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Representation theorem of composite odd numbers indices

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Abstract

We study the composite odd numbers via their indices with a function ranging through all of them. This mapping allows us to demonstrate that the set of these indices are described with two families of finite sequences with arithmetic differences. Composite odd numbers are then shown to be obtained as differences of two squares. We then conjecture odd primes do not appear randomly.

Keywords: composite odd numbers; indices; sequences with arithmetic differences; finite sequence; differences of two squares; odd primes; not appear randomly; reference points; remarkable index.

Introduction

The distribution of prime numbers may seem random. However, in 2016, Robert Lemke Oliver and Kannan Soundararajan [1] of Stanford University (California) discovered that links exist between consecutive prime numbers. Prime numbers might be distributed according to a structure, as well as composite odd numbers. We will demonstrate in this paper that the set of composite odd numbers is characterized by two families of finite sequences with arithmetic differences. The study is performed by indexing the odd numbers greater than 1 with the family $N_k = 2k + 3, k \in \mathbb{N}$.

The integers are represented by quadratic forms [2] [3] [4] [5]. From the two precedent sequences, we deduce that every composite odd number is a difference of two squares.

I- Indices of composite odd numbers: the set W

We will define the function k mapping to the set of indices of composite odd numbers. These indices form the set W. We will show that these indices are obtained with a sequence of arithmetic sequences.

I.1 Preliminaries

Introduction of some notations:

1. Let I be the set of odd integers greater than 1, i.e.:

$$I = \{N_k = 2k + 3 | k \in \mathbb{N}\} \text{ with } k \text{ the index of the odd number } N_k$$

2. Let P be the set of prime numbers.

3. Let C be the set of composite odd integers greater than 1, i.e.:

$$C = I \backslash P = \{N_k \in I | \exists (a,b) \in I, N_k = ab\}$$

The function $f{:}k \in \mathbb{N} \longmapsto N_k \in I$ is bijective. The inverse function is $f^{-1}{:}N_k \in I \longmapsto \ k = \frac{N_k - 3}{2}$. The preimage of the set C is named W:

$$W = f^{-1}(C) = \{k \in \mathbb{N} | \ N_k \in C\}$$

I.2 The map W as the image of the function k

By definition, there is a surjection of $I \times I$ on:

$$s{:}\begin{cases} I \times I \longmapsto C \\ (a, b) \longmapsto ab \end{cases}$$

We match this surjection with a surjection of $\mathbb{N}^* \times \mathbb{N}$ over the set of indices of composite odd numbers:

<u>Proposition/definition 1.2</u>: Let k be defined by:

$$(n, j) \in \mathbb{N}^* \times \mathbb{N} \longmapsto k(n,j) = (2j + 3)n + j.$$

This map returns the index $k(n,j)$ of a composite odd number N_k.

More precisely, k is surjective on W:

$$W = \{k(n,j); (n, j) \in \mathbb{N}^* \times \mathbb{N}\}$$

In other words, when $k \in W$, $N_k \in C$ and when $k \notin W$, $N_k \in P$.

Moreover, we have: $k(n,j) = k(j + 1, n - 1)$.

<u>Proof</u>: Let us note first that $N_k = 2k + 3$ et $k(n,j) = (2j + 3)n + j$.

Thus $N_{k(n,j)} = 2((2j + 3)n + j) + 3 = (2j + 3) \times (2n + 1)$

Thus, for all $j \in \mathbb{N}, n \in \mathbb{N}^*$, $N_{k(n,j)}$ is a composite odd number.

Conversely if $c = ab$ is a composite odd number, in writing $a = 2j + 3$ and $b = 2n + 1$ with $j \in \mathbb{N}, n \in \mathbb{N}^*$, we find that $c = N_{k(n,j)}$ whence the surjectivity of k.

The equality $k(n,j) = k(j + 1, n - 1)$ is straightforward because $(2j + 3)(2n + 1) = (2n + 1)(2j + 3) = (2(n - 1) + 3)(2(j + 1) + 1)$.

