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Generalized Fibonacci Polynomials and its Properties

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Abstract

The Fibonacci polynomials and Lucas polynomials are famous for possessing wonderful and amazing properties and identities. In this paper, Generalized Fibonacci polynomials are introduced and study its properties. Further, some recurrence relations state and derive for generalized Fibonacci polynomials.

Keywords: Fibonacci Polynomial, Generalized Fibonacci Polynomial

1. Introduction

It is well-known that the Fibonacci polynomials and Lucas polynomials are closely related and widely investigated. Fibonacci polynomials appear in different frameworks. These polynomials are of great importance in the study of many subjects such as algebra, geometry, combinatorics, approximation theory, statistics and number theory itself. Moreover these polynomials have been applied in every branch of mathematics. Fibonacci polynomials are special cases of Chebyshev polynomials and have been studied on a more advanced level by many mathematicians.

The Fibonacci polynomials [6] are defined by

$$f_{n+1}(x)=xf_n(x)+f_{n-1}(x),\ n\geq 1 \text{ with } f_0(x)=0,\ f_1(x)=1. \qquad \text{(Eq. 1.1)}$$

Similarly, The Lucas polynomials [6] are defined by

$$l_{n+1}(x)=xl_n(x)+l_{n-1}(x),\ n\geq 1 \text{ with } l_0(x)=2,\ l_1(x)=x. \qquad \text{(Eq.1. 2)}$$

In this paper, we introduce and define generalized Fibonacci polynomials and study its properties. Further, some recurrence relations state and derive for generalized Fibonacci polynomials.

2. Generalized Fibonacci Polynomials

We define generalized Fibonacci polynomials (GFP) $\{g_n(x)\}_{n=0}^{\infty}$ by recurrence relation:

$$g_n(x)=pxg_{n-1}(x)+qg_{n-2}(x),\ n\geq 2 \text{ with } g_0(x)=a,\ g_1(x)=(a+b)x \qquad \text{(Eq. 2.1)}$$

where a, b, p and q are any integer.

For $x=1$, we obtain generalized Fibonacci sequence [10].

The first few generalized Fibonacci polynomials are as follows:

$$g_1(x)=(a+b)x,$$

$$g_2(x)=(a+b)px^2+aq,$$

$$g_3(x)=(a+b)p^2x^3+apqx+(a+b)qx,$$

$$g_4(x)=(a+b)p^3x^4+ap^2qx^2+2(a+b)pqx^2+aq^2,$$

$$g_5(x)=(a+b)p^4x^5+ap^3qx^3+3(a+b)p^2qx^3+2apq^2x+(a+b)q^2x,\ \ldots$$

The characteristic equation for recurrence relation (eq. 2.1) is $t^2-pxt-q=0$.

This equation has two real roots (say); $\alpha_1 = \frac{px+\sqrt{p^2x^2+4q}}{2}$ and $\beta_1 = \frac{px-\sqrt{p^2x^2+4q}}{2}$.

Also, $\alpha_1\beta_1 = -q,\ \alpha_1+\beta_1 = px,\ \alpha_1-\beta_1 = \sqrt{p^2x^2+4q},\ \alpha_1^2+\beta_1^2 = p^2x^2+2q.$ (Eq. 2.2)

Generalized Fibonacci polynomials generalize many polynomials for different values of a, b, p and q. Examples of such polynomials are Fibonacci polynomials, Lucas polynomials, Pell polynomials, Pell-Lucas polynomials, modified or associated Pell polynomials, Jacobthal polynomials, Jacobsthal-Lucas polynomials, Fibonacci-Like polynomials etc.

