On $E^c_\delta(\delta-\beta_c)$-Compact Spaces and $E^c_\delta(\delta-\beta_c)$-Separation Axioms

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Abstract

The purpose of the present paper is to consider new classes of generalized compact spaces called $E^c_\delta(\delta-\beta_c)$-compact spaces via $E^c_\delta(\delta-\beta_c)$-open sets respectively. Several characterizations and fundamental properties concerning of these forms of spaces are obtained. Moreover, some new types of separation axioms in topological spaces namely $E^c_\delta(\delta-\beta_c)$-separation axioms via $E^c_\delta(\delta-\beta_c)$-open sets are introduced and studied. Several of their fundamental properties and relationships with other well-known types of spaces are discussed.

Keywords: $E^c_\delta(\delta-\beta_c)$-Compact space, $E^c_\delta(\delta-\beta_c)$-open sets, $E^c_\delta$- separation axioms, $\delta$-$\beta_c$- separation axioms.
1. Introduction

The notion of compactness is very useful and fundamental notion of general topology also in the other advanced branches of mathematics. Many researchers have investigated the basic properties of compactness. In literature, different classes of generalized compactness such as [1, 2] are studied. Benchallì and Patil [3] introduced and studied a new class of closed sets called aw-closed sets and continuous maps in topological space. In [4] Hariwan Z. Ibrahim presented a new class of space named B, compact and gave some properties of B, compact space by using B, open sets. As well, P. G. Patil, in [5] introduced the concept of ωa-compactness in topological spaces and gave some characterization of ωa-compactness by using ωa-closed sets. On the other hand, recently, Sarika and Rayanagoudar [6] introduced a new concept called ag*s-compactness in topological spaces and obtained some of their properties by using ag*s-closed sets. In recent literature, we find many topologists had focused their research in the direction of investigating different types of separation axioms. Some of these have been found to be useful in computer science and digital topology [see for example [7, 8]. Dontcheve and Ganster [7] proved that the digital line is T 3/4 space but not T 1. Also, Navalagi [9] introduce semi generalized- T i spaces, i = 0, 1, 2. In addition, in 2011, Ahu Açıkgöz [10] defined two new separation axioms called β*T1/2 and β**T1/2 spaces as applications of β*g-closed sets. Hariwan Z. Ibrahim in [11] presented and investigated some weak separation axioms by using the notions of B, open sets and the B, closure operator. As well, in the same year Hussein A. Khaleefah[12] studied new types of separation axioms termed by, generalized b- R i, i = 0, 1 and generalized b-T i, i = 0, 1, 2 by using generalized b-open sets, Relations among these types are investigated, and several properties and characterizations are provided. A.I. EL-Maghrabi and M.A. AL-Juhani [13] introduced and investigated a new class of separation axioms called M-Ti-spaces, i = 0, 1, 2. Also, the M-regularity and the M-normality are examined in the context of these new concepts. This work is devoted to introduce a new classes of generalized compact spaces called E-(resp. δ-βi)-compact spaces. Several characterizations and fundamental properties concerning of these forms of spaces are obtained. As well as to introduce and study new notions of separation axioms called E-(resp. δ-βi)-separation axioms. Several fundamental properties and preservation properties concerning of these kinds of weak separation axioms are provided. Furthermore, the relationships among these types of separation axioms and other well-known types of spaces are discussed.
2. Preliminaries

Throughout this paper, \((X, T)\) and \((Y, T')\) (or simply \(X\) and \(Y\)) mean topological spaces on which no separation axioms are assumed unless explicitly stated. For any subset \(A\) of \(X\), the closure and interior of \(A\) are denoted by \(\text{Cl}(A)\) and \(\text{Int}(A)\), respectively.

We recall the following required results, which will be used often throughout this paper.

**Definition 2.1:** Let \((X, T)\) be a topological space. A subset \(A\) of \(X\) is said to be:

a) Regular open (resp. regular closed) \([14]\) if \(A = \text{Int}(\text{Cl}(A))\) (resp. \(A = \text{Cl}(\text{Int}(A))\)).

b) \(\delta\)-open \([15]\) if for each \(x \in A\) there exists a regular open set \(V\) such that \(x \in V \subseteq A\). The \(\delta\)-interior of \(A\) is the union of all regular open sets contained in \(A\) and is denoted by \(\text{Int}_\delta(A)\). The subset \(A\) is called \(\delta\)-open \([15]\) if \(A = \text{Int}_\delta(A)\). A point \(x \in X\) is called a \(\delta\)-cluster points of \(A\) \([15]\) if \(A \cap \text{Int}(\text{Cl}(V)) \neq \varnothing\), for each open set \(V\) containing \(x\). The set of all \(\delta\)-cluster points of \(A\) is called the \(\delta\)-closure of \(A\) and is denoted by \(\text{Cl}_\delta(A)\). If \(A = \text{Cl}_\delta(A)\), then \(A\) is said to be \(\delta\)-closed \([15]\). The complement of \(\delta\)-closed set is said to be \(\delta\)-open set. A subset \(A\) of a Topological space \(X\) is called \(\delta\)-open \([15]\) if for each \(x \in A\) there exists an open set \(G\) such that, \(x \in G \subseteq \text{Int}(\text{Cl}(G)) \subseteq A\). The family of all \(\delta\)-open sets in \(X\) is denoted by \(\delta\Sigma(X, T)\).

**Definition 2.2:** Let \((X, T)\) be a Topological space. Then:

a) A subset \(A\) of a space \(X\) is called \(E\)-open \([16]\) if \(A \subseteq \text{Cl}(\text{Int}(A)) \cup \text{Int}(\delta\text{-Cl}(A))\). The complement of an \(E\)-open set is called \(E\)-closed. The intersection of all \(E\)-closed sets containing \(A\) is called the \(E\)-closure of \(A\) \([16]\) and is denoted by \(\text{E-Cl}(A)\). The union of all \(E\)-open sets of \(X\) contained in \(A\) is called the \(E\)-interior \([16]\) of \(A\) and is denoted by \(\text{E-Int}(A)\).

b) A subset \(A\) of a space \(X\) is called \(\delta\beta\)-open \([17]\) or \(e^*\)-open \([18]\), if \(A \subseteq \text{Cl}(\text{Int}(\delta\beta\text{-Cl}(A)))\), the complement of a \(\delta\beta\)-open set is called \(\delta\beta\)-closed. The intersection of all \(\delta\beta\)-closed sets containing \(A\) is called the \(\delta\beta\)-closure of \(A\) \([17]\) and is denoted by \(\delta\beta\text{-Cl}(A)\). The union of all \(\delta\beta\)-open sets of \(X\) contained in \(A\) is called the \(\delta\beta\)-interior \([17]\) of \(A\) and is denoted by \(\delta\beta\text{-Int}(A)\).

