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Fixed point theorems for generalized concave operators and applications to fractional differential equation boundary value problems

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Abstract

In this paper, by introducing the concept of a generalized concave operator and the properties of cone and monotone iterative technique in ordered Banach spaces, some new existence and uniqueness theorems of fixed points for the operator under more extensive conditions are obtained. Finally, as applications, we apply the results obtained in this paper to study the existence and uniqueness of positive solutions for nonlinear fractional differential equation boundary value problems.

Keywords: fixed point; generalized concave operator; normal cone; positive solution; fractional differential equation; boundary value problems

1. Introduction and Preliminaries

It is well known that a concave operator is important for studying positive solutions of nonlinear differential and integral equations. In applications, in order to prove existence or uniqueness for solution of such equations, one usually considers the fixed points of some related operators. There are many fixed point theorems in ordered Banach spaces, see [1–3]. In the past several decades, many of the fixed point theory for non-compact increasing operators with concavity have been extensively studied and applied to the study of various nonlinear equations (see [1,3–8] and the references therein). Krasnosel'skii studied u_0-concave operator (see [4]) with $u_0 > \theta$, Potter introduced α-concave operator (see [8]), and Guo established the existence of the unique fixed point for α-concave operator (see [3]) with $\alpha \in (0,1)$ and Zhai studied generalized concave and convex operators and established the existence and uniqueness of positive fixed points (see [9]). In 2010, Zhai and Cao introduced the definition of $\tau - \varphi -$concave operators and existence and uniqueness theorems of fixed points for increasing operators with such concavity are established (see [10]).

Suppose $(E, \|\cdot\|)$ is a real Banach space which is partially ordered by a cone $P \subset E$, i.e. $x \le y$ if and only if $y - x \in P$.If $x \le y$ and $x \ne y$, then we denote $x < y$. We denote the zero element of E by θ.Recall that a non-empty closed convex set $P \subset E$ is a cone if it satisfies

(i) $x \in P, \lambda \ge 0 \Rightarrow \lambda x \in P$;(ii) $x \in P, -x \in P \Rightarrow x = \theta$.

Putting $P^0 = \{x \in P \mid x \text{ is an interior point of } P\}$, a cone P is said to be solid if P^0 is non-empty. Moreover, P is called normal if there exists a constant $N > 0$ such that, for all $x, y \in E, \theta \le x \le y$ implies $\|x\| \le N\|y\|$; in this case N is called the normality constant of P.

We say that an operator $A: E \to E$ is increasing if $x \le y$ implies $Ax \le Ay$. Element $x \in P$ is called a fixed point of A if $Ax = x$.

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \le y \le \mu x$.Clearly $\sim$ is an equivalence relation. Given $w > \theta$ (i.e. $w \ge \theta$ and $w \ne \theta$), we denote the set $P_w = \{x \in E \mid x \sim w\}$ by P_w.It is easy to see that $P_w \subset P$ for $w \in P$.

All the concepts discussed above can be found in [3, 4, 7, 11]. For more results about a concave operator and other related concepts, the reader is referred to [12, 13] and some of the references therein.

In [10], Zhai and Cao introduced the following definition of $\tau - \varphi -$concave operators.

Definition 1.1(see [10]) Let E be a real Banach space and P be a cone in E .We say an operator $A:P\to P$ is $\tau-\varphi-$ concave if there exist two positive-valued functions $\tau(t),\varphi(t)$ on interval (a,b) such that

(H1) $\tau:(a,b)\to(0,1)$ is a surjection;

(H2) $\varphi(t)>\tau(t)$, for all $t\in(a,b)$;

(H3) $A(\tau(t)x)\geq\varphi(t)Ax$, for all $t\in(a,b), x\in P$.

Now, we improve and generalize the definition of $\tau-\varphi-$concave operator.

Definition 1.2 Let E be a real Banach space and P be a cone in E .We say an operator $A:P\to P$ is a generalized $\tau-\varphi-$concave operator if there exist a positive-valued function $\tau(t)$ defined on an interval (a,b) and a map $\varphi:(a,b)\times P\to(0,+\infty)$ such that

(C1) $\tau:(a,b)\to(0,1)$ is a surjection;

(C2) $\varphi(t,x)>\tau(t)$, for all $t\in(a,b),\forall x\in P$;

(C3) $A(\tau(t)x)\geq\varphi(t,x)Ax$, for all $t\in(a,b),\forall x\in P$.

