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Additive Pulsating Fibonacci Sequences and Some Results

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Abstract

In this paper, we consider additive pulsating Fibonacci sequences. First, we are introduced additive pulsating Fibonacci sequences of first and second kind. Further, explicit formulas in the form of its members are formulated and proved for additive pulsating Fibonacci sequences of first and second kind.

Keywords: Fibonacci Sequence, Additive Pulsating Fibonacci Sequences

1. Introduction

Fibonacci sequence stands as a kind of super sequence with fabulous properties. Fibonacci numbers are a sequence of numbers in which each successive number is the sum of two previous numbers: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, …

Fibonacci sequence is defined by the recurrence relation:

$$F_{n+1} = F_n + F_{n-1} \text{ with } F_0 = 0,\ F_1 = 1 \text{, where } n \geq 1. \tag{1.1}$$

The Fibonacci sequence has been generalized in a number of ways [1], [2], [4]. In last decade, the coupled difference equations or recurrence relations are popularized. The coupled Fibonacci sequences are new direction in generalization of Fibonacci sequence. They involve two sequences of integers in which the elements of one sequence are part of the generalization of the other and vice versa. We can say that these are generalization of ordinary recursive sequences and many results can be developed for considering the two sequences are identical. They can be considered as the complementary concept of the intersections of linear sequences. The coupled sequences provides visual pattern of relationship.

Atanassov is the first person who introduced the concept of coupled Fibonacci sequences. He also discussed many curious properties and new direction of generalization of Fibonacci sequence in [5], [6], [8] and [9] with his friends. He defined additive coupled Fibonacci sequences of second order in four different ways and called them 2-Fibonacci sequence (or 2-F sequences). Further Atanassov *et al.* [5], [6], [9] has been studied the properties of additive coupled Fibonacci sequences of second order under two schemes. Singh *et al.* [10], presented coupled Fibonacci sequences of fifth order with some properties for positive and negative integers. Sikhwal [11] presented properties of additive and multiplicative coupled Fibonacci sequences of second order and higher order.

In 2013, Atanassov [7] presented another new idea in generalization of Fibonacci sequences in the case of one or more sequences. They constructed pulsating Fibonacci sequences of second order in additive form and established explicit formulas for pulsating Fibonacci sequences in the form of its members. Morover, Suvarnamani and Koyram [3] defined multiplicative pulsated Fibonacci sequences and formulated explicit formulas in the form of its members for multiplicative pulsated Fibonacci sequences.

In present paper, we will consider pulsating Fibonacci sequence as additive pulsating Fibonacci sequence and will call it additive pulsating Fibonacci sequences of first kind. Further, we introduced additive pulsating Fibonacci sequences of second kind. Moreover, we will establish explicit formulae in the form of its members for additive pulsating Fibonacci sequences of first and second kind.

2. Additive Pulsating Fibonacci Sequences of First Kind

In 2013, Atanassov introduced a new type of Fibonacci–like sequence and called it pulsating Fibonacci sequence.

Let $\{\alpha_k\}_{k=0}^{\infty}$ and $\{\beta_k\}_{k=0}^{\infty}$ be two infinite sequences and Let a and b be two fixed real numbers. Let us construct the following two sequences

$$\alpha_0 = a,\ \beta_0 = b,$$

$$\alpha_{2k+1} = \beta_{2k+1} = \alpha_{2k} + \beta_{2k},$$

$$\alpha_{2k+2} = \alpha_{2k+1} + \beta_{2k},$$

$$\beta_{2k+2} = \beta_{2k+1} + \alpha_{2k},$$

for natural number $k \geq 0$. This pair of sequences we called additive Pulsating Fibonacci sequences of first kind.

