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Existence, multiplicity, and non-existence of positive solutions for singular mixed boundary value problems

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In this work, we study the existence, multiplicity, and non-existence of positive solutions for singular mixed boundary value problems. Our argument is based on the fixed point index theory together with the generalized Picone identity.

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1 Introduction

In this work, we study the existence, multiplicity, and non-existence of positive solutions for singular mixed boundary value problems

$$(P_\mu) \qquad \begin{cases} (\varphi_p(u'(t)))' + f(t,u(t)) = 0, \ t \in (0,1), \\ u'(0) = 0, u(1) = \mu, \end{cases}$$

where $\varphi_p(s) = |s|^{p-2}s, p > 1, \mu$ is a nonnegative real parameter, $f \in C((0,1) \times (0,\infty),(0,\infty))$. Here, $f(t,u)$ may be singular at $t = 0$ and/or 1.

The problem under consideration arises in the study of radially symmetric solutions to quasi-linear partial differential equations defined on radially symmetric domains such as unit ball, annular domain, exterior domain and whole space $\Re^n$. This kind of problem has been studied by many authors with various methods and techniques. We refer for instance to the papers [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] and the references therein.

For convenience, we give a list of hypotheses which are used in this paper.

(F_1) for all $M > 0$, there exists $h_M \in \mathcal{A}$ such that $f(t,u) \le h_M(t)$, for all $t \in (0,1)$ and $u \in [0,M]$, where

$$\mathcal{A} = \left\{ h \in L^1_{loc}[0,1) \mid \int_0^1 \varphi_p^{-1}\left(\int_0^s h(\tau)d\tau\right)ds < \infty \right\},$$

(F_2) there exist $r > 0$ and $h_r \in \mathcal{A}$ such that, for all $t \in (0,1)$ and $u \in [0,r]$,

$$f(t,u) \le r^{p-1}h_r(t)$$

and h_r satisfies

$$\int_0^1 \varphi_p^{-1}\left[\int_0^s h_r(\tau)d\tau\right]ds < 1,$$

(F_3) there exists an interval $[\alpha,\beta] \subset (0,1)$ such that

$$\lim_{u\to\infty} \frac{f(t,u)}{u^{p-1}} = \infty \quad uniformly\ on\ [\alpha,\beta],$$

(F_4) $f = f(t,u)$ is a nondecreasing function with respect to u, i.e.

$$f(t,u) \le f(t,v) \ \ for\ all\ \ t \in (0,1)\ \ and\ \ 0 \le u \le v.$$

By a positive solution to problem (P_μ), we understand a function $u \in C[0,1] \cap C^1[0,1)$ with $\varphi_p(u') \in W^{1,1}_{loc}[0,1)$ which satisfies (P_μ) and $u > 0$ in $[0,1)$.

The following is the main result in this work.

Theorem 1.1 *Assume* (F_1),(F_2),(F_3) *and* (F_4). *Then there exists* $\mu^* > 0$ *such that* (P_μ) *has at least two positive solutions for* $\mu \in (0,\mu^*)$, *at least one positive solution for* $\mu \in \{0,\mu^*\}$ *and no positive solution for* $\mu > \mu^*$.

The rest of this work is organized as follows. In Section 2, we give preliminaries which are essential for proving our results in this paper. In Section 3, we give the proof of Theorem 1.1 and supply examples to illustrate our results.

2 Preliminaries

Let $C[0,1]$ denote the real Banach space of continuous functions u with norm

$$\| u \| = \max_{t\in[0,1]} |u(t)|.$$

Put

$$\mathcal{K} = \{u \in C[0,1] \cap C^1[0,1) | u \; is \;\; a \;\; non-negative \;\; concave \;\; function\, , u'(0) = 0\}.$$

Then, $\mathcal{K}$ is a cone in $C[0,1]$.

We usually use integration by parts twice for semi-linear problem to obtain the useful information for solutions. However, it does not work for p-Laplacian problem. Thus, we use the following generalized Picone identity to overcome this difficulty.