When $\tau(t)=t$, we say that A is a generalized $\varphi-$concave operator.

Remark 1.1 The main difference between Definition 1.1 and Definition 1.2 is that, for Definition 1.1, a function $\varphi(t)$ is only a positive-valued function defined on interval (a,b), but for Definition 1.2, a map $\varphi(t,x)$ is defined by $\varphi:(a,b)\times P\to(0,+\infty)$ .Hence they are different concepts.

2. Main results

In this section, we present our main results.

Theorem 2.1 Let P be a normal cone in a real Banach space E, and let an operator $A:P\to P$ be an increasing and generalized $\tau-\varphi-$concave operator. In addition, suppose that there exists $w\in P\setminus\{\theta\}$ such that $Aw\in P_w$ and for any $t\in(a,b)$ and $x\in P_w$, a map $\varphi(t,x)$ is non-increasing in x .Then

(i) there are $u_0,v_0\in P_w$ and $r\in(0,1)$ such that $rv_0\leq u_0<v_0, u_0\leq Au_0\leq Av_0\leq v_0$;

(ii) the operator A has a unique fixed point x^* in $[u_0, v_0]$;

(iii) for any initial $x_0 \in P_w$, constructing successively the sequence $x_n = Ax_{n-1}, n = 1,2,\ldots$, we have $\| x_n - x^* \| \to 0 \ (n \to \infty)$.

Proof: (1) Proof of (i).since $Aw \in P_w$, we can choose a sufficiently small number $\mu_0 \in (0,1)$ such that

$$\mu_0 w \le Aw \le \frac{1}{\mu_0} w.$$

It follows from (C1) that there exists $t_0 \in (a,b)$ such that $\tau(t_0) = \mu_0$, and hence

$$\tau(t_0) w \le Aw \le \frac{1}{\tau(t_0)} w. \tag{2.1}$$

From (C2), we know that $\varphi(t_0, w) > \tau(t_0)$.Thus, $\frac{\varphi(t_0, w)}{\tau(t_0)} > 1$ and we can take a sufficiently large positive integer k such that

$$\left(\frac{\varphi(t_0, w)}{\tau(t_0)} \right)^k \ge \frac{1}{\tau(t_0)}, \tag{2.2}$$

which implies

$$\left(\frac{\tau(t_0)}{\varphi(t_0, w)} \right)^k \le \tau(t_0). \tag{2.3}$$

Set $u_0 = [\tau(t_0)]^k w, v_0 = \frac{1}{[\tau(t_0)]^k} w$.Clearly, $u_0, v_0 \in P_w$ and $u_0 = [\tau(t_0)]^{2k} v_0 < v_0$. Take any $r \in (0, [\tau(t_0)]^{2k}]$ then $r \in (0,1)$ and $u_0 \ge r v_0$.By the monotonicity of A, we have $Au_0 \le Av_0$. Further, combining condition (C3) with (2.1) and (2.2), we obtain

$$
\begin{aligned}
Au_0 &= A([\tau(t_0)]^k w) = A(\tau(t_0)\cdot[\tau(t_0)]^{k-1}w) \\
&\ge \varphi(t_0,[\tau(t_0)]^{k-1}w)A([\tau(t_0)]^{k-1}w) \\
&= \varphi(t_0,[\tau(t_0)]^{k-1}w)A(\tau(t_0)\cdot[\tau(t_0)]^{k-2}w) \\
&\ge \varphi(t_0,[\tau(t_0)]^{k-1}w)\varphi(t_0,[\tau(t_0)]^{k-2}w)A([\tau(t_0)]^{k-2}w) \\
&\ge \cdots \\
&\ge \varphi(t_0,[\tau(t_0)]^{k-1}w)\varphi(t_0,[\tau(t_0)]^{k-2}w)\cdots\varphi(t_0,\tau(t_0)w)\varphi(t_0,w)A(w) \\
&\ge [\varphi(t_0,w)]^k\cdot\tau(t_0)w \\
&\ge [\tau(t_0)]^k w = u_0.
\end{aligned}
$$