- For $a=0,\ b=1,\ p=1$ and $q=1$, we obtain Fibonacci polynomials [6].
- For $a=2,\ b=-1,\ p=1$ and $q=1$, we obtain Lucas polynomials [6].
- For $a=0,\ b=1,\ p=2$ and $q=1$, we obtain Pell polynomials [4].
- For $a=2,\ b=0,\ p=2$ and $q=1$, we obtain Pell-Lucas polynomials [4].
- For $a=0,\ b=1,\ p=1$ and $q=2$, we obtain Jacobsthal polynomials [5].
- For $a=2,\ b=-1,\ p=1$ and $q=2$, we obtain Jacobsthal-Lucas polynomials [5].
- For $a=2,\ b=0,\ p=1$ and $q=1$, we obtain Fibonacci-Like polynomials [7].

Further, generalized Fibonacci polynomials generalize many sequences for $x=1$ and different values of a, b, p and q. Examples of such sequences are Fibonacci sequence, Lucas sequence, Pell sequence, Pell-Lucas sequence, modified or associated Pell sequence, Jacobthal sequence, Jacobsthal-Lucas sequence, Fibonacci-Like sequences etc (see [1], [2], [8], [3], [8], [9]).

3. Generating Function of Generalized Fibonacci Polynomials

Generating function of generalized Fibonacci polynomials is given by

$$g(x) = \sum_{n=0}^{\infty} g_n(x)t^n = \frac{a+(a+b-ap)xt}{1-pxt-qt^2}. \quad \text{(Eq. 3.1)}$$

Applying power series to the generalized Fibonacci polynomials $\{g_n(x)\}_{n=0}^{\infty}$, we have

$$a+(a+b)xt+\left(apx^2+bpx^2+aq\right)t^2+\ldots=\sum_{n=0}^{\infty}g_n(x)t^n$$

This is called generating series of generalized Fibonacci polynomials. Now, multiplying the generating series by $\left(1-pxt-qt^2\right)$.

$$\left(1-pxt-qt^2\right)\sum_{n=0}^{\infty}g_n(x)t^n=\sum_{n=0}^{\infty}g_n(x)t^n-px\sum_{n=0}^{\infty}g_n(x)t^{n+1}-q\sum_{n=0}^{\infty}g_n(x)t^{n+2}$$

$$=\left[g_0(x)+g_1(x)t+\sum_{n=2}^{\infty}g_n(x)t^n\right]-px\left[g_0(x)t+\sum_{n=2}^{\infty}g_{n-1}(x)t^n\right]-q\sum_{n=2}^{\infty}g_{n-2}(x)t^n$$

$$=g_0(x)+\left[g_1(x)-pxg_0(x)\right]t+\sum_{n=2}^{\infty}\left[g_n(x)-pxg_{n-1}(x)-qg_{n-2}(x)\right]t^n$$

$$=a+\left[(a+b)x-apx\right]t+\sum_{n=2}^{\infty}\left[pxg_{n-1}(x)+qg_{n-2}(x)-pxg_{n-1}(x)-qg_{n-2}(x)\right]t^n=a+(a+b-ap)xt$$

Therefore, $\left(1-pxt-qt^2\right)\sum_{n=0}^{\infty}g_n(x)t^n==a+(a+b-ap)xt$ **.**

Hence, $\sum_{n=0}^{\infty}g_n(x)t^n=\dfrac{a+(a+b-ap)xt}{1-pxt-qt^2}$.

4. Hypergeometric Representation of Generating Function

Hypergeometric representation of generating function of generalized Fibonacci polynomials is given by

$$\sum_{n=0}^{\infty}\frac{g_n(x)}{n!}\,t^n=\left[a+(a+b-ap)xt\right]e^{pxt}\,{}_2F_1\left(n+1,\,1;\,1;\,qt^2\right). \qquad \text{(Eq. 4. 1)}$$

