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Exhaustive Monty Hall Problems and Solutions

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Abstract

This paper at first isolates a group of Monty Hall problems, in which all doors are exhaustively chosen, swapped and revealed one by one, then introduces a new variant of briefer tree diagrams for the conditional probability calculation of three-doors and four-doors, and finally gives the recurrence form for n-doors and its closed generation function. Some resulted defusing advantages and interesting properties have also been discussed.

Key Words: Exhaustive; Monty Hall Problem; Probability Tree Diagram; Recurrence.

1. Introduction

The Monty Hall Problem, about which over 75 papers have been published in academic journals and the popular press, is no doubt one of the most popular probabilistic or statistical topics in mathematic classes of universities and colleges. It was originally posed in a letter by Steve Selvin to the American Statistician (Selvin 1975), and then became famous as a question from a reader's letter quoted in "Ask Marilyn" column in Parade magazine (vos Savant 1990). The question is "*Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1,* *and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?*”

To plausibly describe and solve the problem, many assumptions have been made, for example the door indexing, the ways of a contestant picking a door, and the ways of the host opening another door (vos Savant 1991), and these assumptions in many textbooks and teaching materials did contribute decently to statistic education. However, they have been at the same time distracting quite a few clever minds to consume efforts in appreciating either the role of the contestant or that of the host (Mlodinow 2008), and thus confused many people including some of my students, colleagues and friends.

From the dozens of existing variant Monty Hall problems we isolated a special group of exhaustive problems, including the very original three-door one, in which no doors are indexed, no probabilities or preferences in picking or revealing doors are considered, and all doors are simply chosen, swapped and revealed one after one. In details, for n-doors with one car and n – 1 goats behind, first the player picks one door, second the host reveals another door with a goat behind, third the player swaps her or his chosen door with one from the left n – 2 doors, forth the second and the third steps iterate until there are only two doors left, and finally after the last swap the host reveals the left door and the player’s chosen door to show the result. This isolation surprisingly led to students’ much better understanding and some numeric properties of further interests, as well as solutions of briefer tree diagrams to the conditional probability calculation, the recurrence form for n-doors, and its closed generation function.

2. Solutions to Three-doors and Four-doors

Tree diagrams similar to that in Figure 1 are often employed to explain the direct calculation for conditional probability of three-doors Monty Hall Problem (Carlton 2005). Let *C* denote the event that the car is behind Door #2; the a priori probability of *C* is *P(C)* = 1/3. Let *D* denote the event that Monty opens Door #3; according to Figure 1.

$$P(D|C) = 1 \text{ and } P(D) = \frac{1}{3} * \frac{1}{2} + \frac{1}{3} * 1 + \frac{1}{3} * 0 = \frac{1}{2}$$

Hence, by Bayes' Rule,

$$P(C|D) = {P(C)P(D|C)}/{P(D)} = {\frac{1}{3} * 1}\Big/{\frac{1}{2}} = \frac{2}{3}$$

Therefore, the car is hidden behind the remaining door two-thirds of the time.

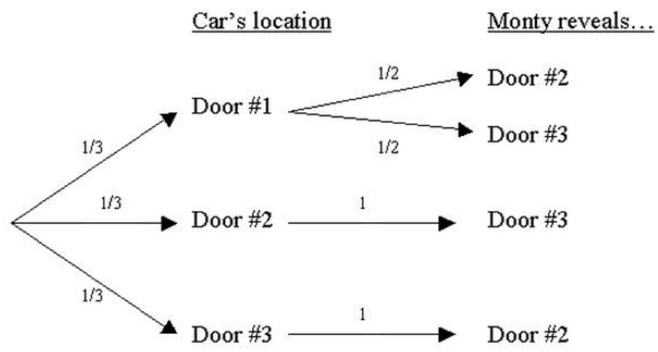

Figure 1: Tree diagram for the three-door Monty Hall Problem

However, arguments remained among students about the probabilistic distribution of Monty revealing the different goat doors even after my proof that Monty's preferences makes no differences if unknown to the contestant. All things turned much better when I attributed this confusion to the door indexes, modified the problem to be an exhaustive version, and introduced the briefer tree diagram in Figure 2, where the doors are not indexed, only the player's picking and swapping are shown, and the role of Monty's revealing doors is simply hidden.