For $t\in(a,b)$, from (C3) we get

$$
Ax = A\left(\tau(t)\cdot\frac{1}{\tau(t)}x\right)\ge\varphi(t,x)A\left(\frac{1}{\tau(t)}x\right),
$$

and hence

$$
A\left(\frac{1}{\tau(t)}x\right)\le\frac{1}{\varphi(t,x)}Ax,\quad \forall t\in(0,1),\ \ x\in P\,.
$$

Thus, we have

$$
\begin{aligned}
Av_0 &= A\left(\frac{1}{[\tau(t_0)]^k}w\right) = A\left(\frac{1}{\tau(t_0)}\cdot\frac{1}{[\tau(t_0)]^{k-1}}w\right) \\
&\le \frac{1}{\varphi(t_0,[\tau(t_0)]^{k-1}w)}A\left(\frac{1}{[\tau(t_0)]^{k-1}}w\right) \\
&= \frac{1}{\varphi(t_0,[\tau(t_0)]^{k-1}w)}A\left(\frac{1}{\tau(t_0)}\cdot\frac{1}{[\tau(t_0)]^{k-2}}w\right) \\
&\le \frac{1}{\varphi(t_0,[\tau(t_0)]^{k-1}w)}\cdot\frac{1}{\varphi(t_0,[\tau(t_0)]^{k-2}w)}A\left(\frac{1}{[\tau(t_0)]^{k-2}}w\right) \\
&\le \cdots \\
&\le \frac{1}{\varphi(t_0,[\tau(t_0)]^{k-1}w)}\cdot\frac{1}{\varphi(t_0,[\tau(t_0)]^{k-2}w)}\cdot\cdots\cdot\frac{1}{\varphi(t_0,\tau(t_0)w)}\cdot\frac{1}{\varphi(t_0,w)}Aw \\
&\le \frac{1}{[\varphi(t_0,w)]^k}\cdot\frac{1}{\tau(t_0)}w \\
&\le \frac{1}{[\tau(t_0)]^k}w = v_0.
\end{aligned}
$$

Hence we get

$$
u_0\le Au_0\le Av_0\le v_0\,. \tag{2.4}
$$

The proof of (i) is complete.

(2) Proof of (ii). Construct successively the sequences

$$u_n = Au_{n-1},\ v_n = Av_{n-1}, \quad n = 1,2,...$$

By the monotonicity of A, we have $u_1 = Au_0 \le Av_0 = v_1$.Similarly we obtain $u_n \le v_n,\ n = 1,2,...$

It follows from (2.4) and the monotonicity of A that

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_1 \le v_0. \qquad (2.5)$$

Note that $u_0 \ge rv_0$, so we can get $u_n \ge u_0 \ge rv_0 \ge rv_n,\ n = 1,2,...$.Set

$$r_n = \sup\{r > 0 \mid u_n \ge rv_n\},\ n = 1,2,...$$

So $r_{n+1} \ge r_n$, namely, $\{r_n\}$ is increasing with $\{r_n\} \subset (0,1]$.Assume $r_n \to r^*$ as $n \to \infty$.Next we prove $r^* = 1$.

Indeed, suppose to the contrary that $0 < r^* < 1$.From (C1), there is $t_1 \in (a,b)$ such that

$\tau(t_1) = r^*$.We distinguish two cases:

Case one: There exists a positive integer N such that $r_N = r^*$.In this case, we get $r_n = r^*$ for all $n \ge N$.Therefore, for all $n \ge N$, we have

$$u_{n+1} = Au_n \ge A(r^* v_n) = A(\tau(t_1)v_n) \ge \varphi(t_1, v_n)Av_n \ge \varphi(t_1, w)Av_n = \varphi(t_1, w)v_{n+1}.$$

By the definition of r_n, we obtain $r_{n+1} = r^* \ge \varphi(t_1, w) > \tau(t_1) = r^*$, which is a contradiction.