**Remark 2.3:** The family of all \(E\)-open (resp. \(E\)-closed, \(\delta\beta\)-open, \(\delta\beta\)-closed) subsets of \(X\) containing a point \(x \in X\) is denoted by \(E\Sigma(X, x)\) (resp. \(EC(X, x), \delta\beta\Sigma(X, x), \delta\beta\text{C}(X, x)\)). The family of all \(E\)-open (resp. \(E\)-closed, \(\delta\beta\)-open, \(\delta\beta\)-closed) sets in \(X\) are denoted by \(E\Sigma(X, T)\) (resp. \(EC(X, T), \delta\beta\Sigma(X, T), \delta\beta\text{C}(X, T)\)).
**Definition 2.4:** Let \((X, T)\) be a Topological space. A subset \(A\) of \(X\) is said to be:

i) \(E_c\)-open set if for each \(x \in A \in E\Sigma(X, T)\), there exists a closed set \(F\) such that, \(x \in F \subseteq A\). The family of all \(E_c\)-open subsets of \((X, T)\) is denoted by \(E\Sigma(X, T)\) \(OR\) \(E\Sigma(X)\).

ii) \(\delta-\beta_c\)-open set if for each \(x \in A \in \delta-\beta\Sigma(X, T)\), there exists a closed set \(F\) such that, \(x \in F \subseteq A\). The family of all \(\delta-\beta_c\)-open subsets of \((X, T)\) is denoted by \(\delta-\beta\Sigma(X, T)\) \(OR\) \(\delta-\beta\Sigma(X)\).

A subset \(F\) of a space \((X, T)\) is said to be \(E_c\) (resp. \(\delta-\beta_c\))-closed set when, \(E\Sigma(X, T)\) (resp. \(?\delta-\beta\Sigma(X, T))\).

**Remark 2.5:** The family of all \(E_c\) (resp. \(\delta-\beta_c\))-closed subsets of \((X, T)\) is denoted by \(E\Sigma(X, T)\) \(OR\) \(E\Sigma(X)\) (resp. \(\delta-\beta\Sigma(X, T)\) \(OR\) \(\delta-\beta\Sigma(X)\)).

**Theorem 2.6:** Every regular closed sub-set in a space \((X, T)\) is \(\delta-\beta_c\)-open set. [19].

**Theorem 2.7:** Let \(X\) be a Topological space, if \(X\) is Regular space. Then every open set is a \(E_c\) (resp. \(\delta-\beta_c\))-open set. [19].

**Remark 2.8:** We have the following figure in which the converses of implications need not be true, see the examples in [20], [16] and [18].

\[\text{Figure (1): The relationships among some well-known generalized open sets in Topological Spaces}\]

### 3. Characterizations of \((E_c)\) and \((\delta-\beta_c)\)-Compact spaces

In this section, several characterizations concerning of new classes of spaces named \(E_c\) (resp. \(\delta-\beta_c\))-compact spaces are introduced and investigated utilizing new generalized open sets.

**Definition 3.1:** A topological space \(X\) is called:

a) \(E_c\) (resp. \(\delta-\beta_c\))-Compact if for every \(E_c\) (resp. \(\delta-\beta_c\))-open cover \(\{\mathcal{V}_\lambda: \mathcal{V} \in \mathcal{\mathcal{V}} \text{ of } X\}\) there exists \(A\) finite sub-set \(\mathcal{V} \subseteq \mathcal{\mathcal{V}}(s.t)\ X = \bigcup\{\mathcal{V}_\lambda: \mathcal{V} \in \mathcal{\mathcal{V}}\}.\)
b) Nearly compact [21] if for each open cover \( \mathcal{U} \) of \( X \), there exists a finite sub collection \( \mathcal{B} \subseteq \mathcal{U} \) \( (s,t) \cup \{ \text{Int} (\text{Cl} (V)) \mid V \in \mathcal{B} \} = X \).

c) Almost regular [22] If for each \( T \)- regularly closed subset \( A \) of \( X \) and \( \forall \) point \( \kappa \notin A \exists \) disjoint \( T_{\delta} \)-open sets \( \mathcal{U} \) and \( \mathcal{V} \) \( (s,t) \) \( A \subseteq \mathcal{U} \) and \( \kappa \in \mathcal{V} \).

d) \( E \)- (resp. \( \delta-\beta \))-Compact [23] if every \( E\) (resp. \( \delta-\beta \))-open cover of \( X \) has a finite sub-cover.

**Remark 3.2:** The family of all regular open sets constitutes a base for a Topology \( T_{\delta} \) on \( X \). This topology \( T_{\delta} \) is known as the semi-regularization of \( T \). We note that \( T_{\delta} \subseteq T \) [22]

**Definition 3.3:** A filter base \( \mathcal{F} \) in a space \( X \) is \( E_{\epsilon} \) (resp. \( \delta-\beta_{c} \))- Convergence to a point \( \kappa \in X \) if \( \forall E_{\epsilon} \) (resp. \( \delta-\beta_{c} \))-open set \( \mathcal{V} \) \( (s,t) \) \( \kappa \in \mathcal{V} \exists F \in \mathcal{F} \) \( (s,t) \) \( F \subseteq \mathcal{V} \).

**Definition 3.4:** A filter base \( \mathcal{F} \) in a space \( X \) is \( E_{\epsilon} \) (resp. \( \delta-\beta_{c} \))- Accumulation of a point \( \kappa \in X \) if \( F \cap \mathcal{V} \neq \emptyset \) \( \forall \mathcal{F} \) – open set \( \mathcal{V} \) \( (s,t) \) \( \kappa \in \mathcal{V} \) \( \& \forall F \in \mathcal{F} \).

**Theorem 3.5:** Let \( \mathcal{F} \) be filter base of a space \( X \). If \( \mathcal{F} \) is \( \delta-\beta_{c} \)- Convergence to a point \( \kappa \in X \), then \( \mathcal{F} \) is \( rc \) – Convergence to \( \kappa \).

**Proof:** Assume that \( \mathcal{F} \) \( \delta-\beta_{c} \)-Converges to \( \kappa \in X \). Let \( \mathcal{V} \) be any regular closed set \( (s,t) \), \( \kappa \in \mathcal{V} \Rightarrow \mathcal{V} \subseteq \delta-\beta \mathcal{C}(X) \). Since \( \mathcal{F} \) is \( \delta-\beta_{c} \)-Convergence to \( \kappa \in X \), thus \( \exists \) an \( F \in \mathcal{F} \) \( (s,t) \) \( F \subseteq \mathcal{V} \). This explains that \( \mathcal{F} \) is \( rc \) – Convergence to \( \kappa \).

