SCIREA Journal of Mathematics
http://www.scirea.org/journal/Mathematics
September 25, 2020
Volume 5, Issue 3, June 2020

The narrow border between a continuous function and a discontinuous function

Alicia C. Sánchez
"Santa Eulalia" High School. Spain
Email address:
csanchezalicia@gmail.com

Abstract

The RAFU functions have been studied in [1, 2, 3, 4, 5 and 6]. They are useful in Approximation Theory to approach continuous functions and to reconstruct continuous functions. In this work we will use the RAFU functions to turn a discontinuous function into a continuous function. We will show the explicit expression of a sequence of continuous functions that converges to a discontinuous function given except at its discontinuity points. The paper shows examples for different types of discontinuities.

Keywords: RAFU functions; continuous functions; discontinuous functions.

Introduction

In the high school students start studying the concept of a continuous function from an intuitive point of view. The points of discontinuity of a discontinuous function can be classified into removable discontinuity, jump discontinuity and essential discontinuity.

Usually teachers say that a discontinuous function with a removable discontinuity point can be turned into a continuous function by placing the corresponding value of the function at the appropriate place. But, what can we do to turn a discontinuity function with a jump discontinuity point or an essential discontinuity point into a continuous function?

Our proposal in this work is to provide a precise and explicit answer to the previous question. Given a function with a discontinuity point, we will give the mathematical expression of a sequence of continuous functions that converge to the discontinuous function in all its domain of definition except at its discontinuity point. To do this, we will use the RAFU functions and we will try to explain why the RAFU functions are appropriate to solve this problem. The paper is illustrated with a few examples.

The readers who are interested in knowing about the RAFU functions can see [1, 2, 3, 4, 5 and 6]

One way to do it

To simplify, we suppose the step function

$$h(x) = \begin{cases} k_1 & if\ x_0 \le x \le x_1 \\ k_2 & if\ x_1 < x \le x_2 \end{cases}$$

If we want to turn $h(x)$ into a continuous functions, it is clear that one way to solve the problem is to connect the constants k_1 with k_2 by means of straight lines that passes through values of $h(x)$ for $x < x_1$ and $x > x_1$ where x is near x_1. It is also clear that the continuous approximation to $h(x)$ will be better the more vertical is the straight line considered.

Without loss of generality, we can assume straight lines that pass through two equidistant points from the discontinuity point $x = x_1$. We will call $r_n(x)$ to these straight lines where n is obtained from $d = \frac{1}{n}$ being d the radius of the neighbourhood centered on $x = x_1$.

Example 1

Given the function

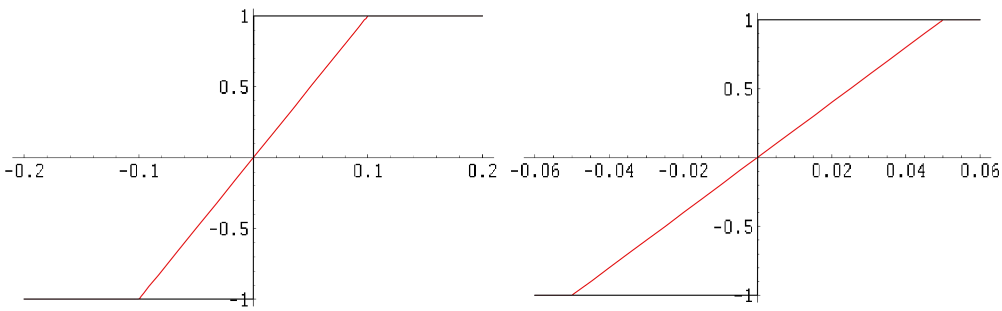
$$f(x)=\begin{cases}-1 & if\ -2\le x\le 0\\ 1 & if\ \ 0<x\le 2\end{cases}\quad (1)$$

and the sequence of continuous approximations defined by

$$fr_n(x)=\begin{cases}-1 & if\ -2\le x<-\frac{1}{n}\\ r_n(x) & if\ -\frac{1}{n}\le x\le\frac{1}{n}\\ 1 & if\ \frac{1}{n}<x\le 2\end{cases}$$

