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The uniform boundedness theorem in b-Banach space

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Abstract

B-Banach space is an extension of Banach space, which provides a suitable framework for studying many analytical problems. The uniform boundedness theorem is is the basic theorem in functional analysis and has many important applications in many field, such as matrix analysis, operator theory, and numerical analysis. In this note, we revisit the concept of b-Banach space, and then establish the uniform boundedness theorem for linear operators. The result may be useful to establish linear operator theory in b-Banach space.

Keywords: B-normed linear space, B-Banach space, Uniform boundedness theorem

Introduction

As an extension of metric space, the concept of b-metric space was given by Bakhtin[1], Czerwik[2] and was announced earlier as quasi-metric by Berinde [3-6].

In the framework of b-metric, we can deal with analytical problems in the spaces $L^p[a, b](0 < p < 1)$ or $l^p(0 < p < 1)$, which are important in theory and applications. For example, Xu[7] proposed the $l^p(0 < p < 1)$ regularization optimization algorithm in signal processing, proved that the optimal condition for the solution was $p = 0.5$, and gave the analytical form of the optimal solution. While in [8], questions of approximation of periodic functions in L^p-spaces for $0 < p < 1$ by certain families of linear polynomial operators were studied.

Since $L^p[a, b](0 < p < 1)$ and $l^p(0 < p < 1)$ not only have topological structure, but also have good linear structure, We have reason to conduct more in-depth research on them.

Recently in [9-10], Monica etc. introduced the concept of **b-Banach space**, which is an extension of Banach space, and a special case of b-metric space. We recognize that the most typical examples of this kind of spaces are $L^p[a, b](0 < p < 1)$ and $l^p(0 < p < 1)$.

As a new type of **normed** linear space, it seems there are many topics can be discussed. For example, in [9-10], a Krasnoselskii type fixed point theorem and a fixed point theorem under a finite number of equality constraints involving a well-known Ciric type mappings were given in the context of b-Banach space, respectively.

While in this paper, we aim to revisit the concept of b-Banach space, then establish the uniform boundedness theorem for linear operators in b-Banach space.

In [11], the author proved Banach-Steinhaus theorem for some families of bounded linear operators from a normed space into a generalized 2-normed space, and in [12], the author gave a elementary proof of the uniform boundedness theorem (i.e., did not use any version of the Baire category theorem). The above results have given us a lot of inspiration. Our uniformly bounded theorem can be seen as another generalization of Banach-Steinhaus's theorem.

This paper is organized as follows. In Section 2, we recall the definition of b-normed linear space and some relevant properties. In Section 3, we establish the uniform boundedness theorem in b-normed linear space.

Preliminaries

Definition 2.1 Let X be a vector space over a field K (either C or R) and let $s \geq 1$ be a given real number. A functional $\|\cdot\|: X \to [0,+\infty)$ is said to be a b-norm if the following conditions are satisfied:

(1) $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$;

(2) $\|\lambda x\| = |\lambda| \|x\|$;

(3) $\|x+y\| \le s(\|x\|+\|y\|)$.

for all $x,y,z \in X$, $\lambda \in K$. A pair $(X, \|\cdot\|)$ is called a b-normed linear space.

Example 2.2 Let $L^p[a, b](0 < p < 1)$ be the set of all real-valued Lebesgue measurable function x on $[a, b]$ for which $\int_{[a,b]} |x(t)|^p dt < \infty$. For each $x \in L^p[a, b]$, define

$$\|x\| = \left[\int_a^b |x(t)|^p dt\right]^{\frac{1}{p}}.$$

Then $(L^p[a,b], \|\cdot\|)$ $(0 < p < 1)$ is a b-normed linear space with $s = 2^{\frac{1}{p}-1}$, see in [13].

Example 2.3[13] Let $l^p := \left\{x = \{x_n\}_{n=1}^{\infty} \mid \sum_{n=1}^{\infty} |x_n|^p < \infty, \quad where\ x_n \in R, n \in N\right\}$, $(0 < p < 1)$. Define

$$\|x\| = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}.$$

for each $x \in l^p$. Then l^p is a b-normed linear space with $s = 2^{\frac{1}{p}-1}$.

Let $(X, \|\cdot\|)$ be a b-normed linear space. For $\forall x, y \in X$, define $d(x,y) = \|x-y\|$, then d is a b-metric on x. We call d a b-metric induced by $\|\cdot\|$.

