



SCIREA Journal of Physics

<http://www.scirea.org/journal/Physics>

November 1, 2016

Volume 1, Issue 1, October 2016

Applications of the Translation Invariant Shell Model and the Variational Monte Carlo Method to the Ground and First Excited-States of ^3H and ^4He Nuclei

S. B. Doma¹ and H. S. El-Gendy²

¹Faculty of Science, Alexandria University, Alexandria, Egypt, sbdoma@alexu.edu.eg

²Faculty of Science & Art, Shaqra University, Shaqra, KSA, hselgendy1@yahoo.com

Abstract

The ground-state nuclear wave functions, the binding energies, the root mean-square radii and the first excited-state energies of the ^3H and ^4He nuclei are investigated by applying the translation invariant shell model and using residual two-body interaction, given by the first author, together with a three-body interaction in the form of a delta force. Furthermore, the binding energies and the root mean-square radii of these nuclei have been calculated by applying the variational Monte Carlo method and using the Reid V_8 two-body potential together with the Urbana model of the three-nucleon interaction.

Keywords: Light nuclei, translation invariant shell model, binding energy, root mean-square radius, nucleon-nucleon interaction, three-body interaction, Monte Carlo variational method.

1. Introduction

The nuclear shell model [1] has achieved wide popularity among nuclear theorists owing to its easily visualizable character and the success in interpreting experimental results. The assumption that, to a first approximation, each nucleon moves in an average potential independent of the motion of the other nucleons is an attractive one, due to one's familiarity with the Hartree fock theory of atomic structure.

The methods of expanding the nuclear wave function in terms of a complete set of orthonormal functions, basis functions, have been used on a large scale especially for the nuclei with $3 \leq A \leq 6$ [2].

The translation invariant shell model [2-5] (TISM) has shown good results for the structure of light nuclei with $A \leq 7$ by using nucleon-nucleon interactions [5-10]. This model considers the nucleus as a system of noninteracting quasi particles and enables us to apply the algebraic methods for studying the general features of matrix elements of operators that correspond to physical quantities. The TISM is based on the group theoretical methods of classifying the basis functions. The basis functions of this model are constructed in such a way that they will have certain symmetry with respect to the interchange of particles and have definite total angular momentum J and isotopic spin T . The basis functions of the TISM are then expanded in the form of products of two types of functions, one corresponding to the set of $A-2$ nucleons and the other corresponding to the last pair of nucleons by means of the two-particle fractional parentage coefficients [2,6]. Using two-body interactions it is then possible to calculate the Hamiltonian matrices of the different nuclear states. In principle, the predicted results for the nuclear characteristics should be independent of the particular chosen basis functions when the number of terms in the expansion is kept large enough. The inclusion of all such bases in the expansion is too difficult since the matrices of the two-particle fractional parentage coefficients corresponding to these basis functions grow rapidly. It is therefore fundamental to have some rules that would allow us to reduce the number of these bases. Some of these rules are adopted for the nucleus ${}^6\text{Li}$ in ref. [6].

On the other hand, microscopic calculations of light nuclei and nuclear matter [11] have indicated that it is difficult to explain the observed binding-energies and densities if we assume a non relativistic nuclear Hamiltonian having only two-nucleon interactions, consistent with the nucleon-nucleon scattering data at low energies ($E_{lab} < \sim 400\text{MeV}$).

Since nucleons are composite objects made up of quarks and gluons, we can not approximate their interactions by a sum of two-body terms. The mesonic degrees of freedom can also generate three- and more-body potentials in the Hamiltonian in which only the nucleon degrees of freedom are retained. Since the energies obtained with Hamiltonian having only two-body potentials are not far from the experiment, we expect that the contribution of the many-body potentials is small compared to that of the two-body interaction in the realm of nuclear physics, and particularly, only three body-potentials may be important.

The effect of the three-nucleon interactions has been studied recently [12], where the effect of the different three-nucleon interactions in p - ^3He elastic scattering at low energies has been calculated for the four-nucleon scattering observables by using the Kohn variational principle and the hyperspherical harmonics technique. On the other hand, the effects of the two-body and the three-body hyperon-nucleon interactions in Λ hypernuclei have been studied by assessing the relative importance of two- and three-body hyperon-nucleon force and by studying the effect of the hyperon-nucleon-nucleon interaction in closed shell Λ hypernuclei from $A = 5$ to 91 [13,14]. Moreover, the authors in [15] extended the formalism of self-consistent Green's function theory to include three-body interactions and applied it to isotopic chains around oxygen for the first time. Furthermore, the authors in [16] used the realistic Argonne v_{18} potential for the two-nucleon interaction and Urbana three-nucleon potentials to generate accurate variational Monte Carlo wave functions for the $A \leq 12$ nuclei.

