*x_i*w$

Assume that w is limited (i.e. smaller than the function specified in [1]). Note that this assumption implies that the j:th derive is 'even more' limited.

Due to Cauchy-Schwarz, the (j+1): th derive, is limited if $v_i= \exp(-r)*x_i$ is limited.

Since exp(-r) is very much decreasing, v_i is obviously limited, but a detailed proof will be given, because it will also provide the regularity for the 0:th derive, which is needed for induction.

The 'largest' term for derive of v_i, is $(x_j/r)x_i \exp(-r)$. Evaluation of the norm and using Cauchy-Schwarz gives

$|(x_j/r)x_i \exp(-r)| \leq |\mathrm{sgn}(x_j/r)|\,|x_i \exp(-r)| \leq |x_i \exp(-r)| = |v_i|$
i.e. this is smaller than the norm of v_i, which proofs that v_i limited.
The 0:th derive is the velocity u_k, which can be decomposed as

$u_k =(\exp(-r)*x_{ji})(\exp(-r)*x_i)$, i.e. a product of two functions of the kind v_i.
To complete the proof, we use induction, and pre-assumed properties for j:th derive.
Thus the norm of the (j+1): th derive, is a product of two norms of functions limited in R^3, and hereby the derive itself has the desired features. *Qed*

Exercise. Analyse the function on spheres, by determination of maximum and a plot.

Remark. To explain a so-called Pile-Up, we consider the restriction to $x_1=0$. The flow is then one-dimensional, and ‘piles up’, at a point, e.g. (0,1,1), since flow from motion of material particles behind at e.g. (0,½, ½), has a larger velocity than present point.

Conclusion

The solution with Pile Up velocity field was published 29th dec 2012, on a web site, but without the detailed proof, and this version as lecture notes in Strömberg (2016). In the Proof of fulfilment of conditions at infinity, regularity was shown with a decomposition of the exponential term. Then we used Cauchy-Schwarz inequality for norms (e.g. 2-norm), and induction. The general substance is given below:

Proof by induction: Assume that the statement holds for j:th derive. Show that then it is valid for the (j+1):th derive. Then (when also valid for 0th) by induction, it is valid for all j.

References

[1] Charles Fefferman. (2000)
http://en.wikipedia.org/wiki/NavierStokes_existence_and_smoothness
http://www.claymath.org/millennium/Navier-Stokes_Equations/

[2] Strömberg L (2007). Formations of matter, light and sound described with Bernoulli's principle 10th ESAFORM Conference on Material Forming, Pts A and B / [ed] Cueto, E; Chinesta, F, MELVILLE: AMER INST PHYSICS, Vol. 907, 53-58 s.

[3] Strömberg L (2016). Noncircular orbits at MC vehicle wobbling, in whirls and for light, LAP Lambert Academic publishing, Germany. ISBN-13 978-3-659-85664-8.