Case two: If for all positive integer n, then $r_n < r^*$.Then we obtain $0 < \frac{r_n}{r^*} < 1$.By (C1), there exist $s_n \in (a,b)$ such that $\tau(s_n) = \frac{r_n}{r^*}$.So we have

$$\begin{aligned} u_{n+1} &= Au_n \ge A(r_n v_n) = A(\frac{r_n}{r^*} r^* v_n) = A(\tau(s_n) r^* v_n) \\ &\ge \varphi(s_n, r^* v_n)A(r^* v_n) = \varphi(s_n, r^* v_n)A(\tau(t_1)v_n) \\ &\ge \varphi(s_n, r^* v_n)\varphi(t_1, v_n)Av_n = \varphi(s_n, r^* v_n)\varphi(t_1, v_n)v_{n+1} \\ &\ge \varphi(s_n, w)\varphi(t_1, w)v_{n+1} \end{aligned}$$

By the definition of r_n, we have

$$r_{n+1} \geq \varphi(s_n, w)\varphi(t_1, w) > \tau(s_n)\varphi(t_1, w) = \frac{r_n}{r^*}\varphi(t_1, w).$$

Let $n \to \infty$, we obtain $r^* \geq (\frac{r^*}{r^*})\varphi(t_1, w) > \tau(t_1) = r^*$, which also is a contradiction.

Hence $\lim_{n\to\infty} r_n = 1$.For any natural number m, we have

$$\theta \leq u_{n+m} - u_n \leq v_n - u_n \leq v_n - r_n v_n = (1 - r_n)v_n \leq (1 - r_n)v_0;$$
$$\theta \leq v_n - v_{n+m} \leq v_n - u_n \leq (1 - r_n)v_0.$$

By the normality of P, we get, as $n \to \infty$,

$$\| u_{n+m} - u_n \| \leq N \| (1 - r_n)v_0 \| \to 0;$$
$$\| v_n - v_{n+m} \| \leq N \| (1 - r_n)v_0 \| \to 0.$$

Here N is the normality constant of P .So $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences. Since E is complete, there exist u^* and v^* such that $u_n \to u^*, v_n \to v^*$ as $n \to \infty$ .From (2.5), we have

$u^*, v^* \in [u_0, v_0]$ and for any n , $u_n \leq u^* \leq v^* \leq v_n$ .therefore,

$$\theta \leq v^* - u^* \leq v_n - u_n \leq (1 - r_n)v_0.$$

Since P is normal, $\| v^* - u^* \| \leq N \| (1 - r_n)v_0 \| \to 0$ as $n \to \infty$ .Thus $u^* = v^*$.

Let $x^* = u^* = v^*$, by the monotocnicity of A, we have $u_{n+1} = Au_n \leq Ax^* \leq Av_n = v_{n+1}$.

Take $n \to \infty$, the normality of P implies $x^* = Ax^*$.Namely, x^* is a fixed point of A in $[u_0, v_0]$.

In the following, we prove that x^* is the unique fixed point of A in P_w.In fact, assume $\tilde{x}$ is a fixed point of A in P_w. By the definition of P_w, there exist positive numbers $\lambda_1, \lambda_2, \eta_1, \eta_2$ such that

$$\lambda_1 w \leq x^* \leq \lambda_2 w, \eta_1 w \leq \tilde{x} \leq \eta_2 w.$$

Then, we have

$$\tilde{x} \geq \eta_1 w \geq \frac{\eta_1}{\lambda_2}\lambda_2 w \geq \frac{\eta_1}{\lambda_2} x^*,$$
$$x^* \geq \lambda_1 w \geq \frac{\lambda_1}{\eta_2}\eta_2 w \geq \frac{\lambda_1}{\eta_2}\tilde{x}.$$

Put $\gamma_1 = \sup\{\gamma > 0 : \tilde{x} \geq \gamma x^*, x^* \geq \gamma\tilde{x}\}$.Clearly, $0 < \gamma_1 < +\infty$. Furthermore, we prove $\gamma_1 \geq 1$.

