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From the Ideas of Edgeworth and Pareto in Exchange Economy to Multi-Objective Mathematical Programming

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Abstract

In this paper we consider the first general theories of multi-objective mathematical programming. They stem from optimization techniques in economics and are attributed to the economists Francis Edgeworth and Vilfredo Pareto. We will focus our attention on these ideas from a mathematical point of view.

Keywords: optimization, multi-objective mathematical programming, Edgeworth-box, Pareto-optimal, equilibrium.

1. Introduction

The key concept of every optimization problem is to seek the best solution that maximizes or minimizes one or more objective functions. During optimization one often needs to consider several objective functions simultaneously. When more than one objective functions are associated, the optimization problem becomes multi-objective, in which case the usual optimization techniques for a scalar function cannot be used. This type of problem is known as either multi-criteria or vector optimization. Mathematically, it is known as multi-objective mathematical programming (MOMP) and the following records exist in 2010 Mathematics Subject Classification: 90Cxx - Mathematical programming, 90C29 - Multi-objective and goal programming, 65Kxx - Mathematical programming, optimization and variational techniques, 65K05 - Mathematical programming methods, 49Mxx - Numerical methods and 49M37 - Methods of nonlinear programming type. The standard form of a MOMP problem is to find a variable $x(x_1, x_2, ..., x_m) \in R^m$, $m \geq 1$, so as to

maximize (or minimize) $f(x) = (f_1(x), f_2(x), ..., f_n(x))$ (1)

subject to $g_i(x) \geq 0 \quad \forall i \in J_p$,

$h_i(x) = 0 \quad \forall i \in J_q$,

$x_i \in [a_i, b_i] \quad \forall i \in J_m$,

where $\{f_i : i \in J_n\}$ are given objective functions, $n \geq 2$, $\{g_i : i \in J_p\}$ are given inequality constraint functions, $p \geq 0$, $\{h_i : i \in J_q\}$ are given equality constraint functions, $q \geq 0$, a_i and b_i are the lower and upper bounds for x_i, $a_i < b_i$, and $J_m = \{1, 2, ..., m\}$ is the index set.

Let the feasible domain (or constraint set) be denoted by

$$X = \{x(x_1, x_2, ..., x_m) \in R^m : \begin{cases} g_i(x) \geq 0 & \forall i \in J_p \\ h_i(x) = 0 & \forall i \in J_q \\ a_i \leq x_i \leq b_i & \forall i \in J_m \end{cases} \}.$$

As usual, let us assume that set X is nonempty, but the index sets J_p and J_q can be either empty or nonempty.

We will discuss only maximization problems. However each minimization problem can be transformed to a maximization problem very easily by considering the negative of the objective function values.

Since the objective functions $\{f_i : i \in J_n\}$ may conflict with each other, it is usually difficult to obtain the global maximum for each objective function at the same time. So, if no single point maximizes all of the several objective functions at once, then $\bigcap_{i=1}^{n} Arg\max(f_i, X)$ is empty. Therefore, the target of the maximization problem is to obtain a set of solutions that are Pareto-optimal.

2. Notions and Definitions

Edgeworth and Pareto are often called the fathers of multi-objective mathematical programming. The notion of Edgeworth-Pareto optimality was introduced by the Irish economist Francis Edgeworth (1845-1926) in 1881. However, this concept is usually attributed to the French-Italian economist Vilfredo Pareto (1848-1923) who in 1896 developed it further, see Figure 1. Edgeworth and Pareto were studying the natural ordering in the image space in mathematical models of a simple exchange market [2]. The first mathematical consideration of this problem was done by Kuhn and Tucker in 1951 [4] [10] [11]. Since that time multi-objective optimization has became an active research field.

Francis Edgeworth (1845-1926)

Vilfredo Pareto (1848-1923)

Figure 1: The fathers of multi-objective mathematical programming

In [10], Harold Kuhn and Albert Tucker cite the 1975 Nobel Laureate in Economics Tjalling Koopmans [8] when they talk about "vector maximum of Koopmans' efficient point type for several functions". Kuhn and Tucker also cite the 1972 Nobel Laureate in Economics Kenneth Arrow [1] who contributed to the foundations of multi-objective optimization.