By generating function (eq. 3.1), we have

$$\sum_{n=0}^{\infty}g_n(x)t^n=\frac{a+(a+b-ap)xt}{1-pxt-qt^2}$$

$$=\left[a+(a+b-ap)xt\right]\left[1-(px+qt)t\right]^{-1}$$

$$=\left[a+\left(a+b-ap\right)xt\right]\sum_{n=0}^{\infty}\left(px+qt\right)^{n}t^{n}$$

$$=\left[a+\left(a+b-ap\right)xt\right]\sum_{n=0}^{\infty}t^{n}\sum_{k=0}^{n}\binom{n}{k}\left(px\right)^{n-k}\left(qt\right)^{k}$$

$$=\left[a+\left(a+b-ap\right)xt\right]\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{n!}{k!\,n-k!}\left(px\right)^{n-k}q^{k}t^{n+k}$$

$$=\left[a+\left(a+b-ap\right)xt\right]\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{n+k!}{k!\,n!}\left(px\right)^{n}q^{k}t^{n+2k}$$

$$=\left[a+\left(a+b-ap\right)xt\right]\sum_{n=0}^{\infty}\frac{\left(pxt\right)^{n}}{n!}\sum_{k=0}^{\infty}\frac{n+k!}{k!}q^{k}t^{2k}$$

$$=\left[a+\left(a+b-ap\right)xt\right]e^{pxt}\sum_{k=0}^{\infty}\frac{n+k!}{k!}\left(qt^{2}\right)^{k}$$

$$\sum_{n=0}^{\infty}\frac{g_{n}\left(x\right)}{n!}t^{n}=\left[a+\left(a+b-ap\right)xt\right]e^{pxt}\sum_{k=0}^{\infty}\frac{n+k!}{n!}\frac{\left(qt^{2}\right)^{k}}{k!}$$

Hence, $\sum_{n=0}^{\infty}\frac{g_{n}\left(x\right)}{n!}t^{n}=\left[a+\left(a+b-ap\right)xt\right]e^{pxt}{}_{2}F_{1}\left(n+1,\,1;\,1;\,qt^{2}\right).$

5. Binet’s Formula for Generalized Fibonacci Polynomials

Binet’s formula for generalized Fibonacci polynomials is given by

$$g_{n}\left(x\right)=A_{1}\alpha_{1}^{n}+B_{1}\beta_{1}^{n}=A_{1}\left(\frac{px+\sqrt{p^{2}x^{2}+4q}}{2}\right)^{n}+B_{1}\left(\frac{px-\sqrt{p^{2}x^{2}+4q}}{2}\right)^{n}. \qquad \text{(Eq. 5.1)}$$

Here, $A_{1}=\frac{(a+b)x-a\beta_{1}}{\alpha_{1}-\beta_{1}}$ and $B_{1}=\frac{a\alpha_{1}-(a+b)x}{\alpha_{1}-\beta_{1}}$.

Also, $A_{1}B_{1}=\frac{a\left(a+b\right)px^{2}+a^{2}q-\left(a+b\right)^{2}x^{2}}{\left(\alpha_{1}-\beta_{1}\right)^{2}}$ (Eq. 5.2)

$$A_1\beta_1 + B_1\alpha_1 = (ap - a - b)x \tag{Eq. 5.3}$$

$$A_1\beta_1^2 + B_1\alpha_1^2 = ap^2x^2 + aq - (a+b)px^2. \tag{Eq. 5.4}$$

6. Some Identities of Generalized Fibonacci Polynomials

In this section, we obtain some identities of generalized Fibonacci polynomials through generating function and Binet's formula.

Theorem (6.1). (Explicit Sum Formula) Let $g_n(x)$ be the n^{th} generalized Fibonacci polynomial. Then

$$g_n(x) = a\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-k}{k}(px)^{n-2k}q^k + \left(\frac{a+b}{p} - a\right)\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n-k-1}{k}(px)^{n-2k}q^k. \tag{Eq. 6.1}$$