The ab initio no-core shell model (NCSM) is a well-established theoretical framework aimed at an exact description of nuclear structure starting from high-precision interactions between the nucleons. Barrett, Navrátil, and Vary [17] discussed, in details, the extension of the ab initio NCSM to nuclear reactions and sketch a number of promising future directions for research emerging from the NCSM foundation, including a microscopic non-perturbative framework for the theory with a core. In the NCSM, C. Forssén and P. Navrátil [18] considered a system of A point-like, non-relativistic nucleons that interact by realistic inter-nucleon interactions. They considered two-nucleon interactions that reproduce nucleon-nucleon phase shifts with high precision, typically up to 350 MeV lab energy. Also, they included three-nucleon interactions with terms, e.g., related to two-pion exchanges with an intermediate delta excitation. Both semi-phenomenological potentials, based on meson-exchange models, as well as modern chiral interactions are considered.

In previous papers Doma et al. [9] applied the TISM with number of quanta of excitations $0 \leq N \leq 8$ to investigate the ground-state wave function, the binding energy, the first excited

state energy and the root mean-square radius of ${}^3\text{H}$ by using the Gogny, Pires and De Tourreil (GPT) interaction [19], the Hu and Massey potential [19] and an effective interaction proposed by Vanagas [2]. Furthermore, Doma [10] applied the TISM with number of quanta of excitations $0 \leq N \leq 8$ to investigate the ground- and excited-states wave functions, the spectrum, the root mean-square radius and the integral cross section of the γ -quanta photo absorption of the nucleus ${}^4\text{He}$ by using the GPT-potential.

In the present paper we have applied two different methods in order to investigate the ground- and first-excited states of the ${}^3\text{H}$ and ${}^4\text{He}$ nuclei. Also, the role of the three-body interactions, together with the two-body interactions, in these nuclei is discussed. For this purpose we have calculated the ground-state energies and wave functions, the first excited state energies and the root mean-square radii.

In the first method we applied the TISM with basis functions corresponding to even number of quanta of excitations in the range $0 \leq N \leq 20$ and used nucleon-nucleon interaction of exactly the same shape and structure as that proposed by Doma et al. [20]. This potential consists of central, tensor, spin-orbit and quadratic spin-orbit forces with Gaussian radial dependences, which are extremely suitable for the calculations of the different matrix elements with respect to the harmonic-oscillator basis functions. New parameters for this potential are calculated for the present paper and are so chosen in such a way that they represent the long-range attraction and the short-range repulsion of the nucleon-nucleon interactions. These parameters are also chosen so as to produce good agreement between the calculated values of the binding energy, the root mean-square radius, the D-state probability, the magnetic dipole moment and the electric quadrupole moment of deuteron.

The methods of constructing the nuclear wave functions in the TISM and of calculating the matrix elements of the nucleon-nucleon interactions and the other one- and two-body operators are well explained in refs. [6,9,10]. Furthermore, we improved the obtained results by adding a three-body interaction of the form of Skyrme III potential [21] to the interaction Hamiltonian.

In the second method we have applied the variational Monte Carlo (VMC) method with suitable trial wave functions. In the calculations we have used two- and three-body interactions. For the two-body interactions we used the Reid V_8 model [22] by assuming that the interaction in all isospin-spin $T, S = 1, 0; 1, 1; 0, 0$ and $0, 1$ states is given by that in the 1S_0 , $({}^3P_2 - {}^3F_2)$, 1P_1 and $({}^3S_1 - {}^3D_1)$ channels. This potential fits the scattering data in 1S_0 , $({}^3S_1 - {}^3D_1)$,

1P_1 and ($^3P_2 - ^3F_2$) channels and it does not have any quadratic spin orbit terms. For the three-body interaction we used the Urbana model three-nucleon potential (UVII), which has been written in the form of sum of a long-range two pion exchange and intermediate-range repulsive terms. The radial functions of this potential are associated with the tensor and Yukawa parts of one pion-exchange part [23].

The variational wave function is taken to be of the form $(S \prod_{i<j} F_{ij})\Phi$, where S is a symmetrizing operator, the pair correlation operators F_{ij} can include central, spin-orbit and tensor correlations, and Φ is a pure spin-isospin function with no spatial dependence [23]. The general form of this wave function can be obtained by multiplying it by triplet-correlation operators, which include components induced by three-nucleon potentials.