After her or his first choice from a set of one car and two goats, the contestant either gets in hand a car with the a priori probability of $P(C) = 1/3$ or a goat with the a priori probability of $P(G) = 2/3$, then Monty removes one goat from the set with a fixed probability of $P(M) = 1$, at last the contestant swaps for either the left goat with a conditional probability of $P(S_{1G}|C,M) = 1$ or the car with $P(S_{1C}|C,M) = 1$. Hence, by always switching the player wins with a joint probability of $P(W) = P(G) * P(M) * P(S_{1C}|C,M) = 2/3 * 1 * 1 = 2/3$, and loses with $P(L) = P(C) * P(M) * P(S_{1G}|C,M) = 1/3 * 1 * 1 = 1/3$.

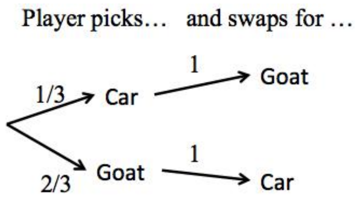

Figure 2: Briefer tree diagram for 3-doors

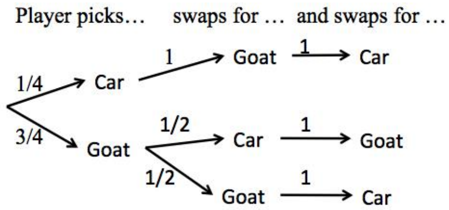

Figure 3: Briefer tree diagram for 4-doors

In the same way, we can construct the briefer tree diagram for the exhaustive four-door Monty Hall Problem in Figure 3, and calculate the probabilities.

$$P(W) = P(C) * P(M) * P(S_{1G}|C, M) * P(M) * P(S_{2C}|S_{1G},C,M)$$
$$+ P(G) * P(M) * P(S_{1G}|G,M) * P(M) * P(S_{2C}|S_{1G},G,M)$$
$$= \frac{1}{4} * 1 * 1 * 1 * 1 + \frac{3}{4} * 1 * \frac{1}{2} * 1 * 1$$
$$= \frac{1}{4} + \frac{3}{8} = \frac{5}{8}$$
$$P(L) = P(G) * P(M) * P(S_{1C}|G,M) * P(M) * P(S_{2G}|S_{1C},G,M)$$
$$= \frac{3}{4} * 1 * \frac{1}{2} * 1 * 1 = \frac{3}{8}$$

3. Solutions to N-doors

We can now generalize exhaustive Monty Hall Problems from three and four doors to n doors, the diagram tree for which is shown in Figure 4. Thanks to its conciseness, this tree could be applied to five-nine doors easily, but the complexity increases exponentially when the door number goes up. However, the tree shows such a clear recurrent structure, illustrated in Figure 4 with dashed and dotted closures, that induction by drawing another diagram tree for n+1 doors should lead to the simple proof for the existence of a Monty Hall recurrence.

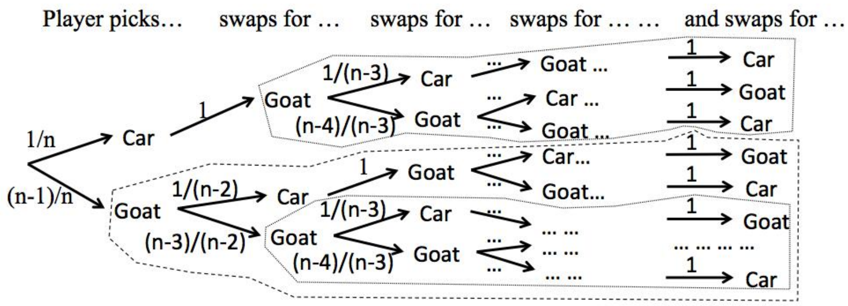

Figure 4: Briefer tree diagram for n-doors

For the convenience of further calculation, let's denote the player's losing probability in the exhaustive n-door Monty Hall game *P(L)* as M_n, the player's winning probability *P(W)* as 1 - M_n, the probability of the player's ending up with a goat in the dashed closure as A_{n-2}, and that in the dotted closure as A_{n-3}, so that we have the Monty Hall recurrence of the form

$$M_n = {n-1}/_{n} A_{n-2} + {1}/_{n} A_{n-3}$$

where $A_n = {n-1}/_{n} A_{n-1} + {1}/_{n} A_{n-2}$.