The origins of the mathematical foundations of multi-objective optimization can be traced back to the period from 1895 to 1906. During that period, Georg Cantor and Felix Hausdorff laid the foundations of infinite dimensional ordered spaces [2] [4] [13].

In countless books and research papers on multi-objective mathematical programming, one can find a mathematical definition essentially as following:

Definition 1. (a) A point $x \in X$ is called an ideal Pareto-optimal solution if and only if $f_i(x) \geq f_i(y)$ for all $y \in X$ and all $i \in J_n$. The set of ideal Pareto-optimal solutions of X is denoted by $IPO(X, f))$ (they are the global maximum of f on X) and is called an ideal Pareto-optimal set.

(b) A point $x \in X$ is called a Pareto-optimal solution if and only if there does not exist a point $y \in X$ such that $f_i(y) \geq f_i(x)$ for all $i \in J_n$ and $f_k(y) > f_k(x)$ for some $k \in J_n$. The set of the Pareto-optimal solutions of X is denoted by $PO(X, f)$ and is called a Pareto-optimal set. The image of a Pareto-optimal set under the objective function is called a Pareto-front set, i.e. $f(PO(X, f)) = PF(X, f)$.

(c) A point $x \in X$ is called a strictly Pareto-optimal solution if and only if there does not exist a point $y \in X$ such that $f_i(y) \geq f_i(x)$ for all $i \in J_n$ and $x \neq y$. The set of strictly Pareto-optimal solutions of X is denoted by $SPO(X, f)$ and is called a strictly Pareto-optimal set.

(d) A point $x \in X$ is called a weakly Pareto-optimal solution if and only if there does not exist a point $y \in X$ such that $f_i(y) > f_i(x)$ for all $i \in J_n$. The set of weakly Pareto-optimal solutions of X is denoted by $WPO(X, f)$ and is called a weakly Pareto-optimal set. The image of a weakly Pareto-optimal set under the objective function is called a weakly Pareto-front set, i.e. $f(WPO(X, f)) = WPF(X, f)$. □

The above definition qualifies Pareto-optimal solutions in the global sense. In literature, the term Pareto-optimal is frequently used synonymously with Edgeworth-Pareto optimal, efficient, non-inferior and non-dominated.

In this paper, let the given functions $\{f_i : i \in J_n\}$, $\{g_i : i \in J_p\}$ and $\{h_i : i \in J_q\}$ be all continuous on X. In this case, it is easy to prove that the feasible domain X is compact.

Remark 1. In our optimization problem, it can be shown that: $PO(X, f)$ and $WPO(X, f)$ are nonempty, but $IPO(X, f) = \bigcap_{i=1}^{n} Arg\max(f_i, X)$ and $SPO(X, f)$ may be empty or nonempty; $IPO(X, f) \subset PO(X, f)$, $SPO(X, f) \subset PO(X, f)$, $PO(X, f) \subset WPO(X, f)$ and $PF(X, f) \subset WPF(X, f) \subset \partial f(X)$; $WPO(X, f)$ and $WPF(X, f)$ are compact, see also [7], [9] and [13]. □

Note that strictly Pareto-optimal solutions are the multi-objective analogue of unique optimal solutions in scalar optimization.

Remark 2. It is well-known that $PO(X, f) = IPO(X, f)$ if and only if $IPO(X, f)$ is nonempty [3]. □

If the ideal Pareto-optimal set $IPO(X, f) = \bigcap_{i=1}^{n} Arg\max(f_i, X)$ is empty, then we will accept the elements into the Pareto-optimal, strictly Pareto-optimal and weak Pareto-optimal sets as three different solutions of our optimization problem (1).

Usually, a Pareto-optimal solution is not necessarily uniquely determined, instead there are several Pareto-optimal solutions.

3. Edgeworth-box Diagram, Exchange and Equilibrium

The general idea of the Edgeworth-box diagram (or geometric interpretation of exchange) is that there are two consumers: consumer A and consumer B, and two goods: good X and good Y. We will assume for mathematical simplicity that every good is perfect homogeneous and perfectly divisible. The divisibility assumption is very convenient in economic analysis since it allows us to use continuity arguments [6].