We consider two approximations. First, $fr_{10}(x)$ *where* $r_{10}(x)$ *is the straight line that passes through* $(-0.1,-1)$ *and* $(0.1,1)$. *Then,* $fr_{20}(x)$ *where* $r_{20}(x)$ *is the straight line that connects* $(-0.05,-1)$ *with* $(0.05,1)$.

In Figure 1a) we represent the graphs of the functions $f(x)$ *and* $fr_{10}(x)$ *for the neighbourhood centered on* $x=0$ *and radius* 0.2 *and in Figure 1b) the graphs of the functions* $f(x)$ *and* $fr_{20}(x)$ *for the neighbourhood centered on* $x=0$ *and radius* 0.06.



a) b)

Figure 1

When we see what happens inside the neighbourhoods, we observe that the continuous approximations $fr_n(x)$ to the discontinuous function $f(x)$ that the straight lines $r_n(x)$ provide can be improvable.

It would be desirable turn $f(x)$ into a continuous function by means of other types of functions nearer to it on both sides of the $x=0$ and more vertical when pass through the discontinuity point $x=0$.

An answer with the RAFU functions

Example 2

Given the previous function $f(x)$ *in Example 1, (1) we consider the RAFU function*

$$c_n(x) = M_n + N_n \cdot \sqrt[2n+1]{x - x_1} \quad (2)$$

where $x_1 = 0$ *is the discontinuity point and* M_n *and* N_n *are constants to calculate.*

Now we obtain the function $c_{10}(x)$ *taking into account the conditions* $c_{10}(-0.1) = -1$ *and* $c_{10}(0.1) = 1$*. So, the values* M_{10} *and* N_{10} *can be obtained by solving the system*

$$\begin{cases} -1 = M_{10} + N_{10} \cdot \sqrt[21]{-0.1 - 0} \\ 1 = M_{10} + N_{10} \cdot \sqrt[21]{0.1 - 0} \end{cases}$$

In this case,

$$N_{10} = \frac{2}{\sqrt[21]{0.1 - 0} + \sqrt[21]{0 + 0.1}}$$

and

$$M_{10} = -1 + \frac{2 \cdot \sqrt[21]{0 + 0.1}}{\sqrt[21]{0.1 - 0} + \sqrt[21]{0 + 0.1}}$$

So,

$$c_{10}(x) = -1 + \frac{2 \cdot \left(\sqrt[21]{0 + 0.1} + \sqrt[21]{x - 0}\right)}{\sqrt[21]{0.1 - 0} + \sqrt[21]{0 + 0.1}}$$

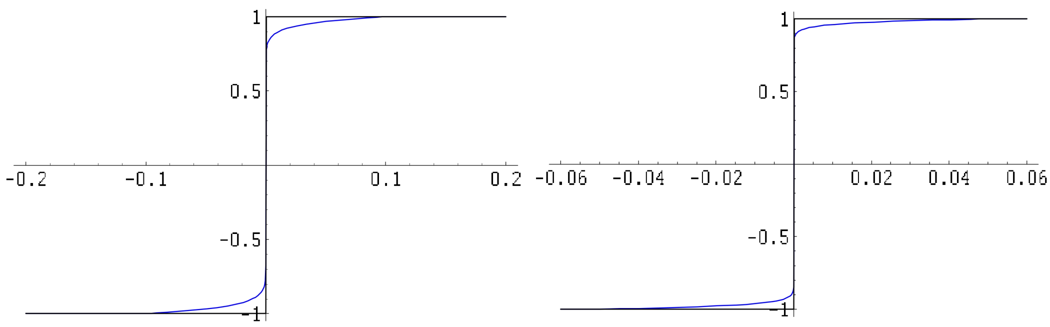
In general, the calculations for $c_n(x)$ *in this case provide*