Suppose to the contrary that $0<\gamma_1<1$, by (C1), there exists $t_2 \in (a,b)$ such that $\tau(t_2)=\gamma_1$.thus

$$\tilde{x} = A\tilde{x} \geq A(\gamma_1 x^*) = A(\tau(t_2)x^*) \geq \varphi(t_2, x^*)Ax^* = x^*.$$

Because $\varphi(t_2,x^*) > \tau(t_2) = \gamma_1$, this contradicts the definition of γ_1.Thus $\gamma_1 \geq 1$ and then we have $\tilde{x} \geq x^*$ and $x^* \geq \tilde{x}$ .Hence $\tilde{x} = x^*$, that is A has a unique fixed point x^* in P_w .Note that $[u_0, v_0] \subset P_w$, so we know that x^* is the unique fixed point of A in $[u_0, v_0]$.

(3) Proof of (iii). For any initial $x_0 \in P_w$, we can choose a small number $e_1 \in (0,1)$ such that

$$e_1 w \leq x_0 \leq \frac{1}{e_1} w.$$

By (C1) again, there exist $t_3 \in (a,b)$ such that $\tau(t_3) = e_1$.Hence,

$$\tau(t_3) w \leq x_0 \leq \frac{1}{\tau(t_3)} w.$$

We can choose a sufficiently large positive integer m such that

$$\left[\frac{\varphi(t_3, w)}{\tau(t_3)}\right]^m \geq \frac{1}{\tau(t_3)}.$$

Put $\tilde{u}_0 = [\tau(t_3)]^m w, \tilde{v}_0 = \dfrac{1}{[\tau(t_3)]^m} w$.Then $\tilde{u}_0, \tilde{v}_0 \in P_w$ and $\tilde{u}_0 < x_0 < \tilde{v}_0$.Let

$$\tilde{u}_n = A\tilde{u}_{n-1}, \quad \tilde{v}_n = A\tilde{v}_{n-1}, \quad \tilde{x}_n = A\tilde{x}_{n-1}, \quad n = 1,2,\ldots.$$

By a proof similar to that of the existence of x^*, we can prove that there exists $y^* \in P_w$ such that

$\lim_{n\to\infty} \tilde{u}_n = \lim_{n\to\infty} \tilde{v}_n = y^*$ and $Ay^* = y^*$ .By the uniqueness of fixed point of A in P_w ,we get $y^* = x^*$. It easy to get $\tilde{u}_n < x_n < \tilde{v}_n,\ n = 1,2,\ldots$.Since P is normal, we have $\lim_{n\to\infty} x_n = x^*$.

Remark 2.1 Under the conditions of Theorem 2.1, from the proof of (i), we can prove the existence of upper and lower solutions for generalized $\tau-\varphi-$concave operator A. From the proof of (ii), we can easily see that operator A has a unique fixed point in P_w .If we suppose that operator $A: P_w \to P_w$ or $A: P^0 \to P^0$ with P is a solid cone, and then $Aw \in P_w$ is automatically satisfied.

This proves the following corollaries.

Corollary 2.1 Let P be a normal cone in a real Banach space E, suppose $w > \theta$ and let an operator $A: P_w \to P_w$ be an increasing and generalized $\tau - \varphi -$concave operator. In addition, suppose for any $t \in (a,b)$ and $x \in P_w$, a map $\varphi(t,x)$ is non-increasing in x.Then

(i) there are $u_0, v_0 \in P_w$ and $r \in (0,1)$ such that $rv_0 \le u_0 < v_0, u_0 \le Au_0 \le Av_0 \le v_0$;

(ii) the operator A has a unique fixed point x^* in P_w;

(iii) for any initial $x_0 \in P_w$, constructing successively the sequence $x_n = Ax_{n-1}, n = 1,2,\ldots$, we have $\| x_n - x^* \| \to 0$ $(n \to \infty)$.

Corollary 2.2 Let P be a normal, solid cone in a real Banach space E, and let an operator $A: P^0 \to P^0$ be an increasing and generalized $\tau - \varphi -$concave operator. In addition, suppose for any $t \in (a,b)$ and $x \in P_w$, a map $\varphi(t,x)$ is non-increasing in x. Then

(i) there are $u_0, v_0 \in P^0$ and $r \in (0,1)$ such that $rv_0 \le u_0 < v_0, u_0 \le Au_0 \le Av_0 \le v_0$;

(ii) the operator A has a unique fixed point x^* in P^0;

(iii) for any initial $x_0 \in P^0$, constructing successively the sequence $x_n = Ax_{n-1}, n = 1,2,\ldots$, we have $\| x_n - x^* \| \to 0$ $(n \to \infty)$.