The Edgeworth-box diagram gives us a graph of different distributions of goods between consumers. Now suppose that consumers A and B have utility functions U^A and U^B (strictly increasing, concave and continuous), respectively, consumer A has indifference curves U_1^A, U_2^A and U_3^A, and consumer B has indifference curves U_1^B, U_2^B and U_3^B. A locus of allocations of goods for which the utility function is constant is called an indifference curve. In order to create an Edgeworth-box diagram for consumer A and consumer B, we must place them within the same barter exchange market. To do this, we simply take one of the consumers, for example consumer B, reflect their indifference curves about the x and y axes, and place them on the same graph as the other consumer, i.e. consumer A, see Figure 2.

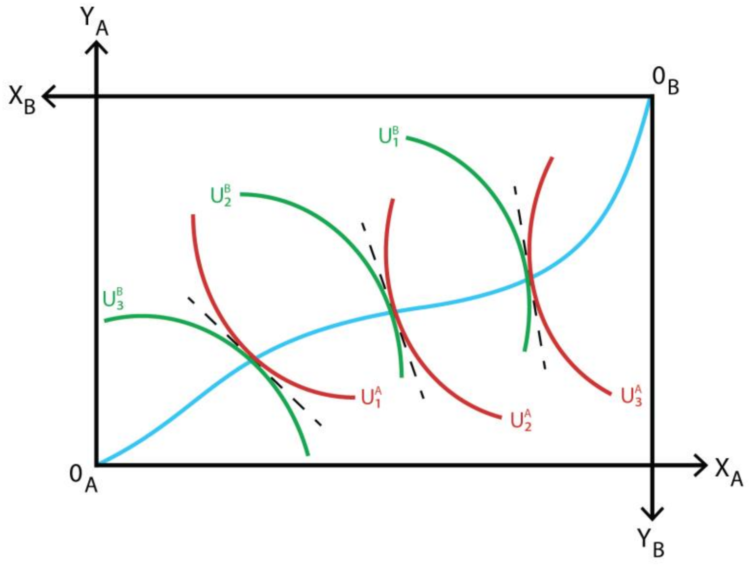

Figure 2: Edgeworth-box diagram and Pareto-optimal set

3.1. Barter Exchange

Now we have to achieve optimal distributions of the two goods between the two consumers. In this case we get an optimization problem: maximize the utility functions U^A and U^B of both consumers A and B subject to a limited total amount of goods X and Y. Analytically, for fixed $x_{total} > 0$ and $y_{total} > 0$ we obtain a multi-objective optimization problem as follows:

maximize $U(x_A, x_B, y_A, y_B) = \left(U^A(x_A, y_A), U^B(x_B, y_B)\right)$ (2)

subject to $x_A + x_B \leq x_{total}$,

$y_A + y_B \leq y_{total}$,

$0 \leq x_A \leq x_{total}$, $0 \leq x_B \leq x_{total}$,

$0 \leq y_A \leq y_{total}$, $0 \leq y_B \leq y_{total}$.

Observe x_A and x_B represent the quantity of good X property of consumers A and B, respectively, and y_A and y_B represent the quantity of good Y property of consumers A and B, respectively.

From the properties of utility functions U^A and U^B it follows that problem (2) is equivalent to problem (3) written as:

maximize $U(x_A, x_B, y_A, y_B) = \left(U^A(x_A, y_A), U^B(x_B, y_B)\right)$ (3)

subject to $x_A + x_B = x_{total}$,

$y_A + y_B = y_{total}$,

$0 \le x_A \le x_{total}$, $0 \le x_B \le x_{total}$,

$0 \le y_A \le y_{total}$, $0 \le y_B \le y_{total}$.

The basic idea is as follows: The indifference curves join all the points that give consumers the same level of utility. By connecting all points of tangency between the indifference curves of both consumers, we construct the contract curve and it represents all Pareto-optimal allocations, i.e. curve $O_A O_B$ is a geometrical image of the Pareto-optimal set. Thus, the Pareto-optimal allocation is the allocation where it is not possible to make one consumer better off without making another worse off, see also Figure 2. From a mathematical point of view, the Pareto-optimal allocations are solutions of multi-objective optimization problem (3). Here, it is true that there are several Pareto-optimal solutions. The Pareto-optimal set does not depend on the initial distribution of goods among consumers, but it depends on the initial total qualities of the various goods, see also [12]. It can also be seen that the Pareto-optimal allocations are not related to a prices system. This means that we analyzed a barter exchange, i.e. mathematical model of exchange without prices.