<u>Remark</u>: In the encyclopedia of integer sequences, the sequence A153238 enumerates increasingly all elements of W.

I.3 The map W as a union of arithmetic sequence supports

1.3.1 The sets of indices

<u>Proposition/definition 1.3.1</u> : With j fixed in $\mathbb{N}$, let $k_j : n \in \mathbb{N}^* \longmapsto k_j(n) = k(n,j) = (2j + 3)n + j \in W$, and W_j its image.

Then W_j is the set of indices of odd multiples of $2j + 3$.

The set W can be written as

$$W = \bigcup_{j\in\mathbb{N}} W_j$$

Proof: We know that $N_{k_j(n)} = N_{k(n,j)} = (2j+3)(2n+1)$

So W_j consists of the indices of odd multiples of $2j+3$.

Conversely if $c = (2j+3)b$ is an odd multiple of $2j+3$, then writing $b = 2n+1$, we get immediately $c = N_{k_j(n)}$.

Finally, according to proposition 1.2, the set W is the union of sets W_j with $j \in \mathbb{N}$.

1.3.2 A sequence of arithmetic sequences

Proposition 1.3.2: W is the countable union of ranges of arithmetic sequences. More precisely, for a given parameter j the set W_j is the range of the arithmetic progression $\left(S_n^{(j)}\right)_{n\in\mathbb{N}^*}$ with common difference $2j+3$ and initial term $3j+3$.

Proof: with definition 1.3.1 we get for all $n \in \mathbb{N}^*$:

$$S_n^{(j)} = k_j(n)$$

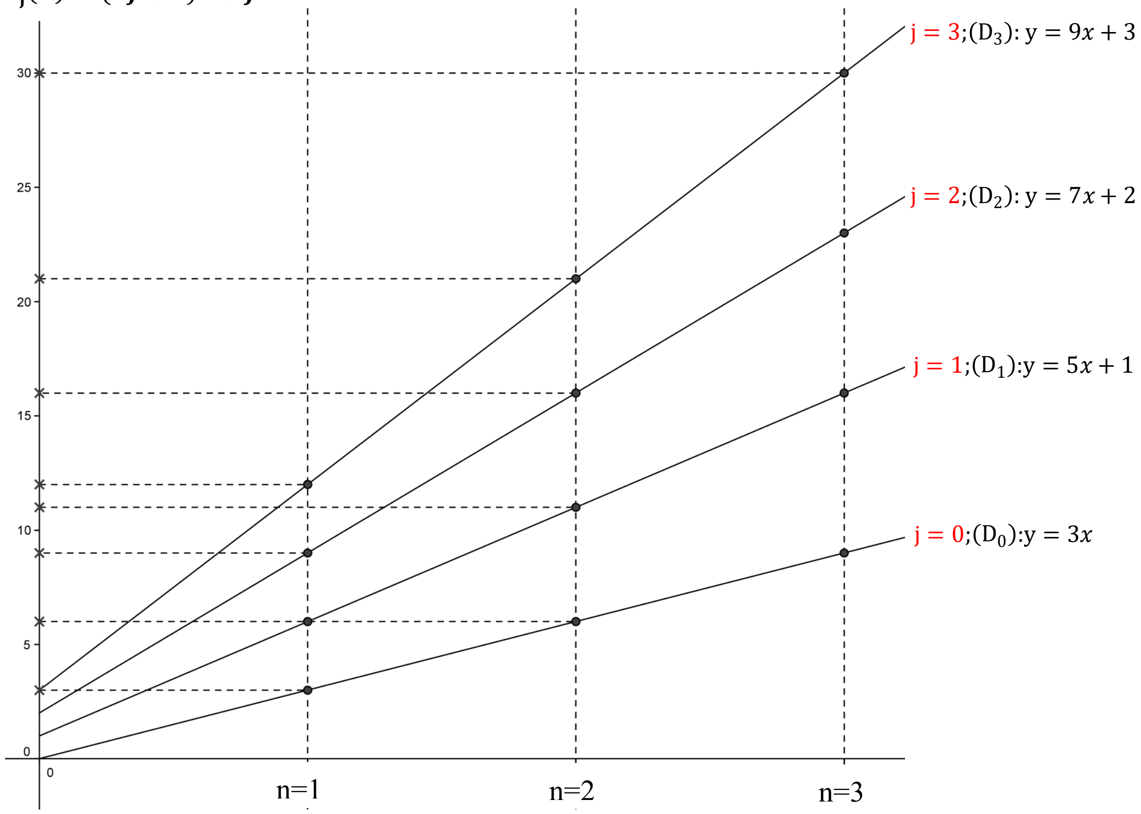
So W is the union of ranges of sequences $S^{(j)}$ for j varying in $\mathbb{N}$.