**Remark 3.6:** The converse of Theorem-(3. 5) is not necessarily true, as shown in the following example:

**Example 3.7:** Consider \( (\mathbb{R}, T_{\delta}) \) the space of usual Topology and let
\[
\mathcal{F} = \{[0, \varepsilon], [0, 0 + \varepsilon] : \varepsilon > 0 \in \mathbb{R} \}. \hspace{1cm}
\]
So \( \mathcal{F} \) is \( rc \) – Convergence to \( 0 \), but \( \mathcal{F} \) doesn’t \( \delta-\beta_{c} \)–Convergence to \( 0 \), because the set of \( (0 - \varepsilon, 0 + \varepsilon) \) is \( \delta-\beta_{c} \)-open set containing \( 0 \), but \( \exists F \in \mathcal{F} \), such that, \( F \subseteq (0 - \varepsilon, 0 + \varepsilon) \).

**Corollary 3.8:** Let \( \mathcal{F} \) be filter base of a space \( X \). If \( \mathcal{F} \) \( \delta-\beta_{c} \)-Accumulation of a point a point \( \kappa \in X \), then \( \mathcal{F} \) is \( rc \) – Accumulation to \( \kappa \).

**Proof:** Similar to that of Theorem-(3.5).

**Theorem 3.9:** Let \( \mathcal{F} \) be filter base of a space \( X \) and \( \mathcal{H} \) be any closed set containing \( \kappa \). If \( \exists F \in \mathcal{F} \) \( (s,t) \) \( F \subseteq \mathcal{H} \). Then \( \mathcal{F} \) is \( E_{\epsilon} \) (resp. \( \delta-\beta_{c} \))- Convergence to a point \( \kappa \in X \).
**Proof:** suppose that \( \mathcal{V} \) be any \( E_c \) (resp. \( \delta-\beta_c \))-open set containing \( \varkappa \), so \( \forall \varkappa \in \mathcal{V} \exists \) a closed set \( \mathcal{H} \) (s. t) \( \varkappa \in \mathcal{H} \subseteq \mathcal{V} \). via hypothesis, \( \exists F \in \mathcal{F} \) (s.t) \( F \subseteq \mathcal{H} \subseteq \mathcal{V} \Rightarrow F \subseteq \mathcal{V} \). Thus, \( \mathcal{F} \) is \( E_c \) (resp. \( \delta-\beta_c \))- Converges to \( \varkappa \in X \).

**Theorem 3.10:** Let \( \mathcal{F} \) be filter base of a space \( X \) and \( \mathcal{H} \) be any closed set containing \( \varkappa \), \( \mathcal{F} \cap \mathcal{H} \neq \emptyset \) \( \forall F \in \mathcal{F} \). Then \( \mathcal{F} \) is \( E_c \) (resp. \( \delta-\beta_c \))- Accumulation of a point \( \varkappa \in X \).

**Proof:** Similar to that of Theorem-(3.9).

**Theorem 3.11:** Let \((X,T)\) be a Topological space. If every closed cover of \( X \) has a finite sub-cover, then \( X \) is \( E_c \) (resp. \( \delta-\beta_c \))- Compact space.

**Proof:** Let \( \{\mathcal{V}_\lambda: \lambda \in \mathcal{V}\} \) be any \( E_c \) (resp. \( \delta-\beta_c \))-open cover of \( X \), and \( \varkappa \in X \), so \( \forall \varkappa \in \mathcal{V}_\lambda(\varkappa): \lambda \in \mathcal{V} \), there exists a closed set \( \mathcal{H}_\lambda(\varkappa)(s. t) \varkappa \in \mathcal{H}_\lambda(\varkappa) \subseteq \mathcal{V}_\lambda(\varkappa) \). so the collection \( \{\mathcal{H}_\lambda(\varkappa): \varkappa \in X\} \) is a cover of \( X \) via closed set, then by hypothesis, this collection has a finite sub-cover (s. t):

\[
X = \{\mathcal{H}_\lambda(x_i): i = 1, 2, 3, ..., n\} \subseteq \{\mathcal{V}_\lambda(x_i): i = 1, 2, 3, ..., n\} \quad \text{Thus} \quad X = \{\mathcal{V}_\lambda(x_i): i = 1, 2, 3, ..., n\} \text{therefore } X \text{ is } E_c \text{ (resp. } \delta-\beta_c \text{)-Compact space.}
\]

**Theorem 3.12:** If a Topological space \((X,T)\) is \( E \) (resp. \( \delta-\beta \))-Compact, then its \( E_c \) (resp. \( \delta-\beta_c \))-Compact space.

**Proof:** Let \( \{\mathcal{V}_\lambda: \lambda \in \mathcal{V}\} \) be any \( E_c \) (resp. \( \delta-\beta_c \))-open cover of \( X \). So \( \{\mathcal{V}_\lambda: \lambda \in \mathcal{V}\} \) is \( E \) (resp. \( \delta-\beta \))-open cover of \( X \), since \( X \) is \( E \) (resp. \( \delta-\beta \))-Compact, so there exists a finite sub-set \( \mathcal{V}_o \subseteq \mathcal{V} \) \( \{s.t \} \mathcal{X} = \mathcal{U} \{\mathcal{V}_\lambda: \lambda \in \mathcal{V}_o\}\). Thus \( X \) is \( E_c \) (resp. \( \delta-\beta_c \))-Compact space.

**Theorem 3.13:** Every \( E_c \) (resp. \( \delta-\beta_c \))-Compact \( T_1 - \) space is \( E \) (resp. \( \delta-\beta \))-Compact space.

**Proof:** Assume that \( X \) is \( E_c \) (resp. \( \delta-\beta_c \))-Compact and \( T_1 - \) space. Let \( \{\mathcal{V}_\lambda: \lambda \in \mathcal{V}\} \) be any \( E \) (resp. \( \delta-\beta \))-open cover of \( X \). so \( \forall \varkappa \in X \exists \lambda(\varkappa) \in \mathcal{V}(s.t) \varkappa \in \mathcal{V}_\lambda(\varkappa) \). Since \( X \) is \( T_1 - \) space, and \( X \) is \( E_c \) (resp. \( \delta-\beta_c \))-Compact, thus there exists a finite sub-set \( \mathcal{V}_o \subseteq \mathcal{V} \) \( \{s.t \} X = \mathcal{U}\{\mathcal{V}_\lambda: \lambda \in \mathcal{V}_o\}\). Thus \( X \) is \( E \) (resp. \( \delta-\beta \))-Compact space.

The following corollary is directly consequence of Theorem-(3.12 and 3.13).

**Corollary 3.14:** Let \( X \) be a \( T_1 - \) space. Then \( X \) is \( E_c \) (resp. \( \delta-\beta_c \))-Compact if \( fX \) is \( E \) (resp. \( \delta-\beta \))-Compact space.