Theorem 2.1 ([14], $Generalized\ Picone\ identity$) *Let us define*

$$l_p[y] = (\varphi_p(y'))' + b_1(t)\varphi_p(y),$$

$$L_p[z] = (\varphi_p(z'))' + b_2(t)\varphi_p(z).$$

If y and z are any functions such that $y, z, b_1\varphi_p(y'),\ \ b_2\varphi_p(z')$ are differentiable on I and $z(t) \neq 0$ for $t \in I$, the generalized Picone identity can be written as

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{|y|^p\varphi_p(z')}{\varphi_p(z)} - y\varphi_p(y')\right\} \\
&= (b_1 - b_2)|y|^p \\
&\quad - \left[|y'|^p + (p-1)\left|\frac{yz'}{z}\right|^p - p\varphi_p(y)y'\varphi_p\left(\frac{z'}{z}\right)\right] \qquad (2.1) \\
&\quad - yl_p[y] + \frac{|y|^p}{\varphi_p(z)}L_p[z].
\end{aligned}$$

Remark 2.2 *By Young's inequality, we get*

$$|y'|^p + (p-1)\left|\frac{yz'}{z}\right|^p - p\varphi_p(y)y'\varphi_p\left(\frac{z'}{z}\right) \geq 0,$$

and the equality holds if and only if $y' = yz'/z$ in (a,b).

Finally we give a fixed point index theorem.

Theorem 2.3 ([15]) *Let X be a Banach space, $\mathcal{K}$ a cone in X and $\mathcal{O}$ bounded open in X. Let $0 \in \mathcal{O}$ and $A:\overline{\mathcal{O}} \cap \mathcal{K} \to \mathcal{K}$ be completely continuous. Suppose that $Ax \neq \nu x$ for all $x \in \partial\mathcal{O} \cap \mathcal{K}$ and all $\nu \geq 1$. Then $i(A,\mathcal{O} \cap \mathcal{K},\mathcal{K}) = 1$.*

3 Main result

Throughout this section, we assume (F_1). Let us define $H:[0,\infty) \times \mathcal{K} \to \mathcal{K}$ by

$$H(\mu,u)(t) = \mu + \int_t^1 \varphi_p^{-1}\left[\int_0^s f(\tau,u(\tau))d\tau\right]ds, \ 0 \leq t \leq 1.$$

Then H is well defined and (P_μ) has a positive solution u if and only if $H(\mu,\cdot)$ has a fixed point u in $\mathcal{K}\backslash\{0\}$. Furthermore, it is easily verified that H is completely continuous by Ascoli-Arzelà theorem and Lebesgue dominated convergence theorem.

Lemma 3.1 *Assume (F_2). Then there exists $\mu_0 > 0$ such that (P_{μ_0}) has a positive solution.*

Proof. By (F_2), there exists $r > 0$ such that $f(t,u) \leq r^{p-1}h_r(t)$ for $t \in (0,1), u \in [0,r]$, where $h_r \in \mathcal{K}$ is the function satisfying

$$\int_0^1 \varphi_p^{-1}\left[\int_0^s h_r(\tau)d\tau\right]ds < 1.$$

Put $\mu_0 = \frac{r}{2}\left(1 - \int_0^1 \varphi_p^{-1}\left[\int_0^s h_r(\tau)d\tau\right]ds\right) > 0$. Then for $u \in \partial B_r \cap \mathcal{K}$,

$$\begin{aligned}
\| H(\mu_0,u) \| &= H(\mu_0,u)(0) = \mu_0 + \int_0^1 \varphi_p^{-1}\left(\int_0^s f(\tau,u(\tau))d\tau\right)ds \\
&\leq \mu_0 + \int_0^1 \varphi_p^{-1}\left(\int_0^s r^{p-1}h_r(\tau)d\tau\right)ds \\
&= \mu_0 + r\int_0^1 \varphi_p^{-1}\left(\int_0^s h_r(\tau)d\tau\right)ds \\
&< r = \| u \|,
\end{aligned}$$

which implies $i(H(\mu_0,\cdot),B_r \cap \mathcal{K},\mathcal{K}) = 1$ in view of Theorem 2.3. Thus the proof is complete.

□

Using the generalized Picone identity (Theorem 2.1) and the properties of the p-sine function ([16], [17]), we obtain the following lemma.

Lemma 3.2 *Assume* (F_3). *Then there exists* $M > 0$ *such that if* u *is a positive solution of* (P_μ), *then* $\| u \| \leq M$.