From Figure 2 and Figure 3, we know $M_3 = 1/3$ and $M_4 = 3/8$, hence we have

$$\begin{cases} M_3 = {3-1}/_{3} A_{3-2} + {1}/_{3} A_{3-3} \\ M_4 = {4-1}/_{4} A_{4-2} + {1}/_{4} A_{4-3} \end{cases}$$

$$= \begin{cases} {1}/_{3} = {2}/_{3} A_1 + {1}/_{3} A_0 \\ {3}/_{8} = {3}/_{4} A_2 + {1}/_{4} A_1 \end{cases},$$

and by the recurrence we have $A_2 = ½ * A_1 + ½ * A_0$, so we can get the initial conditions for the Monty Hall recurrence form as $A_0 = 1$ and $A_1 = 0$, and start to find the closed generating function for A_n. First, multiplying both sides of $A_n = {n-1}/_{n} A_{n-1} + {1}/_{n} A_{n-2}$ by *n!* yields

$$n!A_n = (n-1)((n-1)!A_{n-1} + (n-2)!A_{n-2})$$

Let $B_n = n!A_n$, thus yielding $B_n = (n-1)(B_{n-1} + B_{n-2})$ with $B_0 = 1$, and $B_1 = 0$, which is the famous derangement relation and with a generation function of $B_n = n!\sum_{i=0}^{n} {(-1)^i}/{i!}$. Therefore, we have $A_n = \sum_{i=0}^{n} {(-1)^i}/{i!}$, and

$$M_n = {n-1}/{n}\sum_{i=0}^{n-2} {(-1)^i}/{i!} + {1}/{n}\sum_{i=0}^{n-3} {(-1)^i}/{i!}$$

$$= {1}/{n}\,(n\sum_{i=0}^{n-2} {(-1)^i}/{i!} - \sum_{i=0}^{n-2} {(-1)^i}/{i!} + \sum_{i=0}^{n-3} {(-1)^i}/{i!}$$

$$= {1}/{n}\,(n\sum_{i=0}^{n-2} {(-1)^i}/{i!} - \sum_{i=0}^{n-3} {(-1)^i}/{i!} - {(-1)^{n-2}}/{(n-2)!} + \sum_{i=0}^{n-3} {(-1)^i}/{i!})$$

$$= {1}/{n}\,(n\sum_{i=0}^{n-2} {(-1)^i}/{i!} - {(-1)^{n-2}}/{(n-2)!})$$

$$= {1}/{n}\,(n\sum_{i=0}^{n-2} {(-1)^i}/{i!} - {(n-1)(-1)^{n-2}}/{(n-1)(n-2)!})$$

$$= \sum_{i=0}^{n-2} {(-1)^i}/{i!} + {(n-1)(-1)^{n-1}}/{n!})$$

$$= \sum_{i=0}^{n-2} {(-1)^i}/{i!} + {(-1)^{n-1}}/{(n-1)!} - {(-1)^{n-1}}/{n!})$$

$$= \sum_{i=0}^{n-1} {(-1)^i}/{i!} + {(-1)^n}/{n!}) = \sum_{i=0}^{n} {(-1)^i}/{i!}.$$

Hence the player's winning probability $P(W) = 1 - M_n = 1 - \sum_{i=0}^{n} {(-1)^i}/{i!}$.

4. Some Interesting Properties

There are quite a few interesting properties in exhaustive Monty Hall Problems, for example the Fibonacci sequences of the nodes in the briefer tree diagram for n-doors. The numbers of expanded nodes before each swap of the contestant form a Fibonacci sequences starting from 2, and the numbers of car nodes and goat nodes expanded for the same time form a pair of adjacent numbers in the Fibonacci sequence. Two other properties especially interested us. First, from the solution process above we simultaneously proved that $M_n = A_n$, which could also be proved inductively.

Problem: Given $M_n = {n-1}/{n} A_{n-2} + {1}/{n} A_{n-3}$ and $A_n = {n-1}/{n} A_{n-1} + {1}/{n} A_{n-2}$, show that $M_n = A_n$ for all $n > 2$.