Proof. By generating function (Eq. 3.1), we have

$$\sum_{n=0}^{\infty} g_n(x)t^n = \frac{a + (a+b-ap)xt}{1 - pxt - qt^2}$$

$$= \left[a + (a+b-ap)xt\right]\left[1 - (px+qt)t\right]^{-1}$$

$$= \left[a + (a+b-ap)xt\right]\sum_{n=0}^{\infty}(px+qt)^n t^n$$

$$= \left[a + (a+b-ap)xt\right]\sum_{n=0}^{\infty} t^n \sum_{k=0}^{n}\binom{n}{k}(px)^{n-k}(qt)^k$$

$$= \left[a + (a+b-ap)xt\right]\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{n!}{k!\,n-k!}(px)^{n-k}q^k t^{n+k}$$

$$= \left[a + (a+b-ap)xt\right]\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{n+k!}{k!\,n!}(px)^{n}q^k t^{n+2k}$$

$$= \left[a + (a+b-ap)xt\right]\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n-k!}{k!\,n-2k!}(px)^{n-2k}q^k t^{n}$$

$$= \sum_{n=0}^{\infty}\left[a\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n-k!}{k!\,n-2k!}\ (px)^{n-2k}\,q^{k}\right]t^{n}+\sum_{n=0}^{\infty}\left[\left(\frac{a+b}{p}-a\right)\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n-k!}{k!\,n-2k!}\ (px)^{n-2k+1}\,q^{k}\right]t^{n+1}$$

Equating the coefficient of 't^n' on both sides, we obtain

$$g_n(x) = a\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-k}{k}\ (px)^{n-2k}\,q^{k}+\left(\frac{a+b}{p}-a\right)\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n-k-1}{k}\ (px)^{n-2k}\,q^{k}.$$

Theorem (6.2). (Sum of First n terms) Sum of first n terms of generalized Fibonacci polynomials is given by

$$\sum_{k=0}^{n-1} g_k(x) = \frac{g_n(x)+qg_{n-1}(x)-(a+b-ap)x-a}{px+q-1}. \qquad \text{(Eq. 6.2)}$$

Proof. Using the Binet's formula (eq. 5.1), we have

$$\sum_{k=0}^{n-1} g_k(x) = \sum_{k=0}^{n-1}\left[A_1\alpha_1^k + B_1\beta_1^k\right]$$

$$= A_1\sum_{k=0}^{n-1}\alpha_1^k + B_1\sum_{k=0}^{n-1}\beta_1^k$$

$$= A_1\left[\frac{1-\alpha_1^n}{1-\alpha_1}\right]+B_1\left[\frac{1-\beta_1^n}{1-\beta_1}\right]$$

$$= \frac{(A_1+B_1)-(A_1\beta_1+B_1\alpha_1)-\left(A_1\alpha_1^n+B_1\beta_1^n\right)+\alpha_1\beta_1\left(A_1\alpha_1^{n-1}+B_1\beta_1^{n-1}\right)}{1-(\alpha_1+\beta_1)+\alpha_1\beta_1}$$

By using (eq. 5.2), (eq. 5.1) and (eq. 5.3), we obtain required result.

Theorem (6.3). (Sum of First n terms with odd indices) Sum of first n terms with odd indices of generalized Fibonacci polynomials is given by

$$\sum_{k=0}^{n-1} g_{2k+1}(x) = \frac{g_{2n+1}(x)-q^2 g_{2n-1}(x)+(a+b-ap)qx-(a+b)x}{p^2x^2-q^2+2q-1}. \qquad \text{(Eq. 6.3)}$$