Proof. Assume on the contrary that there exists a sequence $\{u_n\}$ of positive solutions of (P_{μ_n}) such that $||u_n|| \to \infty$ as $n \to \infty$. It follows from the concavity of u_n that

$$u_n(t) \geq (1-\beta)||u_n||, t \in [\alpha,\beta], \tag{3.1}$$

where $[\alpha,\beta]$ is the interval in the condition (F_3). Put

$$C = 2\left(\pi_q/(\beta-\alpha)\right)^p > 0.$$

By (F_3), there exists $K > 0$ such that, for $u > K$ and $t \in [\alpha,\beta]$,

$$f(t,u) > C\varphi_p(u).$$

Since $\| u_n \| \to \infty$ as $n \to \infty$, we get $||u_N|| > (1-\beta)^{-1}K$ for sufficiently large N. Therefore, by (3.1), we have

$$f(t,u_N(t)) > C\varphi_p(u_N(t)),\ t \in (\alpha,\beta)$$

and hence

$$(\varphi_p(u_{N'}(t)))' + C\varphi_p(u_N(t)) < 0,\ t \in (\alpha,\beta). \tag{3.2}$$

It is easy to check that $w(t) = S_q(\pi_q(t-\alpha)/(\beta-\alpha))$, where S_q is the q-sine function and $\frac{1}{p} + \frac{1}{q} = 1$, is a solution of

$$(\varphi_p(w'(t)))' + \left(\frac{\pi_q}{\beta-\alpha}\right)^p \varphi_p(w(t)) = 0,\ t \in (\alpha,\beta). \tag{3.3}$$

Taking $y = w,\ z = u_N,\ b_1 = \left(\pi_q/(\beta-\alpha)\right)^p$ and $b_2 = C$ in Theorem 2.1 and integrating (??) from α to β, by Remark 2.2 we have

$$\int_\alpha^\beta \left(\left(\frac{\pi_q}{\beta-\alpha}\right)^p - C\right)|w|^p dt \geq 0$$

since $w \in C_0^1[\alpha,\beta]$ and $u_N \in C^1[\alpha,\beta]$. Thus

$$C \leq \left(\frac{\pi_q}{\beta-\alpha}\right)^p.$$

This contradicts the choice of C, and thus the proof is complete. □

The following lemma directly follows from Lemma 3.2.

Lemma 3.3 *Assume* (F_3). *Then there exists* $\bar{\mu} > 0$ *such that if* (P_μ) *has a positive solution, then* $\mu \leq \bar{\mu}$.

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemma 3.1, there exists $\mu_0 > 0$ such that (P_{μ_0}) has a positive solution u_0. We claim that (P_μ) has at least two positive solutions for $\mu \in (0,\mu_0)$ and at least one positive solution for $\mu = 0$. For fixed $\mu \in [0,\mu_0)$, let us consider the following modified problem;

$$(M_\mu) \quad \begin{cases} (\varphi_p(u'(t)))' + f(t,\gamma(t,u(t))) = 0, \ t \in (0,1), \\ u'(0) = 0, u(1) = \mu, \end{cases}$$

where $\gamma:(0,1) \times \Re \rightarrow \Re$ is defined by

$$\gamma(t,u) = \begin{cases} u_0(t) & if \ \ u > u_0(t), \\ u & if \ \ 0 \leq u \leq u_0(t), \\ 0 & if \ \ u < 0. \end{cases}$$

Then, solutions of (M_μ) are concave and nonnegative. Define $T_\mu:\mathcal{K} \rightarrow \mathcal{K}$ by

$$(T_\mu u)(t) = \mu + \int_t^1 \varphi_p^{-1}\left[\int_0^s f(\tau,\gamma(\tau,u(\tau)))d\tau\right]ds, 0 \leq t \leq 1.$$

It is easy to check that T_μ is completely continuous on $\mathcal{K}$, and u is a solution of (M_μ) if and only if $u = T_\mu u$. It follows from the definition of γ that there exists $R_1 > 0$ such that $||T_\mu u|| < R_1$ for all $u \in \mathcal{K}$. Applying Theorem 2.3 with $\mathcal{O} = B_{R_1}$, we get

$$i(T_\mu,B_{R_1} \cap \mathcal{K},\mathcal{K}) = 1. \tag{3.4}$$

Here B_{R_1} is the open ball with radius R_1 and center 0 in $C[0,1]$. Put

$$\Omega = \{u \in C[0,1] | -1 < u(t) < u_0(t), t \in [0,1]\}.$$

Then Ω is bounded and open in $C[0,1]$. Let u be a solution of (M_μ). By (F_4), for $t \in (0,1)$

$$-(\varphi_p(u'(t)))' = f(t,\gamma(t,u(t))) \le f(t,u_0(t)) = -(\varphi_p(u_{0'}(t)))'.$$