The first few values of the additive pulsating Fibonacci sequences of first kind are given in the following Table 1.

k	α_k	$\alpha_k = \beta_k$	β_k
0	a		b
1		$a+b$	
2	$a+2b$		$2a+b$
3		$3a+3b$	
4	$5a+4b$		$4a+5b$
5		$9a+9b$	
6	$13a+14b$		$14a+13b$
7		$27a+27b$	
8	$41a+40b$		$40a+41b$
9		$81a+81b$	
10	$121a+122b$		$122a+121b$

Table 1. Additive Pulsating Fibonacci Sequences of First Kind

Theorem(1). For every integer $k \geq 0$, prove that

$$\alpha_{2k+1} = \beta_{2k+1} = 3^k a + 3^k b,$$

$$\alpha_{2k} = \left(\frac{3^k + (-1)^k}{2} \right) a + \left(\frac{3^k + (-1)^{k+1}}{2} \right) b,$$

$$\beta_{2k} = \alpha_{2k} = \left(\frac{3^k + (-1)^{k+1}}{2} \right) a + \left(\frac{3^k + (-1)^k}{2} \right) b.$$

Proof. We will prove by mathematical induction over $k \geq 0.$

Let P_k:

$$\alpha_{2k+1} = \beta_{2k+1} = 3^k a + 3^k b, \qquad \text{(Eq. 1)}$$

$$\alpha_{2k} = \left(\frac{3^k + (-1)^k}{2} \right) a + \left(\frac{3^k + (-1)^{k+1}}{2} \right) b, \qquad \text{(Eq. 2)}$$

$$\beta_{2k} = \alpha_{2k} = \left(\frac{3^k + (-1)^{k+1}}{2} \right) a + \left(\frac{3^k + (-1)^k}{2} \right) b. \qquad \text{(Eq. 3)}$$

for every integer $k \geq 0.$

If $k = 0,$ then

$$\alpha_{2(0)+1} = \beta_{2(0)+1} = \alpha_{2(0)} + \beta_{2(0)}$$

$$= \alpha_0 + \beta_0$$

$$= a + b$$

$$= 3^0 a + 3^0 b.$$

$$\alpha_{2(0)} = \alpha_0$$

$$= a$$

$$= a + 0b$$

$$= \left(\frac{3^0 + (-1)^0}{2} \right) a + \left(\frac{3^0 + (-1)^{0+1}}{2} \right) b.$$

$$\beta_{2(0)} = \beta_0$$

$$= b$$

$$= 0a + b$$

$$= \left(\frac{3^0 + (-1)^{0+1}}{2} \right) a + \left(\frac{3^0 + (-1)^0}{2} \right) b.$$

Thus P_0 is true.

Next, we assume that P_k is true for some integer number $k \geq 0$, i.e.,

$$\alpha_{2k+1} = \beta_{2k+1} = 3^k a + 3^k b, \qquad \text{(Eq. 4)}$$

$$\alpha_{2k} = \left(\frac{3^k + (-1)^k}{2} \right) a + \left(\frac{3^k + (-1)^{k+1}}{2} \right) b, \qquad \text{(Eq. 5)}$$

$$\beta_{2k} = \alpha_{2k} = \left(\frac{3^k + (-1)^{k+1}}{2} \right) a + \left(\frac{3^k + (-1)^k}{2} \right) b. \qquad \text{(Eq. 6)}$$

Then we will show that P_{k+1} is true.

$$\alpha_{2(k+1)+1} = \beta_{2(k+1)+1}$$

$$= \alpha_{2(k+1)} + \beta_{2(k+1)}$$

$$= \alpha_{2k+2} + \beta_{2k+2}$$

$$= \alpha_{2k+1} + \beta_{2k} + \beta_{2k+1} + \alpha_{2k}$$

$$= (\alpha_{2k+1} + \beta_{2k+1}) + (\alpha_{2k} + \beta_{2k})$$

$$= \alpha_{2k+1} + \alpha_{2k+1} + \alpha_{2k+1}$$

$$= 3\alpha_{2k+1}$$

$$= 3(3^k a + 3^k b)$$

$$= 3^{k+1} a + 3^{k+1} b.$$

$$\alpha_{2(k+1)} = \alpha_{2k+2}$$

$$=\alpha_{2k+1}+\beta_{2k}$$

$$=\alpha_{2k}+\beta_{2k}+\beta_{2k}$$

$$=\alpha_{2k}+2\beta_{2k}$$

$$=\left(\frac{3^k+(-1)^k}{2}\right)a+\left(\frac{3^k+(-1)^{k+1}}{2}\right)b+2\left[\left(\frac{3^k+(-1)^{k+1}}{2}\right)a+\left(\frac{3^k+(-1)^k}{2}\right)b\right]$$