Definition 2.4 Let $(X, \|\cdot\|)$ be a b-normed linear space. Then a sequence $\{x_n\}_{n=1}^{\infty}$ in X is called

(a) Convergent if and only if there exists $x \in X$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. In this case we write $\lim_{n\to\infty} x_n = x$.

(b) Cauchy if and only if $\|x_m - x_n\| \to 0$ as $m, n \to \infty$.

Definition 2.5 The b-normed linear space $(X, \|\cdot\|)$ is complete if and only if every Cauchy sequence in X is convergent in X. A complete b-normed linear space is called a b-Banach space.

It is verified that $\left(L^p[a,b],\|\cdot\|\right)(0<p<1)$ and $\left(l^p[a,b],\|\cdot\|\right)(0<p<1)$ are all b-Banach spaces.

The uniform boundedness theorem in b-Banach space

Theorem 3.1 Let $\{T_\alpha \mid \alpha\in I\}$ be a family of bounded linear operators from Banach space X to normed linear space Y. If $\{T_\alpha \mid \alpha\in I\}$ is pointwise bounded (i.e. for $\forall x\in X$, $\sup\limits_{\alpha}\|T_\alpha x\|<\infty$), then $\{T_\alpha \mid \alpha\in I\}$ is norm-bounded, i.e.($\sup\limits_{\alpha}\|T_\alpha x\|<\infty$).

In order to generalize the classical the uniform boundedness theorem from Banach space to b-Banach space, we need the following definitions and lemmas.

Definition 3.2 Let X and Y be b-normed linear spaces, $T:X\to Y$ is a linear operator, we called T as bounded linear operator if there exist $M>0$ such that for $\forall x\in X$, $\|Tx\|\le M\|x\|$.

Lemma 3.3 Let X and Y be b-normed linear spaces, $T:X\to Y$ is a bounded linear operator.

Define

$$\|T\|=\inf\left\{M:\|Tx\|\le M\|x\|,\forall x\in X\right\},$$

$$\|T\|_1=\sup_{\|x\|=1}\|Tx\|,$$

$$\|T\|_2=\sup_{\|x\|\le 1}\|Tx\|.$$

Then $\|T\|=\|T\|_1=\|T\|_2$.

Proof. On one hand, by the definition of $\|T\|_1$ and $\|T\|_2$, we have

$$\|T\|_1\le\|T\|_2=\sup_{\|x\|\le 1}\|Tx\|=\sup_{\|x\|\le 1}\|T\|=\|T\|.$$

On the other hand, due to the definition of $\|T\|_1$, we get

$$\|Tx\|=\|x\|\cdot\left\|T(\frac{x}{\|x\|})\right\|\le\|x\|\cdot\|T\|_1,\forall x\neq 0,$$

which implies that

$$\|T\| \le \|T\|_1 .$$

Hence we have $\|T\| = \|T\|_1 = \|T\|_2$.

Lemma 3.4 Let X and Y be b-normed linear spaces. Let $B(X,Y)$ be all bounded linear operators from X to Y. Then for $\forall T \in B(X,Y)$ and $r > 0$,

$$\sup_{\|x\|\le r} \|Tx\| = r\|T\| .$$

Proof. Note that $\|T\| = \sup_{\|y\|\le 1}\|Ty\|$, let $y = xr^{-1}$,then

$$\sup_{\|x\|\le r}\|Tx\| = \sup_{\|xr^{-1}\|\le r}\|rT(xr^{-1})\| = \sup_{\|xr^{-1}\|\le r}\|T(xr^{-1})\|r = \sup_{\|y\|\le 1}\|Ty\|r = \|T\|r .$$

Remark If X is a b-normed linear space with $s = s_1$, Y is a b-normed linear space with $s = s_2$, then X and Y are all b-Banach space with $s = s_0$, where $s_0 = \max\{s_1, s_2\}$. Hence, we always assume that two b-normed linear space X, Y have the same constant s.