3.2. Competitive Equilibrium

In addition, let goods X and Y have prices p_X and p_Y, respectively, and let consumers A and B have an initial allocation w_A^X , w_B^X , w_A^Y and w_B^Y of goods X and Y , respectively. From an economic point of view, an exchange economy is characterized by initial allocation of the goods, prices system and utility functions of the consumers. In this case we also have to achieve the optimal distribution of the two goods between the two consumers. Hence, we get a new optimization problem. Analytically, for fixed $w_A^X \ge 0$, $w_B^X \ge 0$, $w_A^Y \ge 0$ and $w_B^Y \ge 0$ such that $x_{total} = w_A^X + w_B^X > 0$ and $y_{total} = w_A^Y + w_B^Y > 0$, and two positive parameters $p_X = p_X(w_A^X, w_B^X, w_A^Y, w_B^Y)$ and $p_Y = p_Y(w_A^X, w_B^X, w_A^Y, w_B^Y)$ we obtain a multi-objective optimization problem written as:

maximize $U(x_A, x_B, y_A, y_B) = \left(U^A(x_A, y_A), U^B(x_B, y_B)\right)$ (4)

subject to $x_A + x_B \le w_A^X + w_B^X$,

$y_A + y_B \leq w_A^Y + w_B^Y$,

$p_X.x_A + p_Y.y_A \leq p_X.w_A^X + p_Y.w_A^Y$,

$p_X.x_B + p_Y.y_B \leq p_X.w_B^X + p_Y.w_B^Y$,

$0 \leq x_A \leq x_{total}$, $0 \leq x_B \leq x_{total}$,

$0 \leq y_A \leq y_{total}$, $0 \leq y_B \leq y_{total}$.

It is important to note that prices p_X and p_Y of the goods is an additional result in the process to solving the multi-objective optimization problem (4), i.e. the prices system is derivative and a very important product of maximization of the utility functions of the consumers in an exchange economy.

Again, from the properties of utility functions U^A and U^B it follows that problem (4) is equivalent to problem (5) written as:

maximize $U(x_A, x_B, y_A, y_B) = \left(U^A(x_A, y_A), U^B(x_B, y_B)\right)$ (5)

subject to $x_A + x_B = w_A^X + w_B^X$,

$y_A + y_B = w_A^Y + w_B^Y$,

$p_X.x_A + p_Y.y_A = p_X.w_A^X + p_Y.w_A^Y$,

$p_X.x_B + p_Y.y_B = p_X.w_B^X + p_Y.w_B^Y$,

$0 \leq x_A \leq x_{total}$, $0 \leq x_B \leq x_{total}$,

$0 \leq y_A \leq y_{total}$, $0 \leq y_B \leq y_{total}$.

It is known that problem (3) has a solution, but obtaining a solution is difficult for problem (5). Let us assume that there exist parameters $p_X > 0$ and $p_Y > 0$ such that problem (5) has a unique solution, i.e. problem (5) has an ideal Pareto-optimal solution. In other words, we assume that we have a competitive equilibrium allocation, this is at point E in Figure 3, and the equilibrium prices are p_X and p_Y.