1.3.3 Some definitions

Illustration 1: In a coordinate system (n,k), sequences $S^{(j)}$ are plotted for j = 0 to 3 and values are represented by points with positive integer abscissa on the line (D_j):$y = (2j+3)x + j$.

The ordinates of these points are the indices of the set W.

$\mathbf{k_j(n) = (2j+3)n + j}$



<u>Definition 1.3.3</u> :

1. A *point* designates a pair of integer values (n,j).

2. We call $k_j(n) \in W$ the *index* of the point (n,j). Each index is associated with one or more points.

3. For a given j, the set W_j is the range of a sequence of indices associated with a particular set of points. In the set W_j, each index is associated with a unique point with second coordinate j.

I.4 Remarkable index of the sets of indices

Among the composite odd numbers, the squares will be said to be remarkable. By extension, their index and the matching point are also said to be remarkable. They will be used in the proof of the main theorem as reference points.

<u>Proposition/definition 1.4 :</u> In the set W_j, there is an unique index corresponding to the number $(2j+3)^2$ that is:

$$k_j(j+1) = 2j^2 + 6j + 3.$$

The matching remarkable point is $(j+1,j)$.

Proof: We have: $(2j+3)^2 = (2j+3)(2(j+1)+1) = N_{k_j(j+1)}$

So $k_j(j+1)$ is the remarkable index associated with remarkable point $(j+1,j)$.

We compute:

$$k_j(j+1) = (2j+3)(j+1) + j = 2j^2 + 6j + 3$$

II- Representation theorem of composite odd numbers indices

We will show that the set W of indices of composite odd numbers is described by two families of finite sequences with *arithmetic differences* via two families of reference points.

II.1 Some definitions

Definition 2.1:

a- We name finite sequence with *arithmetic difference* any finite sequence $(u_n)_{0\leq n\leq N}$ verifying:

$$\exists (a,b) \in \mathbb{Z}^2, \forall n \in [\![0,N-1]\!], u_{n+1} = u_n + an + b$$

b- A *reference point of the first, second and third type* is a point of the form $(j+1,j)$, $(j+2,j)$ and $(j+1,j+1)$ respectively. We simply call *reference point* a reference point of any of these three types.

c- We say that the point $(n+1,j-1)$ *precedes* the point (n,j) (provided that $j > 0$), while the point $(n-1,j+1)$ *succeeds* it (provided that $n > 1$). Any given point has a finite number of predecessors and successors.

Remark 1: reference points of the first type are therefore remarkable points.

Remark 2: we can notice that a sequence with arithmetic difference can also be described as a quadratic progression.

II.2 Sequences with arithmetic difference and symmetries

We will demonstrate in this section that the indices of points that precede or succeed a remarkable point form a finite sequence with arithmetic difference.