$$=\left(\frac{3^k+(-1)^k}{2}\right)a+\left(\frac{3^k+(-1)^{k+1}}{2}\right)b+\left(3^k+(-1)^{k+1}\right)a+\left(3^k+(-1)^k\right)b$$

$$=\left[\frac{3^k+(-1)^k+2\cdot 3^k+2\cdot(-1)^{k+1}}{2}\right]a+\left[\frac{3^k+(-1)^{k+1}+2\cdot 3^k+2\cdot(-1)^k}{2}\right]b$$

$$=\left[\frac{3\cdot 3^k+(-1)^{k+1}}{2}\right]a+\left[\frac{3\cdot 3^k+(-1)^k}{2}\right]b$$

$$=\left[\frac{3^{k+1}+(-1)^{k+1}}{2}\right]a+\left[\frac{3^{k+1}+(-1)^{(k+1)+1}}{2}\right]b.$$

$$\beta_{2(k+1)}=\beta_{2k+2}$$

$$=\beta_{2k+1}+\alpha_{2k}$$

$$=\alpha_{2k}+\beta_{2k}+\alpha_{2k}$$

$$=2\alpha_{2k}+\beta_{2k}$$

$$=2\left[\left(\frac{3^k+(-1)^k}{2}\right)a+\left(\frac{3^k+(-1)^{k+1}}{2}\right)b\right]+\left(\frac{3^k+(-1)^{k+1}}{2}\right)a+\left(\frac{3^k+(-1)^k}{2}\right)b$$

$$=\left(3^k+(-1)^k\right)a+\left(3^k+(-1)^{k+1}\right)b+\left(\frac{3^k+(-1)^{k+1}}{2}\right)a+\left(\frac{3^k+(-1)^k}{2}\right)b$$

$$=\left[\frac{2\cdot 3^k+2\cdot(-1)^k+3^k+(-1)^{k+1}}{2}\right]a+\left[\frac{2\cdot 3^k+2\cdot(-1)^{k+1}+3^k+(-1)^k}{2}\right]b$$

$$=\left[\frac{3\cdot 3^k+(-1)^k}{2}\right]a+\left[\frac{3\cdot 3^k+(-1)^{k+1}}{2}\right]b$$

$$= \left[\frac{3^{k+1} + (-1)^{(k+1)+1}}{2} \right] a + \left[\frac{3^{k+1} + (-1)^{k+1}}{2} \right] b.$$

So, P_{k+1} is true.

By mathematical induction P_k is true for all integer number $k \geq 0$.

3. Additive Pulsating Fibonacci Sequences of Second Kind

Now, we introduce additive pulsating Fibonacci sequences of second kind.

Let $\{\alpha_k\}_{k=0}^{\infty}$ and $\{\beta_k\}_{k=0}^{\infty}$ be two infinite sequences and Let a, b and c be three fixed real numbers. Let us construct the following two sequences

$$\alpha_0 = a,\ \beta_0 = b,$$

$$\alpha_1 = \beta_1 = c,$$

$$\alpha_{2k} = \alpha_{2k-1} + \beta_{2k-2},$$

$$\beta_{2k} = \beta_{2k-1} + \alpha_{2k-2},$$

$$\alpha_{2k+1} = \beta_{2k+1} = \alpha_{2k} + \beta_{2k},$$

for natural number $k \geq 1$. This pair of sequences we called additive Pulsating Fibonacci sequences of second kind.