Proof. Using the Binet's formula (eq. 5.1), we have

$$\sum_{k=0}^{n-1} g_{2k+1}(x) = \sum_{k=0}^{n-1}\left[A_1\alpha_1^{2k+1} + B_1\beta_1^{2k+1}\right]$$

$$= A_1\sum_{k=0}^{n-1}\alpha_1^{2k+1} + B_1\sum_{k=0}^{n-1}\beta_1^{2k+1}$$

$$= A_1\left[\frac{\alpha_1\left(1-\alpha_1^{2n}\right)}{1-\alpha_1^2}\right] + B_1\left[\frac{\beta_1\left(1-\beta_1^{2n}\right)}{1-\beta_1^2}\right]$$

$$= \frac{\left(A_1\alpha_1 + B_1\beta_1\right) - \alpha_1\beta_1\left(A_1\beta_1 + B_1\alpha_1\right) - \left(A_1\alpha_1^{2n+1} + B_1\beta_1^{2n+1}\right) + \left(\alpha_1\beta_1\right)^2\left(A_1\alpha_1^{2n-1} + B_1\beta_1^{2n-1}\right)}{1-\left(\alpha_1^2+\beta_1^2\right)+\left(\alpha_1\beta_1\right)^2}$$

By using (eq. 5.2), (eq. 5.1) and (eq. 5.3), we obtain required result.

Theorem (6.4). (Sum of First *n* terms with even indices) Sum of first *n* terms with even indices of generalized Fibonacci polynomials is given by

$$\sum_{k=0}^{n-1} g_{2k}(x) = \frac{g_{2n}(x) - q^2 g_{2n-2}(x) - \left[(a+b)px^2 - a\left(p^2x^2+q\right)\right] - a}{p^2x^2 - q^2 + 2q - 1}. \qquad \text{(Eq. 6.4)}$$

Proof. Using the Binet's formula (eq. 5.1), we have

$$\sum_{k=0}^{n-1} g_{2k}(x) = \sum_{k=0}^{n-1}\left[A_1\alpha_1^{2k} + B_1\beta_1^{2k}\right]$$

$$= A_1\sum_{k=0}^{n-1}\alpha_1^{2k} + B_1\sum_{k=0}^{n-1}\beta_1^{2k}$$

$$= A_1\left[\frac{\left(1-\alpha_1^{2n}\right)}{1-\alpha_1^2}\right] + B_1\left[\frac{\left(1-\beta_1^{2n}\right)}{1-\beta_1^2}\right]$$

$$= \frac{\left(A_1 + B_1\right) - \left(A_1\beta_1^2 + B_1\alpha_1^2\right) - \left(A_1\alpha_1^{2n} + B_1\beta_1^{2n}\right) + \left(\alpha_1\beta_1\right)^2\left(A_1\alpha_1^{2n-2} + B_1\beta_1^{2n-2}\right)}{1-\left(\alpha_1^2+\beta_1^2\right)+\left(\alpha_1\beta_1\right)^2}$$

By using (eq. 5.2), (eq. 5.1) and (eq. 5.4), we obtain required result.

Theorem (6.5). (Catalan's Identity) Let $g_n(x)$ be the n^{th} generalized Fibonacci polynomial. Then

$$g_n^2(x) - g_{n+r}(x)\,g_{n-r}(x) = \frac{(-q)^{n-r}}{\left[(a+b)^2 x^2 - a(a+b)px^2 - a^2 q\right]}\left[(a+b)xg_r(x) - ag_{r+1}(x)\right]^2,\ n > r \geq 1. \qquad \text{(Eq. 6.5)}$$

Proof. Using Binet's formula (eq. 5.1) to left hand side, we have

$$\text{LHS } = \left(A_1\alpha_1^n + B_1\beta_1^n\right)^2 - \left(A_1\alpha_1^{n+r} + B_1\beta_1^{n+r}\right)\left(A_1\alpha_1^{n-r} + B_1\beta_1^{n-r}\right)$$

$$= A_1B_1\left(\alpha_1\beta_1\right)^n\left(2 - \alpha_1^r\beta_1^{-r} - \alpha_1^{-r}\beta_1^r\right)$$

$$= -A_1B_1(-q)^{n-r}\left(\alpha_1^r - \beta_1^r\right)^2$$

$$= \frac{\left[(a+b)^2 x^2 - a(a+b)px^2 - a^2q\right]}{\left(\alpha_1 - \beta_1\right)^2}(-q)^{n-r}\left(\alpha_1^r - \beta_1^r\right)^2$$

$$= \left[(a+b)^2 x^2 - a(a+b)px^2 - a^2q\right](-q)^{n-r}\left(\frac{\alpha_1^r - \beta_1^r}{\alpha_1 - \beta_1}\right)^2$$