Basis: From the briefer tree diagrams in Figure 2 we know $M_3 = 1/3$; replacing $n - 2$ with n in the dashed closure in Figure 4 yields the following tree diagram in Figure 5 for A_n;

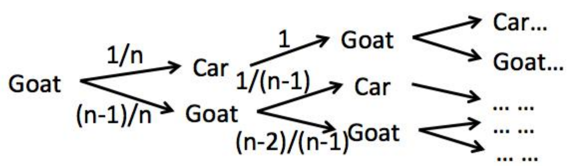

Figure 5: Tree diagram for A_n

Substituting 3 for n, we get $A_3 = 1/3$, which is calculated along the losing path; thus we have $M_3 = A_3 = 1/3$

Inductive step: Assume that $M_k = A_k$ holds, and now show $M_{k+1} = A_{k+1}$ also holds.

$A_k = {}^{k-1}/_{k}\, A_{k-1} + {}^{1}/_{k}\, A_{k-2}$ yields $A_{k-2} = kA_k - (k-1)A_{k-1}$,

hence, $M_{k+1} = {}^{k}/_{k+1}\, A_{k-1} + {}^{1}/_{k+1}\, A_{k-2}$

$$= {}^{1}/_{k+1}\, (kA_{k-1} + kA_k - (k-1)A_{k-1})$$

$$= {}^{1}/_{k+1}\, (kA_k + A_{k-1}) = {}^{k}/_{k+1}\, A_k + {}^{1}/_{k+1}\, A_{k-1}) = A_{k+1}$$

Thereby showing $M_{k+1} = A_{k+1}$ holds, and by mathematical induction $M_n = A_n$ holds for all natural $n > 2$.

The other interesting property is that with n approaching $+\infty$ the limits of M_n could be calculated according the ratio of derangement to permutation,

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} \sum_{i=0}^{n} {}^{(-1)^i}/_{i!} = {}^{1}/_{e} \approx 0.3679\ldots$$

This means that with a large enough number of doors in an exhaustive Monty Hall problem the contestant tends to lose the game by a probability of $P(L)$ = 0.3679 while winning the game by $P(W)$ = 1 – 0.3679 = 0.6321.

5. Conclusion

In this paper we isolated a group of Monty Hall problems, which in fact have also been resorted to by other mathematic teachers while teaching though not explicitly defined or generalized to n-doors. For example, Carlton (2005) shared his experiences.

Before presenting a formal solution to the Monty Hall Problem to my students, I find that it helps to give an intuitive explanation for the 1/3 - 2/3 solution. Imagine you plan to play Let's Make a Deal and employ the "switching strategy." As long as you initially pick a goat prize, you can't lose: Monty Hall must reveal the location of the other goat, and you switch to the remaining door - the car. In fact, the only way you can lose is if you guessed the car's location correctly in the first place and then switched away. Hence, whether the strategy works just depends on whether you initially picked a goat (2 chances out of 3) or the car (1 chance out of 3).

Our experiences with exhaustive Monty Hall problems, especially with n-doors ($n > 3$), have indicated even greater defusing advantages, and the new variant of briefer tree diagrams provides students with a simpler and more feasible manual method to calculate the conditional probability of three till nine door exhaustive problems. The recurrence form for n-doors and its closed generation function lay down the basis for more efficient computing by means of computer software, and the Appendix Table gives most of the results of both *P(W)* and *P(L)* for three to one billion doors. Furthermore, some interesting properties, such as the Fibonacci sequences, the hidden Derangement numbers, and limits of $P(W) = 1 - {}^{1}/_{e}$ and $P(L) = {}^{1}/_{e}$ as n approaching $+\infty$, all implicate the mathematical beauty of exhaustive Monty Hall Problems and greater significance for further academic research.

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Appendix Table: Some sampled exhaustive Monty Hall probabilities

Number of doors	Winning probability	Losing probability
3	0.666666667	0.333333333

4	0.625	0.375
5	0.633333333	0.366666667
6	0.631944444	0.368055556
7	0.632142857	0.367857143
8	0.632118056	0.367881944
9	0.632120811	0.367879189
10	0.632120536	0.367879464
20	0.632120559	0.367879441
30	0.632120559	0.367879441

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