The first few values of additive pulsating Fibonacci sequences of second kind are given in the following Table 2.

k	α_k	$\alpha_k = \beta_k$	β_k
0	a		b
1		c	
2	$b+c$		$a+c$
3		$a+b+2c$	
4	$2a+b+3c$		$a+2b+3c$
5		$3a+3b+6c$	

k	α_k	$\alpha_k = \beta_k$	β_k
6	$4a+5b+9c$		$5a+4b+9c$
7		$9a+9b+18c$	
8	$14a+13b+27c$		$13a+14b+27c$
9		$27a+27b+54c$	
10	$40a+41b+81c$		$41a+40b+81c$

Table 2. Additive Pulsating Fibonacci Sequences of Second Kind

Theorem(2). For every integer $k \geq 1$, prove that

$$\alpha_{2k} = \left(\frac{3^{k-1}+(-1)^k}{2}\right)a + \left(\frac{3^{k-1}+(-1)^{k-1}}{2}\right)b + 3^{k-1}c,$$

$$\beta_{2k} = \left(\frac{3^{k-1}+(-1)^{k-1}}{2}\right)a + \left(\frac{3^{k-1}+(-1)^k}{2}\right)b + 3^{k-1}c,$$

$$\alpha_{2k+1} = \beta_{2k+1} = 3^{k-1}a + 3^{k-1}b + 3^{k-1}c.$$

Proof. We will prove by mathematical induction over $k \geq 1$.

Let P_k:

$$\alpha_{2k} = \left(\frac{3^{k-1}+(-1)^k}{2}\right)a + \left(\frac{3^{k-1}+(-1)^{k-1}}{2}\right)b + 3^{k-1}c, \qquad \text{(Eq. 7)}$$

$$\beta_{2k} = \left(\frac{3^{k-1}+(-1)^{k-1}}{2}\right)a + \left(\frac{3^{k-1}+(-1)^k}{2}\right)b + 3^{k-1}c \qquad \text{(Eq. 8)}$$

$$\alpha_{2k+1} = \beta_{2k+1} = 3^{k-1}a + 3^{k-1}b + 3^{k-1}c. \qquad \text{(Eq. 9)}$$

for every integer $k \geq 1$.

If $k = 1$, then

$\alpha_{2(1)} = \alpha_2$

$= b + c$

$$=\left(\frac{3^{1-1}-1}{2}\right)a+\left(\frac{3^{1-1}+1}{2}\right)b+3^0c.$$

$$\beta_{2(1)}=\beta_2$$

$$=a+c$$

$$=\left(\frac{3^{1-1}+1}{2}\right)a+\left(\frac{3^{1-1}-1}{2}\right)b+3^0c.$$

$$\alpha_{2(1)+1}=\alpha_3$$

$$=a+b+2c$$

$$=3^{1-1}a+3^{1-1}b+2\cdot 3^{1-1}c.$$

$$\beta_{2(1)+1}=\beta_3$$

$$=a+b+2c$$

$$=3^{1-1}a+3^{1-1}b+2\cdot 3^{1-1}c.$$

Thus P_1 is true.

Next, we assume that P_k is true for some integer number $k\geq 1$, i.e.,

$$\alpha_{2k}=\left(\frac{3^{k-1}+(-1)^k}{2}\right)a+\left(\frac{3^{k-1}+(-1)^{k-1}}{2}\right)b+3^{k-1}c, \qquad \text{(Eq. 10)}$$

$$\beta_{2k}=\left(\frac{3^{k-1}+(-1)^{k-1}}{2}\right)a+\left(\frac{3^{k-1}+(-1)^k}{2}\right)b+3^{k-1}c \qquad \text{(Eq. 11)}$$

$$\alpha_{2k+1}=\beta_{2k+1}=3^{k-1}a+3^{k-1}b+3^{k-1}c. \qquad \text{(Eq. 12)}$$

Then we will show that P_{k+1} is true.

$$\alpha_{2(k+1)}=\alpha_{2k+2}$$

$$=\alpha_{2k+1}+\beta_{2k}$$

$$=\alpha_{2k}+\beta_{2k}+\beta_{2k}$$

$$=\alpha_{2k}+2\beta_{2k}$$

$$=\left(\frac{3^{k-1}+(-1)^{k}}{2}\right)a+\left(\frac{3^{k-1}+(-1)^{k-1}}{2}\right)b+3^{k-1}c+2\left[\left(\frac{3^{k-1}+(-1)^{k-1}}{2}\right)a+\left(\frac{3^{k-1}+(-1)^{k}}{2}\right)b+3^{k-1}c\right]$$