Remark 2.2 (1) In Theorem 2.1, Corollaries 2.1 and 2.2, we do not suppose that the operator A is compact and continuous, which is usually done in [2, 3]. The existence of a unique solution is proved only in the case where the cone P is normal and the operator A is generalized $\tau - \varphi$-concave. Moreover, we give the iterative forms. (2) We also remove the condition of upper and lower solutions which is common in many known results and is difficult to verify. The condition that the cone P is normal is automatically satisfied for many Banach spaces. Hence our results in essence improve and generalize relevant results in [2, 3].

Remark 2.3 When $\tau(t) = t, t \in (0,1)$ and $\varphi(t,x) = t(1 + v(t))$ or $t^{\alpha(t)}$ with $v(t) > 0, 0 < \alpha(t) < 1$ for $t \in (0,1)$,Theorem 2.1, Corollaries 2.1 and 2.2 also hold. The corresponding results in [3,5,6,9] turn out to be special cases of our main results, see [3, Theorem 2.2.6], [5, Corollary 2, Corollary3],[6, Corollary 2]and[9, Theorem 2.1,Corollary 2.2, Corollary 2.3].

Remark 2.4 When $\varphi(t,x)=\varphi(t)$ for $t\in(a,b), x\in P$, Theorem 2.1, Corollaries 2.1 and 2.2 also hold. The corresponding results in [10] turn out to be special cases of our main results, see [10, Theorem 2.1, Corollary 2.2 and Corollary 2.3].

3. Applications

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various sciences such as physics, mechanics, chemistry, engineering, etc. For details, see [14-18] and references therein. In this section, we use Theorem2.1 to study the existence of a unique positive solution for nonlinear fractional differential equation boundary value problem:

$$\begin{cases} -D_{a+}^{\alpha}u(t)=f(t,u(t)), \ \ a<t<b \\ u(a)=u(b)=0 \end{cases} \tag{3.1}$$

where $1<\alpha\le 2$ is a real number and D_{a+}^{α} is the Riemann–Liouville fractional derivative of order $\alpha>0$,defined by

$$D_{a+}^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n}\int_{a}^{t}\frac{u(\tau)}{(t-\tau)^{\alpha-n+1}}\mathrm{d}\tau\,,$$

where $n=[\alpha]+1, [\alpha]$ denotes the integer part of number α ,see[19,20].

In recent years, there are many papers discuss the existence and multiplicity of positive solutions for nonlinear fractional differential equation boundary value problem by the use of Leray–Shauder theory, fixed-point theorems , etc., see [21-25].However, there are few papers consider the existence of a unique positive solution for nonlinear fractional differential equation boundary value problem. In this section, we apply Theorems 2.1 to study the nonlinear fractional differential equation boundary value problem (3.1).

Let $E=C[a,b]$ be a Banach space of continuous functions on $[a,b]$ with the maximum norm

$\|u|=\max\{|u(t)|: t\in[a,b]\}$. $P=\{u\in E \mid u(t)\ge 0, \forall t\in[a,b]\}$, then P is a normal solid cone of which the normality constant is 1 in Banach space E. The partial ordering defined by P is given by $u\le v \Leftrightarrow u(t)\le v(t)$ for all $t\in[a,b]$.

Lemma 3.1 (see [20]) Given $y\in C[a,b]$ and $1<\alpha\le 2$, the unique solution of

$$\begin{cases} -D_{a+}^{\alpha}u(t) = y(t), \ \ a < t < b \\ u(a) = u(b) = 0 \end{cases} \tag{3.2}$$

is

$$u(t) = \int_a^b G(t,s)y(s)\mathrm{d}s ,$$

where

$$G(t,s) = \begin{cases} \dfrac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}\left(\dfrac{b-s}{b-a}\right)^{\alpha-1} - \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & a \le s \le t \le b, \\ \dfrac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}\left(\dfrac{b-s}{b-a}\right)^{\alpha-1}, & a \le t \le s \le b. \end{cases} \tag{3.3}$$

Moreover, $G(t,s) \ge 0$ for $\forall t, s \in (a,b)$.