$$c_n(x) = -1 + \frac{2 \cdot \left(\sqrt[2n+1]{0 + \frac{1}{n}} + \sqrt[2n+1]{x - 0}\right)}{\sqrt[2n+1]{\frac{1}{n} - 0} + \sqrt[2n+1]{0 + \frac{1}{n}}}$$

and the sequence of continuous approximations defined by

$$f_n(x) = \begin{cases} -1 & if\ -2 \le x < -\dfrac{1}{n} \\ c_n(x) & if\ -\dfrac{1}{n} \le x \le \dfrac{1}{n} \\ 1 & if\ \dfrac{1}{n} < x \le 2 \end{cases}$$

In Figure 2 we observe the result of turn the discontinuous function $f(x)$ given in (1) into a continuous function by means of the use of $f_{10}(x)$ and $f_{20}(x)$ that contain the RAFU functions $c_{10}(x)$ and $c_{20}(x)$ respectively. The graphs are only for the neighbourhoods centered on $x = 0$ and radius 0.2 and 0.06.



a) *b)*

Figure 2

Moreover the continuous approximation to the function improves as n increases (see Figure 3).

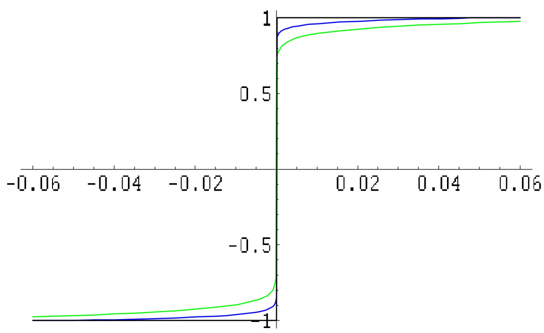
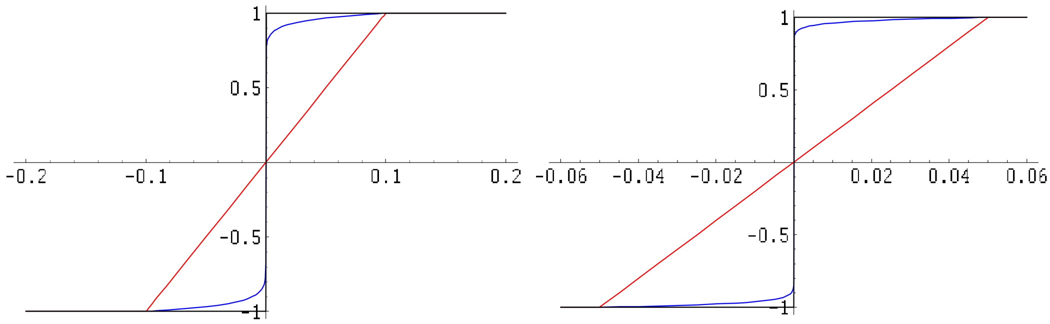


Figure 3

In Figure 4 a) we show the graphs of the functions $f(x)$ given in (1), $fr_{10}(x)$ (red colour) and $f_{10}(x)$ (blue colour) for the neighbourhood centered on $x = 0$ and radius 0.2 and in Figure 1b) the graphs of the functions $f(x)$ and $fr_{20}(x)$ (red colour) and $f_{20}(x)$ (blue colour) for the neighbourhood centered on $x = 0$ and radius 0.06.



a) b)

Figure 4

According to Figure 4, it seems that the use of functions $f_n(x)$ improves the use of functions $fr_n(x)$. Next Section is devoted to explain this assertion.