2. Calculations Using the Translation Invariant Shell Model

The internal Hamiltonian of a nucleus consisting of A nucleons, interacting via two-body potential, can be written in terms of the relative coordinates of the nucleons, in the form [8,10,24]

$$H = H^{(0)} + V', \quad (2.1)$$

where

$$H^{(0)} = \frac{1}{A} \sum_{1=i<j}^A \left[\frac{(\mathbf{p}_i - \mathbf{p}_j)^2}{2m} + \frac{1}{2} m \omega^2 (\mathbf{r}_i - \mathbf{r}_j)^2 \right], \quad (2.2)$$

is the well-known TISM Hamiltonian and

$$V' = \sum_{1=i<j}^A \left[V(|\mathbf{r}_i - \mathbf{r}_j|) - \frac{m\omega^2}{2A} (\mathbf{r}_i - \mathbf{r}_j)^2 \right]. \quad (2.3)$$

is the residual two-body interaction.

The energy eigenfunctions and eigenvalues of the Hamiltonian $H^{(0)}$ are given by [8,10]

$$|A \Gamma; M_L M_S T M_T\rangle \equiv |A N\{\rho\}(v)\alpha[f]LS; M_L M_S T M_T\rangle, \quad (2.4)$$

$$E_N^{(0)} = \left\{ N + \frac{3}{2}(A - 1) \right\} \hbar\omega. \quad (2.5)$$

In eq. (2.4) Γ stands for the set of all irreducible group representations N , $\{\rho\}$, (ν) , α , $[f]$, L and S . The functions (2.4) form a complete set of functions, bases. It is easy to construct bases which have definite total momentum J in the form [8,10]

$$|A \Gamma J M_J T M_T\rangle = \sum_{M_L+M_S=M_J} (L M_L S M_S | J M_J) |A \Gamma; M_L M_S T M_T\rangle, \quad (2.6)$$

where $(L M_L S M_S | J M_J)$ are the Clebsch-Gordan coefficients of the rotational group R_3 . The nuclear wave function of a state with total momentum J , isospin T and parity π can be constructed as follows [8,10]

$$|A J^\pi T M_J M_T\rangle = \sum_{\Gamma} C_{\Gamma}^{J^\pi T} |A \Gamma J M_J T M_T\rangle, \quad (2.7)$$

where $C_{\Gamma}^{J^\pi T}$ are the state-expansion coefficients. In the summation (2.7) the number of quanta of excitations N is permitted to be either even or odd integer depending on the parity of the state π .

The matrix elements of the residual interaction V' with respect to the bases (2.6) are given in details in [8,10,24,25]. The ground-state nuclear wave function, which is obtained as a consequence of the diagonalization of the ground-state energy matrix, is used to calculate the root mean-square radius, R , from the well-known formula, given in [10,24,25]

The two-nucleon interaction $V(|\mathbf{r}_i - \mathbf{r}_j|)$, equation (2.3), which has been used in our calculations, has the well-known form [20]

$$V(r) = {}^{ts}X\{V_C(r) + V_T(r)S_{12} + V_{LS}\{r\}\boldsymbol{\ell} \cdot \mathbf{s} + V_{LL}(r)L_{12}\}, \quad (2.8)$$

where $\boldsymbol{\ell}$, s and t are the orbital angular momentum, the spin angular momentum and the isotopic spin of the two-nucleon state, respectively. The central, tensor, spin-orbit and quadratic spin-orbit terms are standard. The operator ${}^{ts}X$ has the form [20]

$${}^{ts}X = C_W + (-1)^{s+t+1}C_M + (-1)^{s+1}C_B + (-1)^{t+1}C_H, \quad (2.9)$$

where C_W , C_M , C_B and C_H are the Wigner, the Majorana, the Bartlett and the Heisenberg exchange constants, respectively. Each term of the interaction is expressed as a sum of Gaussian functions in the form

$$V_{\alpha}(r) = \sum_{k=1}^4 V_{\alpha k} e^{-\frac{r^2}{r_{\alpha k}^2}}, \quad (2.10)$$

where $\alpha = C, T, LS$ and LL .

A set of values is considered for the exchange constants. This set is as follows: $C_W = -0.41$, $C_M = -0.41$, $C_B = -0.09$ and $C_H = 0.09$, which belong to the Serber case. For the triplet-even state ($t = 0, s = 1$), which is the case for the ground-state of deuteron, and from the normalization condition of the exchange constants the operator ${}^{\text{ts}}X$ equals -1 so that in order to judge the success of such types of potentials one must use it for other nuclei with mass number $A > 2$. Accordingly, this interaction is considered as an effective interaction for deuteron but for other nuclei it is a realistic interaction.

