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The geometric interpretation of some mathematical expressions containing the Riemann ζ-function.

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Abstract

The article discusses some of the mathematical results widely used in practice which contain the Riemann ζ-function, and, at first glance, are in contradiction with common sense. A geometric approach is suggested, based on the concept of the curvature of space, in which is calculated an algorithm that specifies the representation of ζ -function as an infinite diverging series. The analysis is based on the use of Einstein equations to calculate the metric of curved space.

Keywords: Riemann ζ-function, Einstein equations, metric, metric tensor, energy-momentum tensor, Christoffel symbols, algorithm.

Riemann ζ - function is an element of the mathematical apparatus, widely used in various areas of modern science: mathematics, physics, etc. The results expressed by the ζ-function for humans dealing with them for the first time seem puzzling, although experts perceive them as firmly justified. To such results relates the formula representing ζ (-1) in the form of divergent series used, for example, in the string theory [1]: $\varsigma(-1) = 1 + 2 + 3 + 4 + ... = -1/12$ [2].

Although the theory of divergent series is sufficiently developed [3] nevertheless, it is interesting to try to give a different interpretation of known expressions.

Let us formulate a simple physical task - to calculate the distance which is gone by a body for a time *t*, moving along a straight line with constant acceleration.

Mathematically, the answer can be represented as a finite sum

$$S(t) = \sum_{n=1}^{t} n \qquad (1)$$

In the formula (1) acceleration set equal to 1. The time *t* is assumed discrete and dimensionless. We restrict ourselves with only one value of time $t = \infty$. The answer is known and can be expressed with the help of above mention series: S(∞) = ζ(-1)=-1/12 [1, 2]. From a mathematical point of view, it is true, because obtained by analytic continuation of the Riemann ζ-function on the value of the argument u = -1[2]:

$$\varsigma(u) = \sum_{n=1}^{\infty} n^{-u} \qquad (2)$$

However, it is neither obvious nor clear. The reason is that the value of ζ (-1) is represented by divergent series which does not have the sum in the usual sense, inherent in the concept of the sum of convergent series [3]. While the sum of a convergent series, which is understood as the limit of the partial sums of the series is intuitively clear for divergent series intuition fails to work and you have to attribute "for the word "sum" meaning which is different from the usual" [4]. But even agreeing to this extension of the definition, have to deal with the fact that different sums are ascribed for one and the same divergent series [3]. Other words, the value of the sum is dependent on the "context". We will try to give a geometric interpretation of the process of finding the "sum" of a divergent series and, thus, avoid some contradictions that still exist in this regard.

Recall that the uniformly accelerated motion of the body requires according to Newton's second law, that a constant force operates on the body in the direction of its movement. This effect can be achieved by placing at the point $x = 0$ (the direction of movement of the body we take as the axis OX of our frame of reference) an infinite plane coincident with the plane YOZ, having a constant mass density σ. The gravitational potential of this plane is equal to $\varphi(x) = 2\pi\sigma Kx$, and the force acting on a unit point mass is equal to $E = -\nabla\varphi = -2\pi\sigma K$ and is directed along OX, K – is gravitational constant. The expression for the space-time metric we find from solving the Einstein's equations [5]

$$R_{ik} - \frac{1}{2}g_{ik}R = \frac{8\pi K}{c^4}T_{ik} \tag{3}$$

Here R – is the trace of the Ricci tensor R^i_k: $R = R^i_i$, g_{ik} – is the metric tensor; T_{ik} - is the energy-momentum tensor; c – is the speed of light in vacuum; indices i, k have values 0, 1, 2, 3. Let us write the expression for the interval

$$\begin{aligned} & ds^2 = g_{00}c^2dt^2 + g_{11}dx^2 + g_{22}dy^2 + g_{33}dz^2 \\ & x^0 = ct, x^1 = x, x^2 = y, x^3 = z \\ & g_{00} = e^{v}, g_{11} = -e^{\lambda}, g_{22} = g_{33} = -1 \end{aligned} \tag{4}$$

We use the standard notation for g_{00} and g_{11} [5]. The solution is very similar to the Schwarzschild solution of the problem of the finding metric near a point mass [5]. We write the nonzero Christoffel symbols [5]:

$$\begin{aligned} & \Gamma^i_{kl} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^i}\right) \\ & \Gamma^0_{00} = \frac{\dot{v}}{2}, \Gamma^1_{00} = \frac{v'}{2}e^{v-\lambda}, \Gamma^0_{10} = \frac{v'}{2}, \\ & \Gamma^1_{10} = \frac{\dot{\lambda}}{2}, \Gamma^0_{11} = \frac{\dot{\lambda}}{2}e^{\lambda-v}, \Gamma^1_{11} = \frac{\lambda'}{2} \end{aligned} \tag{5}$$