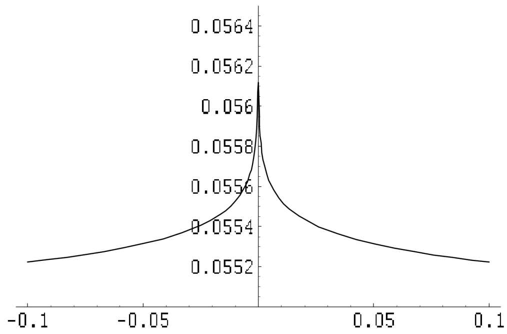
Analysis of the results

The RAFU functions, like $c_n(x)$ defined in (2) have been studied in [1, 2, 3, 4, 5 and 6]. Why are these functions useful to approximate discontinuous functions?

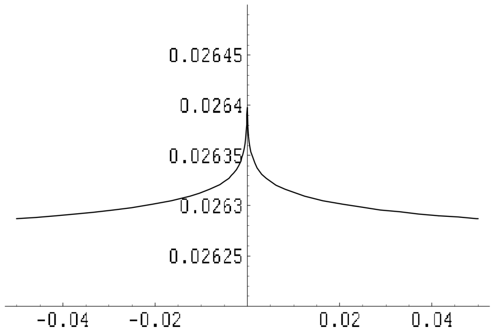
For one explanation, we are going to calculate the derivative of $c_n(x)$.

$$c_n^{'}(x) = \frac{N_n}{(2n+1)\cdot \sqrt[2n+1]{(x-x_1)^{2n}}}$$

We can observe that $c_n^{'}(x)$ exists in all its domain of definition except at the point $x = x_1$. In Figure 5 we show the graphical representations of $c_{10}^{'}(x)$ and $c_{20}^{'}(x)$ when $x_1 = 0$ for some neighbourhoods centered on $x = 0$.



a)



b)

Figure 5

From the study of the monotonicity of the functions $c_n(x)$ one deduces that:

1. The slopes of the tangent lines at the points $x \neq 0$ of its domains of definition are near 0 and the approximation to 0 improves as n increases.

2. At $x = x_1$ all functions $c_n(x)$ have vertical tangent.

Surely all of us agree that if we want to turn an arbitrary discontinuous function into a continuous function, the set of the functions to consider should verify these two conditions.

The straight lines $r_n(x)$ considered in Section 2 have a constant slope for each n and they do not have vertical tangent at $x = x_1$. So, the straight lines are not useful to solve the problem successfully.

Still more, as any other regular function (function with derivative) can not verify condition 2, we can guarantee that RAFU functions are better that any regular function in order to approximate a discontinuous function in the neighbourhoods at its points of discontinuity.

Case of an arbitrary discontinuous function

Case of a jump discontinuity

Let $h(x)$ be a discontinuous function with a jump discontinuity

$$h(x) = \begin{cases} m_1(x) & if \;\; x_0 \leq x \leq x_1 \\ m_2(x) & if \;\; x_1 < x \leq x_2 \end{cases}$$

where $m_i(x)$, $i=1,2$ are continuous functions and $x=x_1$ is a jump discontinuity point. In this case, the points $(-0.1,-1)$, $(0.1,1)$, $(-0.05,-1)$ and $(0.05,1)$ of Sections 2 and 3 become $(x_1-0.1, m_1(x_1-0.1))$, $(x_1+0.1, m_2(x_1+0.1))$, $(x_1-0.05, m_1(x_1-0.05))$ and $(x_1+0.05, m_2(x_1+0.05))$ respectively. All conclusions established in the previous sections are valid in this case.

The sequence of continuous functions $(h_n)_n$ that turn the discontinuous function $h(x)$ into a continuous function is defined by the formula

$$h_n(x)=\begin{cases} m_1(x) & if\ x_0 \le x < x_1-\dfrac{1}{n} \\ c_n(x) & if\ x_1-\dfrac{1}{n} \le x \le x_1+\dfrac{1}{n} \\ m_2(x) & if\ x_1+\dfrac{1}{n} < x \le x_2 \end{cases}$$

If $h(x)$ is a function with several jump discontinuity points, we apply what we have explained in the previous section for each discontinuity point.