$$=\left(\frac{3^{k-1}+(-1)^{k}}{2}\right)a+\left(\frac{3^{k-1}+(-1)^{k-1}}{2}\right)b+3^{k-1}c+\left(3^{k-1}+(-1)^{k-1}\right)a+\left(3^{k-1}+(-1)^{k}\right)b+2\cdot3^{k-1}c$$

$$=\left[\frac{3^{k-1}+(-1)^{k}+2\cdot3^{k-1}+2\cdot(-1)^{k-1}}{2}\right]a+\left[\frac{3^{k-1}+(-1)^{k-1}+2\cdot3^{k-1}+2\cdot(-1)^{k}}{2}\right]b+3\cdot3^{k-1}c$$

$$=\left[\frac{3\cdot3^{k-1}+(-1)^{k+1}}{2}\right]a+\left[\frac{3\cdot3^{k-1}+(-1)^{k}}{2}\right]b+3\cdot3^{k-1}c$$

$$=\left[\frac{3^{k}+(-1)^{k+1}}{2}\right]a+\left[\frac{3^{k}+(-1)^{k}}{2}\right]b+3^{k}c.$$

$$\beta_{2(k+1)}=\beta_{2k+2}$$

$$=\beta_{2k+1}+\alpha_{2k}$$

$$=\alpha_{2k}+\beta_{2k}+\alpha_{2k}$$

$$=2\alpha_{2k}+\beta_{2k}$$

$$=2\left[\left(\frac{3^{k-1}+(-1)^{k}}{2}\right)a+\left(\frac{3^{k-1}+(-1)^{k-1}}{2}\right)b+3^{k-1}c\right]+\left(\frac{3^{k-1}+(-1)^{k-1}}{2}\right)a+\left(\frac{3^{k-1}+(-1)^{k}}{2}\right)b+3^{k-1}c$$

$$=\left(3^{k-1}+(-1)^{k}\right)a+\left(3^{k-1}+(-1)^{k-1}\right)b+2\cdot3^{k-1}c+\left(\frac{3^{k-1}+(-1)^{k-1}}{2}\right)a+\left(\frac{3^{k-1}+(-1)^{k}}{2}\right)b+3^{k-1}c$$

$$=\left[\frac{2\cdot3^{k-1}+2\cdot(-1)^{k}+3^{k-1}+(-1)^{k-1}}{2}\right]a+\left[\frac{2\cdot3^{k-1}+2\cdot(-1)^{k-1}+3^{k-1}+(-1)^{k}}{2}\right]b+3\cdot3^{k-1}c$$

$$=\left[\frac{3\cdot3^{k-1}+(-1)^{k}}{2}\right]a+\left[\frac{3\cdot3^{k-1}+(-1)^{k+1}}{2}\right]b+3\cdot3^{k-1}c$$

$$=\left[\frac{3^{k}+(-1)^{k}}{2}\right]a+\left[\frac{3^{k}+(-1)^{k+1}}{2}\right]b+3^{k}c.$$

$$\alpha_{2(k+1)+1}=\beta_{2(k+1)+1}$$

$$= \alpha_{2(k+1)} + \beta_{2(k+1)}$$

$$= \alpha_{2k+2} + \beta_{2k+2}$$

$$= \alpha_{2k+1} + \beta_{2k} + \beta_{2k+1} + \alpha_{2k}$$

$$= \left(\alpha_{2k+1} + \beta_{2k+1}\right) + \left(\alpha_{2k} + \beta_{2k}\right)$$

$$= \alpha_{2k+1} + \alpha_{2k+1} + \alpha_{2k+1}$$

$$= 3\alpha_{2k+1}$$

$$= 3\left(3^{k-1}a + 3^{k-1}b + 3^{k-1}c\right)$$

$$= 3^k a + 3^k b + 3^k c.$$

So, P_{k+1} is true.

By mathematical induction P_k is true for all integer number $k \geq 1$.

4. Conclusion

In this study, we introduced additive pulsating Fibonacci sequences of first and second kind. Further, we established explicit formulas in the form of its members for additive pulsating Fibonacci sequences of first and second kind. This concept can apply to define and study fundamental properties for pulsating Fibonacci sequences in multiplicative and subtractive form as well.

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