The point means a derivative on ct, prime - on x^1=x. For $x \neq 0$ where T_{ik} = 0 the equations (3) can be reduced to the equations R_{ik}=0, which for R_{00} and R_{11} lead to a single equation

$$\left[\frac{v''}{2} + \frac{v'}{2}\left(\frac{v'}{2} - \frac{\lambda'}{2}\right)\right]e^{v-\lambda} - \left[\frac{\ddot{\lambda}}{2} + \frac{\dot{\lambda}}{2}\left(\frac{\dot{\lambda}}{2} - \frac{\dot{v}}{2}\right)\right] = 0 \tag{6}$$

and the equation for R_{01} is reduced to identity. Assuming λ = -v, and all time derivatives equal zero, the last equation reduces to the form

$$v'' + (v')^2 = 0 \tag{7}$$

which has a solution $e^{\nu} = C_1x+C_2$, $C_{1,2}$ – are constants. Their appropriate choice gives the desired solution

$$e^{\nu} = 1 + \frac{4\pi\sigma K}{c^2}x \tag{8}$$

considering the connection of the metric tensor component g_{00} and the Newtonian potential φ[5]: $g_{00} = 1 + 2\varphi / c^2$. Let write the final form of the interval

$$ds^2 = \left(1 + \frac{4\pi\sigma K}{c^2}x\right)c^2dt^2 - \left(1 + \frac{4\pi\sigma K}{c^2}x\right)^{-1}dx^2 - dy^2 - dz^2 \tag{9}$$

Spatial metric along the direction OX (dx=dy=0) is given by the expression $dl^2 = \dfrac{dx^2}{1 + \dfrac{4\pi\sigma K}{c^2}x}$ (10)

After integration we obtain the relationship between the coordinate x in the system of the remote observer and the distance l, passed by the body, measured in its rest frame

$$l = l_0 + \frac{c^2}{2\pi\sigma K}\sqrt{1 + \frac{4\pi\sigma K}{c^2}x} \tag{11}$$

l_0- is a integrating constant. From the formula (11) we can obtain the relationship between the lengths of the line segments traversed by the body, measured in different systems

$$\frac{dl}{dx} = \left(1 + \frac{4\pi\sigma K}{c^2}x\right)^{-1/2} \tag{12}$$

Let us analyze the resulting expression. In case of $\sigma \neq 0$ the observers in the different systems receives different values. Of particular interest is the case when $x \to x_c = -c^2 / 4\pi\sigma K$, and $dl / dx \to \infty$, although the value of l stays finite. This means that a moving body while approaching the specified point x_c on either side, in view of a distant observer will never achieves it. This is reminiscent of the behavior of a massive body in the vicinity of a black whole, from the point of view of the distant observer. The value of x_c is an analogue of the event horizon in the Schwarzschild problem.

This makes it possible to remove the contradiction, enclosed in the formula (1). According to calculation of an observer[1] moving with the body, he pass increasingly larger segments of the path, while the remote observer believes he stands still and the resulting distance traveled by him stays finite.

We can say that in the formula (1), the right side is a record of calculation algorithm which is performing by an observer or by a calculation device in its rest frame. The algorithm description looks as follows:

1. Take a segment of length 1.

2. Take a segment of length 2 and attach it to the first segment.

3. Follow this procedure infinitely, providing that length of the subsequent segment is greater the length of preceding one by 1.

4. Measure the length of the resulting segment.

The final measurement is performed by the remote observer or device. From the standpoint of general relativity their results differ as described above, which explains the contradiction of the formula (1). The use of such "physical" approach implies that the numbers in the formula (1) are regarded as coordinate values in some different frames of references (coordinate maps) applied to the real numbers axis.

Consider other possible solutions of the equation (6). Put $\nu = -\lambda = \alpha(x) + \beta(t)$. Equation (6) then reduces to a system of two equations (*m* is a parameter of the variables separation)

$$\begin{aligned} &\alpha'' + (\alpha')^2 - me^{-2\alpha} = 0 \\ &\ddot{\beta} - (\dot{\beta})^2 + me^{2\beta} = 0 \end{aligned} \qquad (13)$$

Their solutions look as follows

$$\begin{aligned} &x = x_0 \pm e^{-C_1/2m}\sqrt{\frac{\pi}{2m}}\,erfi\left(\sqrt{\frac{2m\alpha + C_1}{2m}}\right) \\ &ct = ct_0 \mp ie^{-C_2/2m}\sqrt{\frac{\pi}{2m}}\,erf\left(-\sqrt{\frac{2m\beta - C_2}{2m}}\right) \end{aligned} \qquad (14)$$

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[1] We use the terminology adopted in physics to describe the activity on the measurement of some quantity. So, as has often been noted [6] between the processes of measurement and calculation is no fundamental difference, there is reason a person (computing device) that calculates algorithm to name him (it) an observer.

where $C_{1,2}$ – are constants, and *erf* и *erfi* – probability integrals of a real and imaginary argument [7].

$$erf(v) = \frac{2}{\sqrt{\pi}} \int_0^v e^{-u^2} du$$
$$erfi(v) = \frac{2}{\sqrt{\pi}} \int_0^v e^{u^2} du \qquad (15)$$

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