Example 3

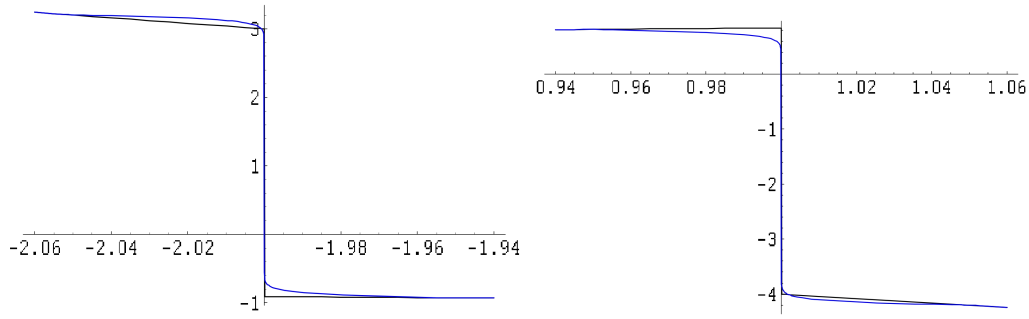
Given the function

$$f(x)=\begin{cases} x^2-1 & if\ -2.2 \le x \le -2 \\ \sin(x) & if\ -2 < x \le 1 \\ 4x & if\ 1 < x \le 1.2 \end{cases}$$

and its corresponding continuous approximation

$$f_{20}(x)=\begin{cases} x^2-1 & if\ -2.2 \le x < -2.05 \\ c_{20}(x) & if\ -2.05 \le x \le -1.95 \\ \sin(x) & if\ -1.95 < x < 0.95 \\ c_{20}(x) & if\ 0.95 \le x \le 1.05 \\ 4x & if\ 1.05 < x \le 1.2 \end{cases}$$

being $c_{20}(x)$ the function defined in (2) but with the corresponding $x=x_1$. In Figure 6 we show $f(x)$ and its approximation $f_{20}(x)$ in the intervals $[-2.06,-1.94]$ and $[0.94,1.06]$ that contain respectively the two jump discontinuity points of $f(x)$.



a) b)

Figure 6

Case of an essential discontinuity

All what we have established in the previous sections is also valid when the discontinuous function presents one or several essential discontinuity points.

Example 4

Given the function

$$f(x)=\begin{cases} x-400 & \text{if } -2\le x\le 0 \\ \dfrac{1}{x} & \text{if } 0<x\le 2 \end{cases}$$

and its continuous approximation

$$f_n(x)=\begin{cases} x-400 & \text{if } -2\le x<-\dfrac{1}{n} \\ c_n(x) & \text{if } -\dfrac{1}{n}\le x\le \dfrac{1}{n} \\ \dfrac{1}{x} & \text{if } \dfrac{1}{n}<x\le 2 \end{cases}$$

being $c_n(x)$ *the function defined in (2). In Figure 7a) we represent* $f(x)$ *and* $f_{20}(x)$.

Example 5

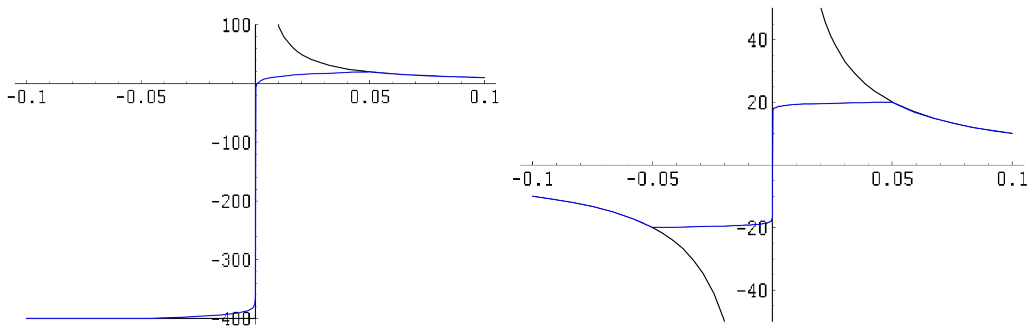
Given the function

$$f(x)=\begin{cases} \dfrac{1}{x} & \text{if } -2\le x<0 \\ \dfrac{1}{x} & \text{if } 0<x\le 2 \end{cases}$$