By $\dfrac{\alpha_1^r - \beta_1^r}{\alpha_1 - \beta_1} = \dfrac{(a+b)xg_r(x) - ag_{r+1}(x)}{(a+b)^2 x^2 - a(a+b)px^2 - a^2q}$, we obtained required identity.

As application of above identity we obtain following result.

Corollary (6.6). (Cassini's Identity) Let $g_n(x)$ be the n^{th} generalized Fibonacci polynomial. Then

$$g_n^2(x) - g_{n+1}(x)\,g_{n-1}(x) = (-q)^{n-1}\left[(a+b)^2 x^2 - a(a+b)px^2 - a^2q\right],\ n \geq 1. \qquad \text{(Eq. 6.6)}$$

Proof. Taking $r = 1$ in the Catalan's identity (eq. 6.5) the proof is complete.

Theorem (6.7). (d'Ocagne's Identity) Let $g_n(x)$ be the n^{th} generalized Fibonacci polynomial. Then

$$g_m(x)\,g_{n+1}(x) - g_{m+1}(x)\,g_n(x) = (-q)^n\left[(a+b)xg_{m-n}(x) - ag_{m-n+1}(x)\right],\ m > n \geq 0. \qquad \text{(Eq. 6.7)}$$

Proof. Using Binet's formula (eq. 5.1) to left hand side, we have

$$\text{LHS} = \left(A_1\alpha_1^m + B_1\beta_1^m\right)\left(A_1\alpha_1^{n+1} + B_1\beta_1^{n+1}\right) - \left(A_1\alpha_1^{m+1} + B_1\beta_1^{m+1}\right)\left(A_1\alpha_1^n + B_1\beta_1^n\right)$$

$$= A_1B_1\left(\alpha_1^m\beta_1^{n+1} + \alpha_1^{n+1}\beta_1^m - \alpha_1^n\beta_1^{m+1} - \alpha_1^{m+1}\beta_1^n\right)$$

$$= A_1B_1\left(\alpha_1\beta_1\right)^n\left[\beta_1\left(\alpha_1^{m-n} - \beta_1^{m-n}\right) - \alpha_1\left(\alpha_1^{m-n} - \beta_1^{m-n}\right)\right]$$

$$= -A_1B_1\left(-q\right)^n\left(\alpha_1 - \beta_1\right)\left(\alpha_1^{m-n} - \beta_1^{m-n}\right)$$

$$= \frac{\left[\left(a+b\right)^2x^2 - a\left(a+b\right)px^2 - a^2q\right]}{\left(\alpha_1 - \beta_1\right)^2}\left(-q\right)^n\left(\alpha_1 - \beta_1\right)\left(\alpha_1^{m-n} - \beta_1^{m-n}\right)$$

By $\dfrac{\alpha_1^{m-n} - \beta_1^{m-n}}{\alpha_1 - \beta_1} = \dfrac{\left(a+b\right)g_{m-n}\left(x\right) - ag_{m-n+1}\left(x\right)}{\left[\left(a+b\right)^2x^2 - a\left(a+b\right)px^2 - a^2q\right]}$, we obtain required identity.

