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On the existence of moments in Cauchy-like distributions induced from the tan function

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Abstract

In this paper we consider cases of the existence of the moments of functions of random variables supported on a bounded interval. Our attention is restricted to the tan function, as a generalization of the Cauchy distribution which is infact the result of applying this function to a uniformly distributed variable.

Keywords: Cauchy distributions, tan function, moments

Preliminaries

The standard Cauchy distribution can be derived as follows - having a uniformly distributed random variable X on $\left(-\frac{\pi}{2}\,;\,-\frac{\pi}{2}\right)$ we define $Y = \tan(X)$. This distribution is famous for its lack of moment existence. [1] [4] [5]

The following question arises naturally - if we allow any distribution for X, perhaps with a well-defined probability density or probability mass function, under what conditions at least some moments of the variable $Y = \tan(X)$ would exist and up to what order? Such questions are important in practice, since real-life modelled variables are generally bounded and all of their moments exist - however, it seems this is not always true for functions of such [3].

As a start, let us consider Y = g(X) where $g : I \to \Re$ is a well-defined strictly increasing diffeomorphism between an interval I with endpoints a and b and $\Re$. In terms of the cumulative distribution functions of Y and X we have

$$F_Y(x) = P[Y \leq x] = P[X \leq g^{-1}(x)] = F_X(g^{-1}(x))$$

via which we have the form

$$E[Y^n] = \int_{-\infty}^{\infty} x^n dF_Y(x) = \int_{-\infty}^{\infty} x^n dF_X(g^{-1}(x)) = \int_{-\infty}^{\infty} x^n f_X(g^{-1}(x))(g^{-1}(x))' dx$$

Another expression, more suitable for our purposes, is the following

$$E[Y^n] = \int_{-\infty}^{\infty} x^n dF_X(g^{-1}(x)) = \int_a^b g^n(y) dF_X(y) = \int_a^b g^n(y) f_X(y) dy$$

The discrete version of the latter is as follows

$$E[Y^n] = \sum_{y \in X} g^n(y) f_X(y)$$

where f is the probability mass function of X. Throughout the rest of the article, we treat $I = \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$ and $g(x) = \tan x$.

General analysis for tan and some examples

Firstly, let us consider the discrete variable case. Here it will be useful that absolute convergence behaves quite well with sums.

Proposition 1. If the set $\{\tan y | y \in X\}$ is bounded, then $E[Y^n]$ exists for all n.

Proof. It suffices to consider $K_0 = \sum_{y \in X} |\tan^n(y) f_X(y)|$. If M is a bound for the given set, then as $\sum_{y \in X} f_X(y) = 1$ and $f_X(y) \geq 0$, we have $K_0 \leq M^n$, thus the sum $E[Y^n]$ is (absolutely) convergent.

Of course, in general this need not be the case.

Example 2. Let X attain the values $y_i = \arctan(2^i)$, $i \geq 1$, with probabilities $f_X(y_i) = \frac{1}{2^i}$ (these are positive and sum exactly to 1). Then the k-th partial sum of the n-th moment is $\sum_{i=1}^{k} 2^{i(n-1)} \geq \sum_{i=1}^{k} 1 = k$ and thus $\tan(X)$ has no moments.

Now we move on to the continuous variable case.

In $E[Y^n] = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^n(y) f_X(y) dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{f_X(y)}{\cos^n y} (\sin^n y) dy$ the sin function is irrelevant at the only possibly problematic points $\pm\frac{\pi}{2}$. Moreover, by Taylor's theorem we have $\cos y = \left(\frac{\pi}{2} \pm y\right) + O\left(\left(\frac{\pi}{2} \pm y\right)^3\right)$. As a consequence we have established the following

Proposition 3. The n-th moment exists if and only if the limits

$$\lim_{R \to \frac{\pi}{2}} \int_{R}^{\frac{\pi}{2}} \frac{f_X(y)}{\left(\frac{\pi}{2} - y\right)^n} dy \quad \text{and} \quad \lim_{R \to -\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{R} \frac{f_X(y)}{\left(\frac{\pi}{2} + y\right)^n} dy$$

both exist.

In particular:

Example 4. In the standard Cauchy distribution we have $f_X \equiv \frac{1}{\pi}$, so neither of the limits above exists and the moments are undefined, as expected.

Example 5. If there exist two points x_1 and x_2 with $-\frac{\pi}{2} < x_1 < x_2 < \frac{\pi}{2}$ and $F_X(x_1) = 0$, $F_X(x_2) = 1$, then all moments exist. Indeed, in this case the density is 0 outside of $[x_1; x_2]$ and thus the above limits are equal to zero.

Example 6. If $f_X(x) = C\left(x + \frac{\pi}{2}\right)^p \left(\frac{\pi}{2} - x\right)^q$ for $p, q > 0$ and an appropriate normalizing constant $C > 0$, then the existing moments are precisely those up to order $\min(p; q)$.

Example 7. If $f_X(x) = C \exp\left(-\frac{M}{\left(x+\frac{\pi}{2}\right)^p \left(\frac{\pi}{2}-x\right)^q}\right)$ for $M, p, q > 0$ and an appropriate normalizing constant $C > 0$, then all moments exist.

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