**Theorem 3.15:** Let \((X,T)\) be a \( \delta-\beta_c \)-Compact space, then \( X \) is \( r \) - Compact space.
**Proof:** Suppose that $\{V_\lambda: \lambda \in \mathcal{V}\}$ is any regular closed cover of $X$, so via Theorem-(2.6), $\{V_\lambda: \lambda \in \mathcal{V}\}$ constitutes a $\delta$-$\beta_c$-open cover of $X$. Since $X$ is $\delta$-$\beta_c$-Compact, hence there exists a finite sub-set $\mathcal{V}_a \subseteq \mathcal{V}$ in $X(s,t) X = \bigcup\{V_\lambda: \lambda \in \mathcal{V}_a\}$. Thus $X$ is $rc$ - Compact space.

**Theorem 3.16:** Let $(X,T)$ be a regular space. If $X$ is $E_c$ (resp. $\delta$-$\beta_c$)-Compact, then $X$ is compact.

**Proof:** Assume that $\{V_\lambda: \lambda \in \mathcal{V}\}$ is any open cover of $X$. Since $X$ is regular, so via Theorem-(2.7), $\{V_\lambda: \lambda \in \mathcal{V}\}$ constitutes a $E_c$ (resp. $\delta$-$\beta_c$)-open cover of $X$ and since $X$ is $E_c$ (resp. $\delta$-$\beta_c$)-Compact, thus there exists a finite sub-set $\mathcal{V}_a \subseteq \mathcal{V}$ in $X(s,t) X = \bigcup\{V_\lambda: \lambda \in \mathcal{V}_a\}$. Thus $X$ is Compact space.

**Theorem 3.17:** Let $(X,T)$ be an almost regular space. If $X$ is $\delta$-$\beta_c$-Compact, then $X$ is nearly compact.

**Proof:** Suppose that $\{V_\lambda: \lambda \in \mathcal{V}\}$ is any regular open cover of $X$. Since $X$ is almost regular, so $\forall \, x \in X$ and regular open $V_\lambda(x) \exists$ an open set $M_x (s,t) x \in M_x \subseteq \text{Cl}(M_x) \subset V_\lambda(x)$, but $\text{Cl}(M_x)$ is regular closed $\forall \, x \in X$, this implies that the collection $\{\text{Cl}(M_x): x \in X\}$ via Theorem (2.6), constitutes a $\delta$-$\beta_c$-open cover of $X$. Since $X$ is $\delta$-$\beta_c$-Compact, so there exists a sub-collection $\{\text{Cl}(M(x_i)) : i = 1, 2, \ldots, n\}, (s,t), X = \bigcup_{i=1}^{n} \text{Cl}(M(x_i)) \subseteq \bigcup_{i=1}^{n} V_\lambda(x_i)$. Thus $X$ is nearly compact.

4. **Fundamental properties of $E_c$ (resp. $\delta$-$\beta_c$)-$T_i$ - Spaces (i = 0, 1, 2)**

In this part, several characterizations and some basic properties concerning of new kinds of separation axioms called $E_c$ (resp. $\delta$-$\beta_c$)- separation axioms such as $E_{cT_0}$ (resp. $\delta$-$\beta_cT_0$), $E_{cT_1}$ (resp. $\delta$-$\beta_cT_1$) $E_{cT_2}$ (resp. $\delta$-$\beta_cT_2$) are given, as well the relationships among these kinds of spaces and other well- known spaces are discussed.

**Definition 4.1:** A mapping $f: (X,T) \to (Y,T^*)$ is said to be:

i) $E_c$-Irresolute, if $f^{-1}(V)$ is $E_c$-open in $X$ for every $E_c$-open sub-set $V$ of $Y$.

ii) $\delta$-$\beta_c$-Irresolute, if $f^{-1}(V)$ is $\delta$-$\beta_c$-open in $X$ for every $\delta$-$\beta_c$-open sub-set $V$ of $Y$.

iii) $E_c$-open, if the image of each open set of $(X, T)$ is $E_c$-open of $(Y, T^*)$.

iv) $\delta$-$\beta_c$-open, if the image of each open set of $(X, T)$ is $\delta$-$\beta_c$-open of $(Y, T^*)$.

v) $E_c$-continuous, if $f^{-1}(V)$ is $E_c$-open in $X$ for every open sub-set $V$ of $Y$. 


vi) $\delta\beta_c$-continuous, if $f^{-1}(\mathcal{V})$ is $\delta\beta_c$-open in $X$ for every open sub-set $\mathcal{V}$ of $Y$.

**Definition 4.2:** A topological space $(X, \mathcal{T})$ is called:

a) $E_c$ (resp. $\delta - \beta_c$) - $T_0$ - Space if for every distinct points $x$ and $y$ of $X$, there is $E_c$ (resp. $\delta\beta_c$)-open set containing one of them but not the other.

b) $E_c$ (resp. $\delta - \beta_c$) - $T_1$ - Space if for each pair of distinct points $x, y \ (x \neq y) \in X$, there exist two $E_c$ (resp. $\delta\beta_c$)-open sets $\mathcal{U} \& \mathcal{V}$ (s.t) $x \in \mathcal{U} \ but \ y \notin \mathcal{U} \ and \ y \in \mathcal{V} \ but \ x \notin \mathcal{V}$.

c) $E_c$ (resp. $\delta - \beta_c$) - $T_2$ - Space or $E_c$ (resp. $\delta - \beta_c$) - Hausdorff Space if for each pair of distinct points $x, y \ (x \neq y) \in X$, there exist two disjoint $E_c$ (resp. $\delta\beta_c$)-open sets $\mathcal{U} \& \mathcal{V}$ (s.t) $x \in \mathcal{U}$ and $y \in \mathcal{V}$.

**Remark 4.3:** From the respective definitions, we have the following diagram. However none of these implications is reversible as shown via examples of [19, 24, 25].

![Diagram illustrating the relationships among other well-known types of generalized open sets in Top-Sp](image)

**Theorem 4.4:** The following properties are hold in a topological space $(X, \mathcal{T})$:

a) Every $E_c$ (resp. $\delta - \beta_c$) - $T_2$ - Space $\Rightarrow E_c$ (resp. $\delta - \beta_c$) - $T_1$ - Space

b) Every $E_c$ (resp. $\delta - \beta_c$) - $T_1$ - Space $\Rightarrow E_c$ (resp. $\delta - \beta_c$) - $T_0$ - Space

c) Every $E_c T_2$ - Space is $\delta - \beta_c T_2$ - Space.

d) Every $E_c T_1$ - Space is $\delta - \beta_c T_1$ - Space.

e) Every $E_c T_0$ - Space is $\delta - \beta_c T_0$ - Space.

**Proof:** The proof is obvious it is follows immediately from their respective definitions.
**Remark 4.5:** From the respective definitions, the relationships among $E_c$ (resp. $\delta$-$\beta_c$)-$T_i$-spaces ($i = 0, 1, 2$) and some other well-known forms of spaces shown in the following figure:

![Relationships among $E_c$ (resp. $\delta$-$\beta_c$)-$T_i$-spaces](image)

Figure (3): Relationships among $E_c$ (resp. $\delta$-$\beta_c$)-$T_i$-spaces ($i = 0, 1, 2$) and some other well-known forms of spaces.