For $t \in (0,1)$, integrating this from 0 to t, we have $u'(t) \ge u_{0'}(t)$. Again integrating it from t to 1, we have $u(t) - u_0(t) \le u(1) - u_0(1) = \mu - \mu_0 < 0$, and thus $u(t) < u_0(t), t \in [0,1]$. This implies $u \in \Omega \cap \mathcal{K}$. By (3.4) and excision property, we get

$$i(T_\mu,\Omega \cap \mathcal{K},\mathcal{K}) = i(T_\mu,B_{R_1} \cap \mathcal{K},\mathcal{K}) = 1. \quad (3.5)$$

Since (P_μ) is equivalent to (M_μ) on $\Omega \cap \mathcal{K}$, we conclude (P_μ) has a nonnegative solution in $\Omega \cap \mathcal{K}$. We may assume $H(\mu,\cdot)$ has no fixed point in $\partial\Omega \cap \mathcal{K}$, otherwise the claim is proved. Then $i(H(\mu,\cdot),\Omega \cap \mathcal{K},\mathcal{K})$ is well defined and by (3.5), we have

$$i(H(\mu,\cdot),\Omega \cap \mathcal{K},\mathcal{K}) = 1. \quad (3.6)$$

It follows from Lemma 3.3 that (P_{μ_1}) has no solution in $\mathcal{K}$ for $\mu_1 > \overline{\mu}$, and

$$i(H(\mu_1,\cdot),O \cap \mathcal{K},\mathcal{K}) = 0 \quad (3.7)$$

for all bounded open set O. By Lemma 3.2, there exists $R_2\,(> R_1)$ such that for all possible solutions u of (P_λ) with $\lambda \in [\mu,\mu_1]$, we have

$$\| u \| < R_2.$$

Define h:$[0,1] \times (\overline{B}_{R_2} \cap \mathcal{K}) \to \mathcal{K}$ by

$$h(\tau,u) = H(\tau\mu + (1-\tau)\mu_1,u).$$

Then h is completely continuous on $[0,1] \times \mathcal{K}$ and $h(\tau,u) \ne u$ for all $(\tau,u) \in [0,1] \times (\partial B_{R_2} \cap \mathcal{K})$. By (3.7) and homotopy invariance property, we obtain

$$i(H(\mu,\cdot),B_{R_2} \cap \mathcal{K},\mathcal{K}) = i(H(\mu_1,\cdot),B_{R_2} \cap \mathcal{K},\mathcal{K}) = 0.$$

Thus by additivity property and (3.6), we get

$$i(H(\mu,\cdot),(B_{R_2}\text{₩}\overline{\Omega}) \cap \mathcal{K},\mathcal{K}) = -1.$$

This implies (P_μ) has another solution in $(B_{R_2}\text{₩}\overline{\Omega}) \cap \mathcal{K}$. Thus the claim is proved.

Put $\mu^* = \sup\{\mu \mid (P_\mu)$ has at least one positive solution$\}$. Then by Lemma 3.3, $\mu^* \in (0,\infty)$, and (P_μ) has at least two positive solutions for $\mu \in (0,\mu^*)$ and at least one positive solution for $\mu = 0$. Furthermore, by Lemma 3.2 and the complete continuity of H, we can show that (P_{μ^*}) has a positive solution, and thus the proof is complete. □

The following corollary directly follows from Theorem 1.1.

Corollary 3.4 *Assume* (F_1),(F_2),(F_3) *and* (F_4). *Assume, in addition,*

$$(F_5)\ f(t_0,0) \neq 0\ \ for\ \ some\ \ t_0 \in [0,1].$$

Then there exists $\mu^* > 0$ such that (P_μ) has at least two positive solutions for $\mu \in [0,\mu^*)$, at least one positive solution for $\mu = \mu^*$ and no positive solution for $\mu > \mu^*$.

Finally we give two examples to illustrate our results.

Example 3.5

(1) Set $f_1(t,u) = (1-t)^{-k}u^q$, where $k < p$ and $q > p-1$. Note that $(1-t)^{-l} \in \mathcal{A}$ if $l < p$, but $(1-t)^{-p} \notin \mathcal{A}$. Then it is easily verified that f_1 satisfies the assumptions of Theorem 1.1.

(2) Set $f_2(t,u) = C(1-t)^{-k}\exp(t+u)$, where $k < p$. Then, for sufficiently small $C > 0$, f_2 satisfies the assumptions of Corollary 3.4.

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