II.2.1 Sequence of indices came from the remarkable index

Proposition 2-2-1: Let (n_x,j_x) be the sequence of predecessors of (n,j). Then indices $k_{j_x}(n_x)$ form a finite sequence with arithmetic difference. Moreover, if (n,j) is a remarkable point, then:

$$k_{j_x}(n_x) = k_j(j+1) - 2x^2$$

Proof: Predecessors of (n,j) can be written as: $(n_x,j_x) = (n+x,j-x)$. So

$$\begin{aligned} k_{j_x}(n_x) &= (2(j-x)+3)(n+x)+j-x \\ &= k_j(n) - 2x^2 + 2(j+1-n)x \end{aligned}.$$

In particular, one gets: $k_{j_{x+1}}(n_{x+1}) - k_{j_x}(n_x) =- 4x + 2(j-n)$. So the sequence of predecessors is a sequence with arithmetic difference.

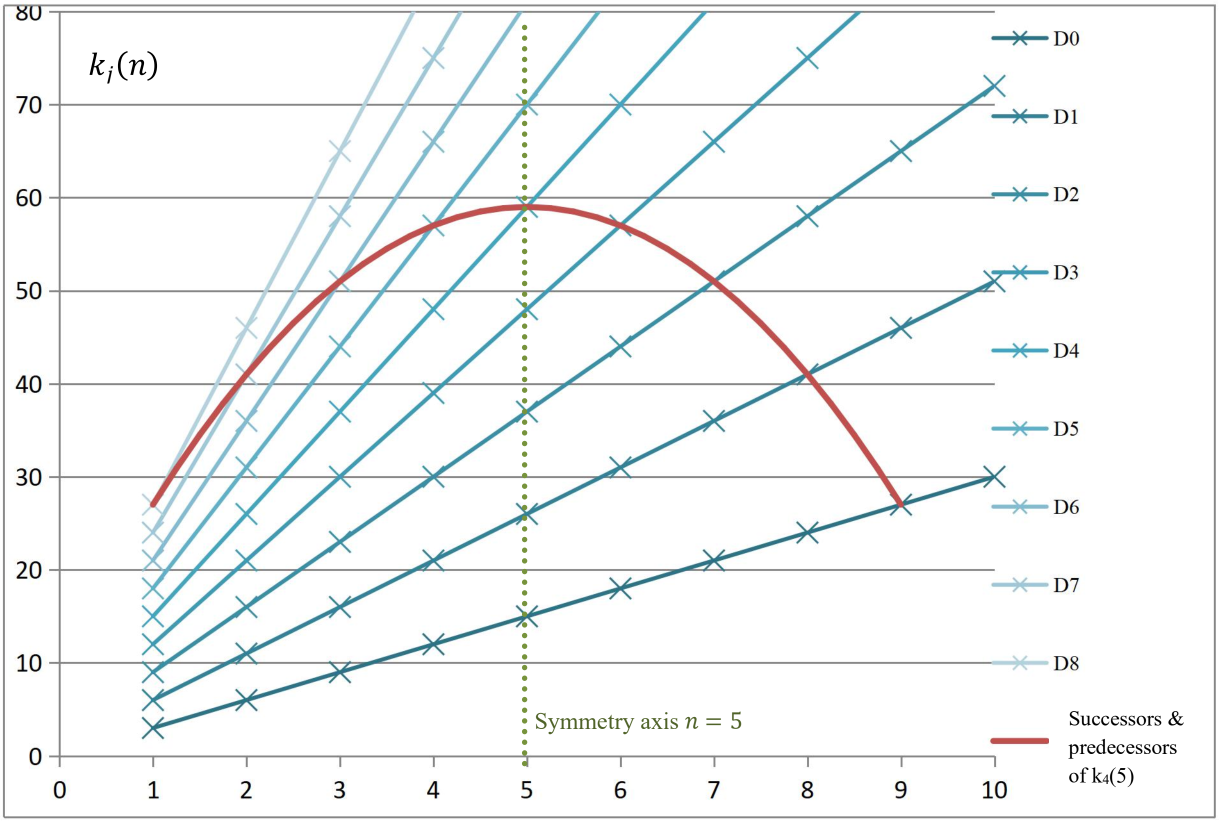
Moreover, if the point (n,j) is remarkable then $n = j+1$ and:

$$k_{j_x}(n_x) = k_{j-x}(j+1+x) = k_j(j+1) - 2x^2$$

Corollary 2-2-1: The sequence of the indices of the successors of a point is also a sequence with arithmetic difference. If this point is remarkable, then the sequence is actually the same sequence as that given by predecessors:

$$k_{j-x}(j+1+x) = k_{j+x}(j+1-x) = k_j(j+1) - 2x^2$$

Remark: There is therefore a symmetrical relationship between the predecessors and successors of a remarkable point. This is shown in the graph below for $k_4(5)$:



II.2.2 Second and third type reference points

Proposition 2-2-2: The sequence of predecessors indices of a reference point of the second type is given by:

$$k_{j-x}(j+2+x) = k_j(j+2) - 2x^2 - 2x$$

This sequence coincides with that of the successors of a reference point of the third type:

$$k_{j+1+x}(j+1-x) = k_{j+1}(j+1) - 2x^2 - 2x$$

Proof: We first check that $k_j(j+2) = k_{j+1}(j+1)$. Indeed, by developing these two expressions, on one hand we get:

$$k_j(j+2) = (2j+3)(j+2) + j = 2j^2 + 8j + 6$$

And on the other hand:

$$k_{j+1}(j+1) = (2(j+1)+3)(j+1) + j + 1 = 2j^2 + 8j + 6.$$

Using the calculations from the prior proposition, we get:

$$k_{j-x}(n+x) = k_j(n) - 2x^2 + 2(j+1-n)x$$

hence, with a reference point of second type $(j+2,j)$:

$$k_{j-x}(j+2+x) = k_j(j+2) - 2x^2 - 2x$$

and with a reference point of third type $(j+1, j+1)$:

$$k_{j+1+x}(j+1-x) = k_{j+1}(j+1) - 2x^2 - 2x.$$

Remark: Similarly, indices of second type reference points successors coincide with those of third type reference points predecessors. There is a symmetry as previously, but with non-integer axis.

II.3 Representation theorem of composite odd numbers indices

Proposition 2.3: Any element from the set W is the index of a predecessor of a reference point of the first or second type.

Proof: Let $k_j(n)$ be an element of W.

We will first assume $j < n$.

If $n - j$ is odd then there exists an integer p such that $n - j = 2p + 1$. In particular, if we let $j_0 = j + p$, then we get:

$$k_j(n) = k_{j_0 - p}(j + 2p + 1) = k_{j_0 - p}(j_0 + 1 + p)$$

so $k_j(n)$ is the index of the p-th predecessor of the remarkable point with index $k_{j_0}(j_0 + 1)$.

On the other hand, if $n - j$ is even (and therefore non-zero since $j < n$) then we write $n - j = 2p + 2$ and we let $j_0 = j + p$.

Then:

$$k_j(n) = k_{j_0 + p}(j + 2p + 2) = k_{j_0 - p}(j_0 + 2 + p)$$

so $k_j(n)$ is the index of the p-th predecessor of the reference point associated with the index $k_{j_0}(j_0 + 2)$.

Now, let us assume $j \geq n$.

If $j = n$ then it is straightforward to get that $k_j(n)$ is the index of a reference point of the third type. It is therefore the index of a reference point of the second type as well as shown in proposition 2-2-2.

If $j = n + 1$ then we get that the point $(j - 1,j)$ succeeds the remarkable point $(j,j - 1)$, its index is therefore the same as that of the predecessor of this remarkable point according to proposition 2-2-1.

More generally, if $j > n + 1$ we can use $k_j(n) = k_{n-1}(j + 1)$ (cf. proposition 1.2) to fall back on the initial case.

Remark: Thanks to the symmetries described in the previous section, we obtain in the same way that every element of W is the index of a successor of a reference point of the first or third type.