However none of these implications is reversible as shown in the following examples.

**Example 4.6:** Let $X = \{a, b, c, d\}$ with a topology $T = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}\}$, $X$. Then, $E \Sigma(X, T) = \{\emptyset, \{a, c\}, \{b, c, d\}, \{a, c, d\}, X\}$.

And, $\delta$-$\beta C \Sigma(X, T) = \{\emptyset, \{a, c\}, \{c, d\}, \{b, c, d\}, \{a, c, d\}, X\}$. Then $X$ is $E_c T_0(\text{resp. } \delta - \beta_c T_3) - Space$, but it is neither $E_c(\text{resp. } \delta - \beta_c) - T_1 - Space$ nor $E_c(\text{resp. } \delta - \beta_c) - T_2 - Space$.

**Example 4.7:** Consider $X$ any infinite set with the co-finite topology $T_c$ (such that the closed sets are $X$ and the finite sub-sets). Since $X \setminus \{x\}$ is $E_c(\text{resp. } \delta - \beta_c)$-open, therefore $X$ is, $E_c(\text{resp. } \delta - \beta_c) - T_1 - Space$. But there is no non empty $E_c(\text{resp. } \delta - \beta_c)$-open sets are disjoint, so $X$ cannot be $E_c(\text{resp. } \delta - \beta_c) - T_2 - Space$.

**Theorem 4.8:** the following statements are equivalent for a space $(X, T)$:

a) $X$ is an $E_c(\text{resp. } \delta - \beta_c) - T_0 - Space$

b) For every two distinct points $x, y$ ($x \neq y$) $\in X$,

$E_c Cl(\{x\}) (\text{resp. } \delta - \beta_c Cl(\{x\})) \neq E_c Cl(\{y\}) (\text{resp. } \delta - \beta_c Cl(\{y\}))$.

**Proof:** **Necessity.** Suppose that $(X, T)$ is $E_c(\text{resp. } \delta - \beta_c) - T_0 - Space$ and for each $x, y$ ($x \neq y$) $\in X$, there exists an $E_c(\text{resp. } \delta - \beta_c)$-open set $\mathcal{U}$ (s.t) $x \in \mathcal{U}$, $y \notin \mathcal{U} \Rightarrow y \notin X \setminus \mathcal{U}$, where $X \setminus \mathcal{U}$ is $E_c(\text{resp. } \delta - \beta_c)$-closed which does not
contain $x$ but contains $y$. Since $E_c Cl(\{y\})$ (resp. $\delta-\beta_c Cl(\{y\})$) is the smallest $E_c$(resp. $\delta-\beta_c$)-closed set containing $y$, so $E_c Cl(\{y\})$ (resp. $\delta-\beta_c Cl(\{y\})$) $\subseteq X \setminus U$ and hence $x \notin E_c Cl(\{y\})$ (resp. $\delta-\beta_c Cl(\{y\})$). Consequently, $E_c Cl(\{x\})$ (resp. $\delta-\beta_c Cl(\{x\})$) $\neq E_c Cl(\{y\})$ (resp. $\delta-\beta_c Cl(\{y\})$).

**Sufficiency.** Assume that $x, y$ ($x \neq y$) $\in X$, and $E_c Cl(\{x\})$ (resp. $\delta-\beta_c Cl(\{x\})$) $\neq E_c Cl(\{y\})$ (resp. $\delta-\beta_c Cl(\{y\})$). Let $z \in X$ (s.t.) $z \in E_c Cl(\{x\})$ (resp. $\delta-\beta_c Cl(\{x\})$) but $z \notin E_c Cl(\{y\})$ (resp. $\delta-\beta_c Cl(\{y\})$). We prove that $x \notin E_c Cl(\{y\})$ (resp. $\delta-\beta_c Cl(\{y\})$). Suppose that $x \in E_c Cl(\{y\})$ (resp. $\delta-\beta_c Cl(\{y\})$), so $\{x\} \subseteq E_c Cl(\{y\})$ (resp. $\delta-\beta_c Cl(\{y\})$), which implies that, $E_c Cl(\{x\})$ (resp. $\delta-\beta_c Cl(\{x\})$) $\subseteq E_c Cl(\{y\})$ (resp. $\delta-\beta_c Cl(\{y\})$) and thus $x \in E_c Cl(\{y\})$ (resp. $\delta-\beta_c Cl(\{y\})$) which is a contradiction with the fact of $z \notin E_c Cl(\{y\})$ (resp. $\delta-\beta_c Cl(\{y\})$), hence $x \notin E_c Cl(\{y\})$ (resp. $\delta-\beta_c Cl(\{y\})$) which implies that, $x \notin X \setminus E_c Cl(\{y\})$ (resp. $\delta-\beta_c Cl(\{y\})$).

Consequently $X \setminus E_c Cl(\{y\})$ (resp. $\delta-\beta_c Cl(\{y\})$) is an $E_c$(resp. $\delta-\beta_c$)-open set containing $x$ but not $y$. Therefore, $X$ is $E_c$(resp. $\delta-\beta_c$) $- T_0 - Space$.

**Theorem 4.9:** Let $(X, T)$ be a topological space. Then the following statements are equivalent:

a) $X$ is an $E_c$(resp. $\delta-\beta_c$) $- T_1 - Space$.

b) For each point $x \in X$ the singleton set $\{x\}$ is $E_c$(resp. $\delta-\beta_c$)-closed set,

c) For each point $x \in X, E_c D(\{x\})$ (resp. $\delta-\beta_c D(\{x\})$) $= \emptyset$.

**Proof:** (a) $\Rightarrow$ (b) Suppose that $X$ is $E_c$(resp. $\delta-\beta_c$) $- T_1 - Space$. For each $x, y$ ($x \neq y$) $\in X$, there exists $E_c$(resp. $\delta-\beta_c$)-open set $U$ (s.t.) $y \in U$ but $x \notin U$. Consequently, $y \in U \subseteq X \setminus \{x\}$. Thus $X \setminus \{x\} = \bigcup \{U: y \in X \setminus \{x\}\}$ which is the union of an $E_c$(resp. $\delta-\beta_c$)-open sets. Then $X \setminus \{x\}$ is an $E_c$(resp. $\delta-\beta_c$)-open sets. Thus $\{x\} E_c$(resp. $\delta-\beta_c$)-closed sets.