Theorem 3.1 Assume that

(P1) $f(t,u) \in C([a,b]\times[0,\infty))$ is increasing in u for fixed t and

$$f(t,u) \ge 0, \forall (t,u) \in [a,b]\times[0,\infty);$$

(P2) there exist a positive-valued functions $\tau(t)$ defined on an interval (a,b) and a map

$\varphi : (a,b)\times P \to (0,+\infty)$ such that $\tau : (a,b) \to (0,1)$ is a surjection and

for any $t \in (a,b)$ and $x \in P_w$, a map $\varphi(t,x)$ is non-increasing in x and $\varphi(t,u) > \tau(t)$, for all $t \in (a,b), \forall u \in P$ which satisfy

$$f(t,\tau(\lambda)u) \ge \varphi(\lambda,u) f(t,u), \text{ for all } t, \lambda \in (a,b), u \in P.$$

(P3) there exist two constants $M_1, M_2 > 0$ and $w > \theta$ such that

$$M_1 w(t) \le \int_a^b G(t,s) f(s,w(s))\mathrm{d}s \le M_2 w(t), \qquad \forall t \in [a,b].$$

Then problem (3.1) has a unique positive solution u^* in P_w .Moreover, for any $u_0 \in P_w$, constructing successively the sequence

$$u_{n+1}(t) = \int_a^b G(t,s) f(s,u_n(s))\mathrm{d}s, \quad n = 0,1,2,\ldots,$$

we have $\| u_n - u^* \| \to 0$ as $n \to \infty$.

Proof: To begin with, from Lemma 3.1, the problem (3.1) has an integral formulation given by

$$u(t)=\int_a^b G(t,s)f(s,u(s))\mathrm{d}s,$$

where $G(t,s)$ is given as in Lemma 3.1.

Define an operator $A:P\to E$ by

$$Au(t)=\int_a^b G(t,s)f(s,u(s))\mathrm{d}s.$$

It is easy to prove that u is the solution of problem (3.1) if and only if $u=Au$ .Note that since $f(t,u(t))\geq 0$, we have $Au(t)\geq 0$ for $t\in(a,b)$ .That is $A:P\to P$ .It follows from (P1) that operator A is increasing in P .For any $\lambda\in(a,b),u\in P$, from (P2) we obtain

$$\begin{aligned}A(\tau(\lambda)u)(t)&=\int_a^b G(t,s)f(s,\tau(\lambda)u(s))\mathrm{d}s\\&\geq\varphi(\lambda,u)\int_a^b G(t,s)f(s,u(s))\mathrm{d}s=\varphi(\lambda,u)Au(t)\end{aligned}$$

That is $A(\tau(\lambda)u)\geq\varphi(\lambda,u)Au$ for any $\lambda\in(a,b),u\in P$. So operator $A:P\to P$ is an increasing and generalized $\tau-\varphi-$concave operator. Further, from (P3) we know that

$$M_1w(t)\leq\int_a^b G(t,s)f(s,w(s))\mathrm{d}s\leq M_2w(t),\qquad\forall t\in[a,b].$$

Then $Aw\in P_w$.Thus all the conditions of Theorem 2.1 are satisfied and the conclusion of Theorem 3.1 follows from Theorem 2.1.

Remark 3.1 There exist many functions which satisfy the conditions of Theorem 3.1.

Example 3.1 We give an example to illustrate Theorem 3.1. Consider the following nonlinear fractional differential equation boundary value problem:

$$\begin{cases}-D_{0+}^{\alpha}u(t)=u^{\frac{1}{5}}(t)+t,\quad 0<t<1,\\u(0)=u(1)=0,\end{cases}\tag{3.4}$$

where $1<\alpha\leq 2$ is a real number. In this example, we have $a=0,b=1,f(t,u)=u^{\frac{1}{5}}(t)+t$. It is easy to show that the nonlinear fractional differential equation boundary value problem satisfy the conditions of Theorem 3.1. So equation (3.2) has a unique positive solution.

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