Representation theorem of composite odd numbers indices: The set W is thus obtained as the ranges of two families of finite sequences with arithmetic difference. More precisely:

$$W = S_1 \cup S_2$$

with:

$$\begin{aligned} S_1 &= \{k_j(j + 1) - 2x^2; j \in \mathbb{N}, x \in [\![0,j]\!]\} \\ S_2 &= \{k_j(j + 2) - 2x^2 - 2x; j \in \mathbb{N}, x \in [\![0,j]\!]\} \end{aligned}$$

The indices $k_j(j + 1)$ of first type reference points (or remarkable points) are thus the initial terms of sequences ranging in S_1. Similarly, the indices $k_j(j + 2)$ of second type reference points are the initial terms of sequences ranging in S_2.

II.4 Composite odd numbers as differences of two squares

Proposition 2-4: S_1 is the set of odd indices from W.

On the contrary, S_2 is the set of even indices from W.

In particular $S_1 \cap S_2 = \emptyset$.

Proof: It is sufficient to check that the indices of the remarkable points are odd, and those of the reference points of the second type are even. Let us remember that $k(n,j) = (2j + 3)n + j$. In particular, the parity of $k(n,j)$ is the same as that of $n - j$, which completes the proof.

Remark: The index is even if and only if the odd number is congruent to 3 modulo 4, otherwise it is congruent to 1.

Proposition 2-5: Any composite odd number can be written as a difference of two squares.

Proof: Let $N_k \in C$. So $k \in W = S_1 \cup S_2$.

If $k \in S_1$ then $k = k_j(j+1) - 2x^2$ with $j \in \mathbb{N}$ and $x \in [\![0,j]\!]$.

Thus $N_k = 2k + 3 = (2j+3)^2 - (2x)^2$.

If $k \in S_2$ then $k = k_j(j+2) - 2x^2 - 2x$ whence $N_k = (2j+3)(2j+5) - 4(x^2+x) = (2j+4)^2 - (4x^2+4x+1) = (2j+4)^2 - (2x+1)^2$.

Corollary 2-5: There is an infinite number of odd squares that can be written as the sum of two squares.

Proof: Let consider a non-remarkable point (n,l) with same index as a remarkable point $(j+1,j)$. To prove that such a point exists, it is enough to take $j \in W$ since we can then write:

$$2j + 3 = ab$$

thus:

$$(2j+3)^2 = (a^2b)\,.b,$$

which corresponds to the point (n,l) with $2n + 1 = a^2b$ and $2j + 3 = b$.

The proof of proposition 2.3 then allows us to state that $k_l(n)$ is the strict predecessor of a reference point, and the proposition 2.4 allows us to specify that the reference point is necessarily that of the first type and so a remarkable point.

Therefore, using the proof of proposition 2.5, we can conclude that there are two positive integers m and x such that $(2j+3)^2 = (2m+3)^2 - (2x)^2$.

Hence: $(2m+3)^2 = (2j+3)^2 + (2x)^2$.

Since we have $(2m+3)^2 > (2j+3)^2$ and we could choose any $j \in W$, we deduce that there is also an infinite number of possible values for m.

Example: $9^2 = 3 \times 27 = (15-12).(15+12) = 15^2 - 12^2$. Thus $15^2 = 12^2 + 9^2$ is the smallest square equal to a sum of two squares we can obtain with this method.

Conclusion

We indexed odd numbers to describe properties of composite odd numbers. First, we described the set of indices of composite odd numbers as the image of a function k with two integer variables, or equivalently as the countable union of arithmetic sequence ranges.

Then, we improved this description with finite sequences with arithmetic difference. Thanks to symmetries underlying function k and reference points which correspond to the centers of these symmetries, we showed that we needed only two families of sequences: one for odd indices and the other for even indices – which constitutes our representation theorem of composite odd numbers indices. We then observed that this representation was equivalent to writing the composite odd numbers as differences of two squares.

Composite odd numbers indices are therefore fully described by two families of finite sequences with arithmetic differences. It therefore seems to us that the set of odd primes, complement of C in I, presents a structure that is not random.

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