(b) $\Rightarrow$ (a) Assume that $\{P\}$ is $E_c$(resp. $\delta-\beta_c$)-closed for each $P \in X$. So via hypothesis for each $x, y$ ($x \neq y$) $\in X$, $\{x\}, \{y\}$ are $E_c$(resp. $\delta-\beta_c$)-closed sets. Hence $X \setminus \{x\}, X \setminus \{y\}$ are $E_c$(resp. $\delta-\beta_c$)-open sets (s. t.) $x \in X \setminus \{y\}, y \notin X \setminus \{y\}$ and $y \in X \setminus \{x\}, x \notin X \setminus \{x\}$.

Therefore $X$ is $E_c$(resp. $\delta-\beta_c$) $- T_1 - Space$.

(b) $\Rightarrow$ (c) Assume that $\{x\}$ is $E_c$(resp. $\delta-\beta_c$)-closed set for each $x \in X$. Thus,

$\{x\} = E_c Cl(\{x\})$ (resp. $\delta-\beta_c Cl(\{x\})$) $= \{x\} \cup E_c D(\{x\})$ (resp. $\delta-\beta_c D(\{x\})$). Therefore,

$E_c D(\{x\})$ (resp. $\delta-\beta_c D(\{x\})$) $= \emptyset$. 

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\((c) \Rightarrow (b)\) Let \(E_c D(\{x\}) \ (\text{resp. } \delta \beta_c D(\{x\})) = \varphi\) for each \(x \in X\). Since
\[E_c Cl(\{x\}) \ (\text{resp. } \delta \beta_c Cl(\{x\})) = \{x\} \cup E_c D(\{x\}) \ (\text{resp. } \delta \beta_c D(\{x\}))\]. Thus,
\[E_c Cl(\{x\}) \ (\text{resp. } \delta \beta_c Cl(\{x\})) = \{x\} \iff \{x\} \ E_c(\text{resp. } \delta \beta_c)\text{-closed set.}\]

\((a) \Rightarrow (c)\) Suppose that \(X\) is \(E_c(\text{resp. } \delta = \beta_c) - T_1 - \text{Space}\) and assume that \(E_c D(\{x\}) \ (\text{resp. } \delta \beta_c D(\{x\})) \neq \varphi\) for some \(x \in X\), then \(\exists y \in E_c D(\{x\}) \ (\text{resp. } \delta \beta_c D(\{x\}))\) and \((x \neq y)\). Since \(X\) is \(E_c(\text{resp. } \delta = \beta_c) - T_1 - \text{Space}\), so \(\exists E_c(\text{resp. } \delta = \beta_c)\)-open set \(U (s.t. ) y \in U \text{ and } x \notin U\) which implies, \(U \cap \{x\} = \varphi\), and thus \(y \notin E_c D(\{x\}) \ (\text{resp. } \delta \beta_c D(\{x\}))\) a contradiction with the hypothesis. Hence, \(\forall x \in X\), \(E_c D(\{x\}) \ (\text{resp. } \delta \beta_c D(\{x\})) = \varphi\).

\((c) \Rightarrow (a)\) Let \(E_c D(\{x\}) \ (\text{resp. } \delta \beta_c D(\{x\})) = \varphi\), \(\forall x \in X\), so
\[E_c Cl(\{x\}) \ (\text{resp. } \delta \beta_c Cl(\{x\})) = \{x\} \cup E_c D(\{x\}) \ (\text{resp. } \delta \beta_c D(\{x\})) = \{x\} \text{ which implies, } \{x\} \text{ is } E_c(\text{resp. } \delta = \beta_c)\text{-closed set and thus via (part (a) \& (b)) } X \text{ is } E_c(\text{resp. } \delta = \beta_c) - T_1 - \text{Space}.\]

**Theorem 4.10:** If \((X, T)\) is a topological space, then the following properties are equivalent:

a) \(X\) is an \(E_c(\text{resp. } \delta = \beta_c) - T_2 - \text{Space}\)

b) If \(x \in X\), then \(\forall (x \neq y), \exists\) an \(E_c(\text{resp. } \delta = \beta_c)\)-open set \(U\) containing \(x\) \((s.t.)\) \(y \notin E_c Cl(\{U\})\) \((\text{resp. } \delta \beta_c Cl(\{U\}))\).

**Proof:** \((a) \Rightarrow (b)\) since \(X\) is \(E_c(\text{resp. } \delta = \beta_c) - T_2 - \text{Space}\) \(\forall (x \neq y) \exists E_c(\text{resp. } \delta = \beta_c)\)-open sets.
\(U \& V (s.t.) x \in U \& y \in V\text{ and } U \cap V = \varphi\). Thus, \(x \in U \subseteq X \setminus V\), put \(X \setminus V = F\), then \(F\) is \(E_c(\text{resp. } \delta \beta_c)\)-closed set, \(U \subseteq F\) and \(y \notin F\) \(\Rightarrow y \in \{F : F\text{ is } E_c(\text{resp. } \delta \beta_c)\text{-closed set } \& U \subseteq F\} = E_c Cl(\{U\})\) \((\text{resp. } \delta \beta_c Cl(\{U\}))\).

\((b) \Rightarrow (a)\) Assume that \(x, y (x \neq y) \in X\), by hypothesis, there exists \(E_c(\text{resp. } \delta \beta_c)\)-open set \(U\) containing \(x\) \((s.t.)\) \(y \in E_c Cl(\{U\})\) \((\text{resp. } \delta \beta_c Cl(\{U\}))\). Hence \(y \in X \setminus (E_c Cl(\{U\})\) \((\text{resp. } \delta \beta_c Cl(\{U\}))\)) which is \(E_c(\text{resp. } \delta \beta_c)\)-open and \(x \notin X \setminus (E_c Cl(\{U\})\) \((\text{resp. } \delta \beta_c Cl(\{U\}))\)). As well, \(U \cap (X \setminus (E_c Cl(\{U\})\) \((\text{resp. } \delta \beta_c Cl(\{U\})\)) = \varphi\). So, \(X\) is \(E_c(\text{resp. } \delta = \beta_c) - T_2 - \text{Space}.\)

**Definition 4.11:** Let \((X, T)\) be a topological space and \(\mathcal{A} \subseteq X\). Then, the intersection of all \(E_c(\text{resp. } \delta \beta_c)\)-open subsets of \(X\) containing \(\mathcal{A}\) is called the \(E_c\)-kernal\((\text{resp. } \delta \beta_c\text{-kernal})\) of \(\mathcal{A}\) and its denoted via \(E_c\ker(\mathcal{A})(\text{resp. } \delta \beta_c\ker(\mathcal{A}))\) of \(\mathcal{A}\) \((i.e.)\):
\(E_c\text{-ker}(\mathcal{A})(\text{resp. } \delta\beta_c\text{-ker}(\mathcal{A})) = \bigcap \{U \in EC\Sigma(X) \text{ (resp. } \delta\beta C\Sigma(X)): \mathcal{A} \subseteq U\}.