Theorem (6.8). (Generalized Identity) Let $g_n\left(x\right)$ be the n^{th} generalized Fibonacci polynomial.Then

$$g_m\left(x\right)g_n\left(x\right) - g_{m-r}\left(x\right)g_{n+r}\left(x\right) =$$

$$\left(-q\right)^{m-r}\frac{\left[\left(a+b\right)xg_r\left(x\right) - ag_{r+1}\left(x\right)\right]\left[\left(a+b\right)xg_{n-m+r}\left(x\right) - ag_{n-m+r+1}\left(x\right)\right]}{\left[\left(a+b\right)^2x^2 - a\left(a+b\right)px^2 - a^2q\right]},\ n > m \geq r \geq 1. \qquad \text{(Eq. 6.8)}$$

Proof. Using Binet's formula (Eq. 5.1) to left hand side, we have

$$\text{LHS} = \left(A_1\alpha_1^m + B_1\beta_1^m\right)\left(A_1\alpha_1^n + B_1\beta_1^n\right) - \left(A_1\alpha_1^{m-r} + B_1\beta_1^{m-r}\right)\left(A_1\alpha_1^{n+r} + B_1\beta_1^{n+r}\right)$$

$$= A_1B_1\left(\alpha_1^r - \beta_1^r\right)\left[\frac{\alpha_1^m\beta_1^n}{\alpha_1^r} - \frac{\alpha_1^n\beta_1^m}{\beta_1^r}\right]$$

$$= A_1B_1\frac{\left(\alpha_1^r - \beta_1^r\right)}{\left(\alpha_1\beta_1\right)^r}\left(\alpha_1^m\beta_1^{n+r} - \alpha_1^{n+r}\beta_1^m\right)$$

$$= -A_1B_1\left(-q\right)^{m-r}\left(\alpha_1^r - \beta_1^r\right)\left(\alpha_1^{n-m+r} - \beta_1^{n-m+r}\right)$$

$$=\frac{\left[(a+b)^2 x^2 - a(a+b)px^2 - a^2 q\right]}{(\alpha_1-\beta_1)^2}(-q)^{m-r}\left(\alpha_1^r-\beta_1^r\right)\left(\alpha_1^{n-m+r}-\beta_1^{n-m+r}\right)$$

By $\frac{\alpha_1^r-\beta_1^r}{\alpha_1-\beta_1}=\frac{\left[(a+b)xg_r(x)-ag_{r+1}(x)\right]}{\left[(a+b)^2 x^2-a(a+b)px^2-a^2q\right]}$

and $\frac{\alpha_1^{n-m+r}-\beta_1^{n-m+r}}{\alpha_1-\beta_1}=\frac{\left[(a+b)xg_{n-m+r}(x)-ag_{n-m+r+1}(x)\right]}{\left[(a+b)^2 x^2-a(a+b)px^2-a^2q\right]}$, we obtain required identity.

- The identity (eq. 6.8) provides Catalan's, Cassini's and d'Ocagne's identities.
- If $m=n$, Catalan's identity (6.5) is obtained.
- If $m=n$ and $r=1$, Cassini's identity (eq. 6.6) is obtained.
- If $n=m$, $m=n+1$ and $r=1$, d'Ocagne's identity (eq. 6.7) is obtained.

7. Some Recurrence Relations

In this section, we state and derive some recurrence relations for generalized Fibonacci polynomials.

Theorem (7.1). Prove that

$$g_n(x)=pxg_{n-1}(x)+qg_{n-2}(x),\ n\geq 2. \qquad \text{(Eq. 7.1)}$$

Proof. Differentiating generating function (eq. 3.1) w.r.t. t, we have

$$\sum_{n=0}^{\infty} ng_n(x)t^{n-1}=(px+2qt)\left[a+(a+b-ap)xt\right]\left(1-pxt-qt^2\right)^{-2}+$$
$$(a+b-ap)x\left(1-pxt-qt^2\right)^{-1}$$

$$\Rightarrow\left(1-pxt-qt^2\right)\sum_{n=0}^{\infty} ng_n(x)t^{n-1}=(px+2qt)\sum_{n=0}^{\infty} g_n(x)t^n+(a+b-ap)x$$