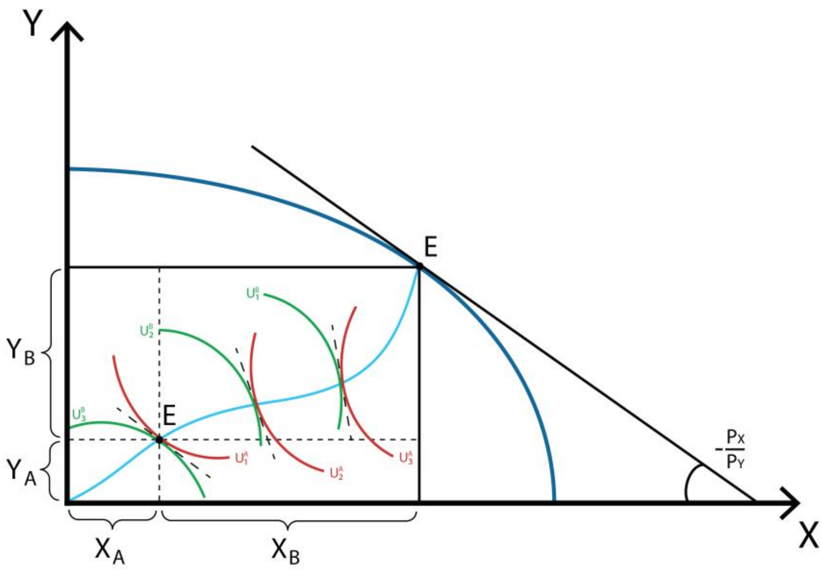

Figure 3: Competitive equilibrium allocation - point E

Remark 3. Note that the slope in absolute value at point E of the contract curves U_1^A and U_3^B is equal to $k = \frac{p_X}{p_Y}$, see Figure 3. Let us assume that utility functions U^A and U^B have first partial derivatives, therefore, we have that $grad(U^A) \parallel grad(U^B) \parallel \vec{p}(p_X, p_Y)$. For $MRS_A = \frac{\partial U^A}{\partial x_A} : \frac{\partial U^A}{\partial y_A}$ (MRS - marginal rate of substitution) and $MRS_B = \frac{\partial U^B}{\partial x_B} : \frac{\partial U^B}{\partial y_B}$ we obtain the tangency between indifference curves is the point where $MRS_A = MRS_B$ (for problem (3)) and if this point is a competitive equilibrium allocation, then $MRS_A = k = MRS_B$ (for problem (5)) [5] [6]. □

Consider the linear system of constraints in problem (5) written as:

$$\left| \begin{array}{l} x_A + x_B = w_A^X + w_B^X \\ y_A + y_B = w_A^Y + w_B^Y \\ p_X.x_A + p_Y.y_A = p_X.w_A^X + p_Y.w_A^Y \\ p_X.x_B + p_Y.y_B = p_X.w_B^X + p_Y.w_B^Y \end{array} \right. .$$

It is easy to show mathematically that the above system is equivalent to the following system:

$$\left| \begin{array}{l} x_A + x_B = w_A^X + w_B^X \\ y_A + y_B = w_A^Y + w_B^Y \\ p_X.x_A + p_Y.y_A = p_X.w_A^X + p_Y.w_A^Y \end{array} \right. .$$

Obviously, we obtain problem (5) is equivalent to problem (6), that is:

maximize $U(x_A, x_B, y_A, y_B) = \left(U^A(x_A, y_A), U^B(x_B, y_B)\right)$ (6)

subject to $x_A + x_B = x_{total}$,

$y_A + y_B = y_{total}$,

$p_X .x_A + p_Y .y_A = p_X .w_A^X + p_Y .w_A^Y$,

$0 \le x_A \le x_{total}$, $0 \le x_B \le x_{total}$,

$0 \le y_A \le y_{total}$, $0 \le y_B \le y_{total}$.

Remark 4. Note that each solution to problem (6) is also a solution to problem (3): therefore, each competitive equilibrium allocation is Pareto-optimal. □

Remark 5. Algebraically, consumer A trades $q^X = \left|x_A - w_A^X\right|$ for $q^Y = \left|y_A - w_A^Y\right|$ and similarly, consumer B trades $q^Y = \left|y_B - w_B^Y\right|$ for $q^Y = \left|x_B - w_B^X\right|$; therefore, $p_X .q^X = p_Y .q^Y$. It is easy to prove that the following statements are equivalent:

(a) $x_A = w_A^X$;

(b) $x_B = w_B^X$;

(c) $y_A = w_A^Y$;

(d) $y_B = w_B^Y$.

In all above cases the consumers do not really trade, i.e. $q^X = 0$ and $q^Y = 0$. If consumers trade goods, then $(x_A - w_A^X)(y_A - w_A^Y) < 0$ and $(x_B - w_B^X)(y_B - w_B^Y) < 0$. □

Of course, we can also consider these ideas in a pure exchange economy with $n \ge 2$ consumers and $m \ge 2$ goods.

3.3. A Numerical Example

Here we consider an illustrative example to demonstrate the concepts of optimality and equilibrium in a simple exchange economy with two consumers and two goods.

Let the consumers have Cobb-Douglas utility functions U^A and U^B given by $U^A(x_A, y_A) = A_0.x_A^{\alpha}.y_A^{1-\alpha}$ and $U^B(x_B, y_B) = B_0.x_B^{\beta}.y_B^{1-\beta}$ where $\alpha, \beta \in (0,1)$ and $A_0, B_0 > 0$.