Lemma 3.5 Let X and Y be b-normed linear spaces, $T \in B(X,Y)$. Then for $\forall x \in X$ and $r > 0$,

$$\sup_{\|x'-x\|\le r}\|Tx'\| \ge \frac{r}{s}\|T\| .$$

Proof. By the inequality (3) in Definition 2.1, we have

$$\begin{aligned}\max\{\|T(x+y)\|, \|T(x-y)\|\} &\ge \frac{1}{2}(\|T(x+y)\| + \|T(x-y)\|) \\ &\ge \frac{1}{2s}\|T(x+y) - T(x-y)\| \\ &\ge \frac{1}{2s}\|Tx + Ty - Tx + Ty\| = \frac{1}{s}\|Ty\|\end{aligned}$$

Let $x' = x - y$. Take the upper bound for $y \in B(0,r)$, then using Lemma 3.4 we obtain

$$\sup_{\|x'-x\|\le r}\|Tx'\| \ge \frac{1}{s}\sup_{\|y\|\le r}\|Ty\| = \frac{r}{s}\|T\| .$$

The main result is given as follows,

Theorem 3.6 Let F be a family of bounded linear operators from b-Banach space X to b-normed linear space Y. If F is pointwise bounded, then F is uniformly bounded. In other words, if $\sup_{T\in F}\|Tx\| < \infty$ for $\forall x \in X$, then $\sup_{T\in F}\|T\| < \infty$.

Proof. Conversely, we assume that $\sup_{T\in F}\|T\|=\infty$.

Firstly, let $\varepsilon_n=\dfrac{1}{(2s)^n\cdot n}$ for $n=1,2,\cdots,$ then $\varepsilon_n\to 0$ as $n\to\infty$. Now, we can choose a sequence of bounded linear operators $\{T_n\}_{n=1}^{\infty}\subseteq F$ satisfying $\|T_n\|\geq \varepsilon_n^{-2}\to\infty$.

Moreover, we can find $x_n\in X$ such that $\|x_n-x_{n-1}\|\leq\varepsilon_n$, $\|T_nx_n\|>\dfrac{\varepsilon_n}{2s}\|T_n\|$.

In fact, by Lemma 3.5, we have

$$\sup_{\|x-x_{n-1}\|\leq\varepsilon_n}\|T_nx\|\geq\frac{\varepsilon_n}{s}\|T_n\|>\frac{\varepsilon_n}{2s}\|T\| .$$

Secondly, let's show that $\{x_n\}$ is a cauchy sequence. In fact, for $m>n$,

$$\begin{aligned}\|x_m-x_n\|&\leq s\|x_{n+1}-x_n\|+s^2\|x_{n+2}-x_{n+1}\|+\ldots+s^m\|x_m-x_{m-1}\|\\&=s\varepsilon_{n+1}+s^2\varepsilon_{n+2}+\ldots+s^m\varepsilon_m\\&=\frac{1}{2^{n+1}\cdot s^n\cdot(n+1)}+\frac{1}{2^{n+2}\cdot s^n\cdot(n+2)}+\ldots+\frac{1}{2^m\cdot s^n\cdot(m)}\\&\leq 2\cdot\frac{1}{2^{n+1}\cdot s^n\cdot(n+1)}=\frac{1}{(2s)^n\cdot(n+1)}\\&\leq\frac{\varepsilon_n}{(2s)^2}(n>2)\to 0\end{aligned}$$

Hence, $\{x_n\}$ is a cauchy sequence. By the completeness of X, there is $x\in X$ such that $1\lim_{n\to\infty}x_n=x$.

Let $m\to\infty$, we have

$$\|x-x_n\|<\frac{\varepsilon_n}{(2s)^2} .$$

Finally, we get

$$
\begin{aligned}
\|T_n x\| &= \|T_n x_n - T_n(x_n - x)\| \\
&\geq \frac{1}{s}\cdot\|T_n x_n\| - \|T_n(x_n - x)\| \\
&> \frac{1}{s}\cdot\frac{\varepsilon_n}{2s}\|T_n\| - \|x_n - x\|\,\|T_n\| \\
&> \left(\frac{1}{s}\cdot\frac{\varepsilon_n}{2s} - \frac{\varepsilon_n}{(2s)^2}\right)\|T_n\| \\
&= \frac{\varepsilon_n}{(2s)^2}\|T_n\| \\
&> \frac{1}{(2s)^2}\cdot\frac{1}{\varepsilon_n} \to \infty.
\end{aligned}
\tag{1}
$$

which is contradiction.

This completes the proof.

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