and its continuous approximation

$$f_n(x) = \begin{cases} \dfrac{1}{x} & if \ -2 \le x < -\dfrac{1}{n} \\ c_n(x) & if \ -\dfrac{1}{n} \le x \le \dfrac{1}{n} \\ \dfrac{1}{x} & if \ \dfrac{1}{n} < x \le 2 \end{cases}$$

being $c_n(x)$ the function defined in (2). In Figure 7b) we represent $f(x)$ and $f_{20}(x)$.



a) b)

Figure 7

Conclusions

In this paper we have check that the RAFU functions can be used to turn any discontinuous function into a continuous function.

In this work, we give the mathematical expression of a sequence of continuous function to do this. In fact, if

$$f(x) = \begin{cases} m_1(x) & if \ x_0 \le x \le x_1 \\ m_2(x) & if \ x_1 < x \le x_2 \\ \dots & \\ m_p(x) & if \ x_{p-1} < x \le x_p \end{cases} \qquad (3)$$

is an arbitrary discontinuous function (where $m_i(x)$ $1 \le i \le p$ are continuous functions), its corresponding sequence of continuous approximations defined by using RAFU functions is defined by the formula

$$f_n(x) = \begin{cases} m_1(x) & if\ x_0 \le x < x_1 - \dfrac{1}{n} \\ c_n(x) & if\ x_1 - \dfrac{1}{n} \le x \le x_1 + \dfrac{1}{n} \\ m_2(x) & if\ x_1 + \dfrac{1}{n} < x < x_2 - \dfrac{1}{n} \\ \dots & \\ c_n(x) & if\ x_{p-1} - \dfrac{1}{n} \le x \le x_{p-1} + \dfrac{1}{n} \\ m_p(x) & if\ x_p + \dfrac{1}{n} < x \le x_p \end{cases} \quad (4)$$

being $c_n(x)$ the function defined like in (2) but changing $x = x_1$ for the corresponding $x = x_j$, $j = 1,\dots, p-1$, in each case.

Finally, if we agree that to turn a discontinuous function (3) into a continuous function (4) the two conditions inside the neighbourhoods $\left(x_j - \frac{1}{n}, x_j + \frac{1}{n}\right)$, $j = 1,\dots, p-1$, have to be

1. The slopes of the tangent lines at the points $x \ne x_j \;\; j = 1,\dots, p-1$, should be near 0 and the approximation to 0 should improve as n increases.

2. At $x = x_j \;\; j = 1,\dots, p-1$, all functions should have vertical tangent.

then the RAFU functions solve the problem successfully.

References

[1] E. Corbacho. Uniform approximation with radical functions. SeEMA Journal 58 (1). 97-122. April 2012.

[2] E. Corbacho. A RAFU linear space uniformly dense in C[a, b]. Applied General Topology 14 (1). 53-60. 2013.

[3] E. Corbacho. The RAFU method on approximation. LAP Lambert Academic Publishing. 2015.

[4] E. Corbacho. Approximation in different smoothness spaces with the RAFU method. Applied General Topology 15 (2). 221--228. 2014

[5] E. Corbacho. Simultaneous approximation with the RAFU method. Journal of Mathematical Inequalities 10 (1). 219-231. 2016

[6] E Corbacho. Uniform reconstruction of continuous functions with the RAFU method. Applied General Topology 18 (2). 361-375. 2017.