Now we will find the Pareto-optimal set and the competitive equilibrium.

Based on Remark 3, it is easy to show that problem (3) is equivalent to the following system:

$$\begin{cases} MRS_A = MRS_B \\ x_A + x_B = x_{total} \\ y_A + y_B = y_{total} \end{cases}.$$

First, we compute $MRS_A = \frac{\alpha}{1-\alpha}\frac{y_A}{x_A}$ and $MRS_B = \frac{\beta}{1-\beta}\frac{y_B}{x_B}$; therefore, $\frac{\alpha}{1-\alpha}\frac{y_A}{x_A} = \frac{\beta}{1-\beta}\frac{y_{total} - y_A}{x_{total} - x_A}$, i.e. $y_A = \frac{x_A}{x_A + \gamma(x_{total} - x_A)} y_{total}$ where $\gamma = \frac{1-\beta}{\beta}\frac{\alpha}{1-\alpha} > 0$.

As a result we obtain that the Pareto-optimal set is equal to $\{(x_A, x_B, y_A, y_B) \in R_+^4 : \quad x_A \in (0, x_{total}), \quad x_B = x_{total} - x_A, \quad y_A = \frac{x_A}{x_A + \gamma(x_{total} - x_A)} y_{total},$ $y_B = y_{total} - y_A\}$ and $O_A O_B : y_A = \frac{x_A}{x_A + \gamma(x_{total} - x_A)} y_{total}$, see Figure 2. In fact, there is an infinite number of Pareto-optimal allocations.

Here there are three cases:

(1) If $\gamma = 1$, then $\alpha = \beta$ and $O_A O_B$ is a straight line.

(2) If $\gamma < 1$, then $\alpha < \beta$ and $O_A O_B$ is a convex curve.

(3) If $\gamma > 1$, then $\alpha > \beta$ and $O_A O_B$ is a concave curve.

In this example, all of the Pareto-optimal allocations are strictly Pareto-optimal and weakly Pareto-optimal, and the ideal Pareto-optimal set is empty.

Clearly, problem (6) is equivalent to the following system:

$$\begin{cases} MRS_A = k \\ MRS_B = k \\ x_A + x_B = x_{total} \\ y_A + y_B = y_{total} \\ k.x_A + y_A = k.w_A^X + w_A^Y \end{cases}.$$

We solve this system and the result is as follows:

$$x_A = \frac{\alpha.w_A^Y.x_{total} + \alpha\beta(w_A^X.w_B^Y - w_A^Y.w_B^X)}{\alpha.w_A^Y + \beta.w_B^Y};$$

$$x_B = \frac{\beta.w_B^Y.x_{total} - \alpha\beta(w_A^X.w_B^Y - w_A^Y.w_B^X)}{\alpha.w_A^Y + \beta.w_B^Y};$$

$$y_A = \frac{\frac{1-\alpha}{\alpha}x_A}{\frac{1-\alpha}{\alpha}x_A + \frac{1-\beta}{\beta}x_B}y_{total};$$

$$y_B = \frac{\frac{1-\beta}{\beta}x_B}{\frac{1-\alpha}{\alpha}x_A + \frac{1-\beta}{\beta}x_B}y_{total};$$

$$k = \frac{\alpha}{1-\alpha}\frac{y_A}{x_A}.$$

If the prices lie on a unit simplex, i.e. $p_X + p_Y = 1$, then the prices are $p_X = \frac{\alpha.y_A}{(1-\alpha)x_A + \alpha.y_A}$ and $p_Y = \frac{(1-\alpha)x_A}{(1-\alpha)x_A + \alpha.y_A}$. In the general case, we obtain $p_X = \frac{\alpha.y_A}{(1-\alpha)x_A + \alpha.y_A}s$ and $p_Y = \frac{(1-\alpha)x_A}{(1-\alpha)x_A + \alpha.y_A}s$ where $s \in R_{++}$ or $p_X = \alpha.y_A.s'$ and $p_Y = (1-\alpha)x_A.s'$ where $s' \in R_{++}$; therefore, $p_X = p_X(\alpha, \beta, w_A^X, w_B^X, w_A^Y, w_B^Y)$ and $p_Y = p_Y(\alpha, \beta, w_A^X, w_B^X, w_A^Y, w_B^Y)$.

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