$$\Rightarrow\left(1-pxt-qt^2\right)\sum_{n=0}^{\infty}ng_n\left(x\right)t^{n-1}=\left(px+2qt\right)\sum_{n=0}^{\infty}g_n\left(x\right)t^n+$$
$$\frac{1}{t}\left[\left(1-pxt-qt^2\right)\sum_{n=0}^{\infty}g_n\left(x\right)t^n-a\right]$$

$$\Rightarrow\sum_{n=0}^{\infty}ng_n\left(x\right)t^n-\sum_{n=0}^{\infty}pnxg_n\left(x\right)t^{n+1}-\sum_{n=0}^{\infty}qng_n\left(x\right)t^{n+2}=\sum_{n=0}^{\infty}pxg_n\left(x\right)t^{n+1}+$$
$$\sum_{n=0}^{\infty}2qg_n\left(x\right)t^{n+2}+\sum_{n=0}^{\infty}g_n\left(x\right)t^n-\sum_{n=0}^{\infty}pxg_n\left(x\right)t^{n+1}-\sum_{n=0}^{\infty}qg_n\left(x\right)t^{n+2}-a$$

$$\Rightarrow\sum_{n=0}^{\infty}\left(n-1\right)g_n\left(x\right)t^n=\sum_{n=0}^{\infty}pnxg_n\left(x\right)t^{n+1}+\sum_{n=0}^{\infty}q\left(n+1\right)g_n\left(x\right)t^{n+2}-a$$

Equating the coefficient of 't^n' on both sides, we have

$$g_n\left(x\right)=pxg_{n-1}\left(x\right)+qg_{n-2}\left(x\right),\ n\geq 2.$$

Theorem (7.2). Prove that

$$g_n'\left(x\right)=pxg_{n-1}'\left(x\right)+qg_{n-2}'\left(x\right)+pg_{n-1}\left(x\right),\ n\geq 2. \qquad \text{(Eq. 7.2)}$$

Proof. By differentiating equation (eq. 7.1) w.r.t. x, we obtain required relation.

Theorem (7.3). Prove that

$$g_n\left(-x\right)=\left(-1\right)^n g_n\left(x\right),\ n\geq 0. \qquad \text{(Eq. 7.3)}$$

Proof. Replacing x by '$-x$' in generating function (eq. 3.1), we have

$$\sum_{n=0}^{\infty}g_n\left(-x\right)t^n=\frac{a-\left(a+b-ap\right)xt}{1+pxt-qt^2}. \qquad \text{(Eq. 7.4)}$$

Now replacing t by '$-t$' in generating function (eq. 3.1), we have

$$\sum_{n=0}^{\infty}\left(-1\right)^n g_n\left(x\right)t^n=\frac{a-\left(a+b-ap\right)xt}{1+pxt-qt^2}. \qquad \text{(Eq. 7.5)}$$

From (eq. 7.4) and (eq. 7.5), it follows that

$$\sum_{n=0}^{\infty}g_n\left(-x\right)t^n=\sum_{n=0}^{\infty}\left(-1\right)^n g_n\left(x\right)t^n.$$

Equating the coefficients of 't^n' on both sides, we obtain required relation.

Theorem (7.4). Prove that

$$g_{2n}(0) = aq^n \text{ and } g_{2n+1}(0) = 0,\ n \geq 0 \qquad \text{(Eq. 7.6)}$$

Proof. Putting $x = 0$ in generating function (eq. 3.1), we have

$$\sum_{n=0}^{\infty} g_n(0)t^n = \frac{a}{1-qt^2} = a\left(1-qt^2\right)^{-1}.$$

$$\Rightarrow \sum_{n=0}^{\infty} g_n(0)t^n = a\sum_{n=0}^{\infty} q^n t^{2n}.$$

Equating the coefficients of 't^{2n}' and 't^{2n+1}' on both sides, we obtain required relations.

Conclusion

In this paper, generalized Fibonacci polynomials are introduced. Further, some of its properties studied through their generating fuction and Binet's formula. Moreover, some recurrence relations are stated and derived.

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