**Theorem 4.12:** Let \((X,T)\) be a topological space and \(x \in X\), then, \(y \in E_c\text{-ker}([x])\) (resp. \(\delta\beta_c\text{-ker}([x])\)) \iff \(x \in E_c\text{-Cl}([y])\) (resp. \(\delta\beta_c\text{-Cl}([y])\)).

**Proof:** Assume that \(y \notin E_c\text{-ker}([x])\) (resp. \(\delta\beta_c\text{-ker}([x])\)). So, there exists \(E_c\text{-open set } U\ containing \ x \ (s.t) \ y \notin U\). Thus we get \(x \notin E_c\text{-Cl}([y])\) (resp. \(\delta\beta_c\text{-Cl}([y])\)). Similarly we can prove the converse case.

**Theorem 4.13:** Let \(\mathcal{A}\) be a sub set of a topological space \((X,T)\). Then,

\(E_c\text{-ker}(\mathcal{A})(\text{resp. } \delta\beta_c\text{-ker}(\mathcal{A})) = \{x \in X: E_c\text{-Cl}([x])\text{ (resp. } \delta\beta_c\text{-Cl}([x])\} \cap \mathcal{A} \neq \varnothing\).

**Proof:** Suppose that \(x \in E_c\text{-ker}(\mathcal{A})(\text{resp. } \delta\beta_c\text{-ker}(\mathcal{A}))\) and

\(E_c\text{-Cl}([x])\text{ (resp. } \delta\beta_c\text{-Cl}([x])\} \cap \mathcal{A} = \varnothing\). Thus, \(x \notin X \setminus (E_c\text{-Cl}([x])\text{ (resp. } \delta\beta_c\text{-Cl}([x])\})\) which is an \(E_c\text{-open set containing } \mathcal{A}\). This case is not possible, since \(x \in E_c\text{-ker}(\mathcal{A})(\text{resp. } \delta\beta_c\text{-ker}(\mathcal{A}))\). Therefore \(E_c\text{-Cl}([x])\text{ (resp. } \delta\beta_c\text{-Cl}([x])\} \cap \mathcal{A} \neq \varnothing\). Now suppose that \(x \in X \ (s.t) \ E_c\text{-Cl}([x])\text{ (resp. } \delta\beta_c\text{-Cl}([x])\} \cap \mathcal{A} = \varnothing\) and \(x \notin E_c\text{-ker}(\mathcal{A})(\text{resp. } \delta\beta_c\text{-ker}(\mathcal{A}))\). So, there exists an \(E_c\text{-open set } U\ containing \ \mathcal{A} and x \notin U\). Let \(y \in E_c\text{-Cl}([x])\text{ (resp. } \delta\beta_c\text{-Cl}([x])\} \cap \mathcal{A} \). Hence, \(U\) is an \(E_c\text{-Neighbourhood of } y\) which does not contain \(x\). So via this contradiction we get \(x \in E_c\text{-ker}(\mathcal{A})(\text{resp. } \delta\beta_c\text{-ker}(\mathcal{A}))\) and this is the request.

**Theorem 4.14:** The following properties hold for the subsets \(\mathcal{A}\) and \(\mathcal{B}\) of a topological space \((X,T)\):

a) \(\mathcal{A} \subseteq E_c\text{-ker}(\mathcal{A})(\text{resp. } \delta\beta_c\text{-ker}(\mathcal{A})).\)

b) \(\mathcal{A} \subseteq \mathcal{B} \implies E_c\text{-ker}(\mathcal{A})(\text{resp. } \delta\beta_c\text{-ker}(\mathcal{A})) \subseteq E_c\text{-ker}(\mathcal{B})(\text{resp. } \delta\beta_c\text{-ker}(\mathcal{B})).\)

c) If \(\mathcal{A}\) is \(E_c\text{-open of } (X,T)\), then \(\mathcal{A} = E_c\text{-ker}(\mathcal{A})(\text{resp. } \delta\beta_c\text{-ker}(\mathcal{A})).\)

\(E_c\text{-ker}(E_c \text{-ker}(\mathcal{A}))\) (resp. \(\delta\beta_c\text{-ker}(\delta\beta_c \text{-ker}(\mathcal{A}))) = E_c\text{-ker}(\mathcal{A})(\text{resp. } \delta\beta_c\text{-ker}(\mathcal{A})).\)

**Proof:** The proof of parts (a), (b) and (c) are directly consequences of definition (4.11). Now we prove part (d), first via parts (a) and (b) we have:

\(E_c\text{-ker}(\mathcal{A})\) (resp. \(\delta\beta_c\text{-ker}(\mathcal{A})) \subseteq \ E_c\text{-ker}(E_c \text{-ker}(\mathcal{A}))\) (resp. \(\delta\beta_c\text{-ker}(\delta\beta_c \text{-ker}(\mathcal{A})))\). If \(x \notin E_c\text{-ker}(\mathcal{A})(\text{resp. } \delta\beta_c\text{-ker}(\mathcal{A})).\) So \(\exists U \in EC\Sigma(X)\) (resp. \(\delta\beta C\Sigma(X))\) (s. t)

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$\mathcal{A} \subseteq \mathcal{U}$ and $x \notin \mathcal{U}$. Thus, $E_c\ker(\mathcal{A})$ (resp. $\delta\beta_c\ker(\mathcal{A})$) $\subseteq \mathcal{U}$, and so we get:

$x \notin E_c\ker(E_c - \ker(\mathcal{A}))$ (resp. $\delta\beta_c\ker(\delta - \beta_c - \ker(\mathcal{A}))$). Therefore,

$E_c\ker(E_c - \ker(\mathcal{A}))$ (resp. $\delta\beta_c\ker(\delta - \beta_c - \ker(\mathcal{A}))$) $= E_c\ker(\mathcal{A})$ (resp. $\delta\beta_c\ker(\mathcal{A})$).

**Theorem 4.15**: for any two distinct points $x$ and $y$ in a topological space $(X,T)$, the following properties hold:

a) $E_c\ker([x])$ (resp. $\delta\beta_c\ker([x])$) $\neq E_c\ker([y])$ (resp. $\delta\beta_c\ker([y])$).

b) $E_c\text{Cl}([x])$ (resp. $\delta\beta_c\text{Cl}([x])$) $\neq E_c\text{Cl}([y])$ (resp. $\delta\beta_c\text{Cl}([y])$).

**Proof**: (a) $\Rightarrow$ (b) Assume that $E_c\ker([x])$ (resp. $\delta\beta_c\ker([x])$) $\neq E_c\ker([y])$ (resp. $\delta\beta_c\ker([y])$). So there exists a point $z \in X$ (s.t) $z \in E_c\ker([x])$ (resp. $\delta\beta_c\ker([x])$) and $z \notin E_c\ker([y])$ (resp. $\delta\beta_c\ker([y])$). Since $z \in E_c\ker([x])$ (resp. $\delta\beta_c\ker([x])$) and $z \notin E_c\ker([y])$ (resp. $\delta\beta_c\ker([y])$), so $E_c\text{Cl}([z])$ (resp. $\delta\beta_c\text{Cl}([z])$) and $\{y\} \subseteq E_c\text{Cl}([z])$ (resp. $\delta\beta_c\text{Cl}([z])$) $= \varnothing$. Thus, it follows that $E_c\text{Cl}([x])$ (resp. $\delta\beta_c\text{Cl}([x])$) $\neq E_c\text{Cl}([y])$ (resp. $\delta\beta_c\text{Cl}([y])$). So, $E_c\ker([x])$ (resp. $\delta\beta_c\ker([x])$) implies that $E_c\text{Cl}([x])$ (resp. $\delta\beta_c\text{Cl}([x])$) $\neq E_c\text{Cl}([y])$ (resp. $\delta\beta_c\text{Cl}([y])$).

(b) $\Rightarrow$ (a) Suppose that $E_c\text{Cl}([x])$ (resp. $\delta\beta_c\text{Cl}([x])$) $\neq E_c\text{Cl}([y])$ (resp. $\delta\beta_c\text{Cl}([y])$). So there exists a point $z \in X$ (s.t) $z \in E_c\text{Cl}([x])$ (resp. $\delta\beta_c\text{Cl}([x])$) and $z \notin E_c\text{Cl}([y])$ (resp. $\delta\beta_c\text{Cl}([y])$). Then, there exists an $E_c$ (resp. $\delta\beta_c$)-open set containing $z$ and $x$ but not $y$, namely, $y \in E_c\ker([x])$ (resp. $\delta\beta_c\ker([x])$) and hence $E_c\ker([x])$ (resp. $\delta\beta_c\ker([x])$) $\neq E_c\ker([y])$ (resp. $\delta\beta_c\ker([y])$).

**Theorem 4.16**: Suppose that $f: (X,T) \to (Y,T^*)$ is an injective $E_c$ (resp. $\delta\beta_c$)-continuous mapping and $Y$ is $T_i$-space, then $X$ is $E_c$ (resp. $\delta\beta_c$)-$T_i$-space, where $(i = 0, 1, 2)$

**Proof**: We prove that the theorem for $E_c$ (resp. $\delta - \beta_c$)-$T_0$-Space, and the other are similar.

Assume that $x, y$ ($x \neq y$) $\in X$, since $f$ is injective, then $f(x) \neq f(y)$ in $Y$. But $Y$ is $T_0$, then there exist an open set $U$ (s.t) $f(x) \in U, f(y) \in U$ OR $f(y) \in U, f(x) \in U$, since $f$ is $E_c$ (resp. $\delta\beta_c$)-continuous, so $f^{-1}(U)$ is $E_c$ (resp. $\delta\beta_c$)-open set of $X$ (s.t) $x \in f^{-1}(U), y \notin f^{-1}(U)$ OR
Thus $x \in f^{-1}(U), y \notin f^{-1}(U)$. Thus $X$ is $E_c(\text{resp. } \delta - \beta_c) - T_0 - Space$.

**Theorem 4.17**: Let $f: (X, T) \to (Y, T')$ be an injective $E_c(\text{resp. } \delta - \beta_c)$-irresolute mapping and $Y$ is an $E_c(\text{resp. } \delta - \beta_c) - T_i - space$, then $X$ is $E_c(\text{resp. } \delta - \beta_c) - T_i - space$, where $(i = 0, 1, 2)$

**Proof**: We prove that the theorem for $E_c(\text{resp. } \delta - \beta_c) - T_2 - Space$ and the other are similar.

Suppose that $x, y (x \neq y) \in X$, since $f$ is injective, then $f(x) \neq f(y)$ in $Y$. But $Y$ is an $E_c(\text{resp. } \delta - \beta_c) - T_2 - Space$ so there exist two disjoint $E_c(\text{resp. } \delta - \beta_c)$-open sets $U$ and $V$ (s.t) $f(x) \in U \& f(y) \in V$. Utilizing $E_c(\text{resp. } \delta - \beta_c)$-irresolute of $f$ we get, $f^{-1}(U) \& f^{-1}(V)$ are $E_c(\text{resp. } \delta - \beta_c)$-open set of $X$ (s.t):

$x \in f^{-1}(U), y \notin f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$.

Hence $X$ is $E_c(\text{resp. } \delta - \beta_c) - T_2 - Space$.

**Theorem 4.18**: Suppose that $f: (X, T) \to (Y, T')$ is a bijective $E_c(\text{resp. } \delta - \beta_c)$-open mapping and $X$ is $T_i - space$, then $Y$ is $E_c(\text{resp. } \delta - \beta_c) - T_i - space$, where $(i = 0, 1, 2)$

**Proof**: We prove that the theorem for $E_c(\text{resp. } \delta - \beta_c) - T_2 - Space$ and the other are similar.

Let $y_1, y_2 (y_1 \neq y_2) \in Y$. since $f$ is bijective, so there exist $x_1, x_2 (x_1 \neq x_2) \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since $X$ is $T_2$, then there exist two disjoint open sets $U$ and $V$ of $X$ (s.t) $x_1 \in U \& x_2 \in V$. Since $f$ is $E_c(\text{resp. } \delta - \beta_c)$-open mapping, then $f(U) \& f(V)$ are $E_c(\text{resp. } \delta - \beta_c)$-open sets of $Y$ with $y_1 \in f(U) \& y_2 \in f(V)$. therefore $Y$ is $E_c(\text{resp. } \delta - \beta_c) - T_2 - Space$.

**Acknowledgements**

I would like to express my sincere gratitude to the referees for their valuable suggestions and comments which improved the paper and I am thankful to professor, Dr. E. EKICI (Turkey) for sending many of his papers as soon as I had requested and Dr. Yasir Al-Ani (University of Anbar-Iraq) for help.

**Conclusion**

Compactness is the generalization to topological spaces of the property of closed and bounded subsets of the real line. The notions of compactness are useful and fundamental notions not only of general topology but also of other advanced branches of mathematics. Many researchers have investigated the basic properties of compactness”. The productivity of
these notions of compactness motivated mathematicians to generalize these notions. In the course of these attempts many stronger and weaker forms of compactness have been introduced and investigated. The class of generalized closed sets has an important role in general topology, especially its suggestion of new separation axioms which are useful in digital topology. "The investigation on generalization of closed set has lead to significant contribution to the theory of separation axioms. In this work we introduced and study new types of spaces called $E_c$(resp. $δ$-$β_c$)-compact spaces. Several characterizations and fundamental properties concerning of these forms of spaces are obtained. Moreover, some new types of separation axioms in topological spaces called $E_c$(resp. $δ$-$β_c$)- separation axioms via $E_c$(resp. $δ$-$β_c$)-open sets are studied.

References