In the present paper we have varied the depth and range parameters $V_{\alpha k}$ and $r_{\alpha k}$, respectively, in order to obtain results for the binding energy, the root mean-square radius, the D-state probability, the magnetic dipole moment and the electric quadrupole moment of deuteron in excellent agreement with the corresponding experimental values. The resulting potential is denoted by Pot-I.

It is well known that three-body forces are important to describe the properties of finite nuclei. The parameters in the nucleon-nucleon potential may not be unique or there may be some redundant parameters in order to reproduce the deuteron properties. In order to investigate these points of view, we have considered the following Hamiltonian operator, which takes into consideration the three-body forces:

$$H = H^{(0)} + V' + V'', \quad (2.11)$$

where the first two terms in (2.11) are given by (2.2), and (2.3) and

$$V'' = \sum_{i,j,k} V(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k) \quad (2.12)$$

is the three-body potential. For the three-body potential we have used the Skyrme-III potential [21]

$$V'' = t_3 \delta(\mathbf{r}_1 - \mathbf{r}_2) \delta(\mathbf{r}_2 - \mathbf{r}_3), \quad (2.13)$$

where $t_3 = 14000.0 \text{ MeV} \times fm^6$.

To calculate the matrix elements of the three-body interactions, given by (2.13), we have used the fact that

$$\delta(\mathbf{r}_1 - \mathbf{r}_2) = \frac{1}{r_1 r_2} \delta(r_1 - r_2) \delta(\cos\theta_1 - \cos\theta_2) \delta(\varphi_1 - \varphi_2) \quad (2.14)$$

3. Calculations with the Variational Monte Carlo Method

In this method the nuclear Hamiltonian has the same form given by (2.11), V' is a two-nucleon interaction that fits nucleon-nucleon (NN) scattering data and deuteron properties and V'' is an explicit three-body nucleon interaction. For the two-nucleon interactions we used the V_8 -model potential [22]. This potential is obtained by assuming that the interaction in all isospin-spin: $T, S = 1, 0; 1, 1; 0, 0$ and $0, 1$ states is given by that in the $^1S_0, (^3P_2 - ^3F_2), ^1P_1$ and $(^3S_1 - ^3D_1)$ channels. This potential was introduced by Reid to fit the scattering data in $^1S_0, (^3S_1 - ^3D_1), ^1P_1$ and $(^3P_2 - ^3F_2)$ channels and is used to construct a two nucleon interaction operator

$$V_{8,ij} = V^C + V^\sigma(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j) + V^\tau(\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j) + V^{\sigma\tau}(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)(\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j) + V^t S_{ij} + V^{t\tau} S_{ij}(\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j) \\ + V^b(\mathbf{L} \cdot \mathbf{S}_{ij}) + V^{b\tau}(\mathbf{L} \cdot \mathbf{S}_{ij}) \cdot (\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j) = \sum_{p=1,8} V^p(r_{ij}) O_{ij}^p, \quad (3.1)$$

where $V^p(r_{ij})$ are functions of the interparticle distance r_{ij} and O_{ij}^p are the operators

$$O_{ij}^{p=1,8} = 1, \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j, \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j, (\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)(\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j), S_{ij}, S_{ij}(\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j), (\mathbf{L} \cdot \mathbf{S}_{ij}), (\mathbf{L} \cdot \mathbf{S}_{ij})(\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j). \quad (3.2)$$

Here S_{ij} is the tensor operator and $\mathbf{L} \cdot \mathbf{S}_{ij}$ is the spin-orbit operator.

Since this model does not have any quadratic spin-orbit terms, the potentials in these channels were assumed to act in all S, T states $0, 1; 0, 0; 1, 0$ and $1, 1$ respectively. This model ignores differences between 1S_0 and 1D_2 potentials, and also assumes that the 3P_0 and 3P_1 potentials can be obtained from the potential operator in the $^3P_2 - ^3F_2$ state.

The Reid potential has a modified one pion exchange part [26]

$$v_\pi = 3.488 \frac{e^{-x}}{x} \left(O^{\sigma\tau} + \left[\left(1 + \frac{3}{x} + \frac{3}{x^2} \right) - \left(\frac{12}{x} + \frac{3}{x^2} \right) e^{-3x} \right] O^{t\tau} \right), \quad x = 0.7r. \quad (3.3)$$

The three-nucleon interaction used here is the Urbana-model three nucleon interaction given in Eqs. (4.1) and (4.2) of section-4.

3.1 The Variational Monte Carlo Method

The VMC techniques have proven to be a very powerful tool for studying quantum systems, particularly within atomic [27-29] and nuclear physics [30]. The most important applications in nuclear physics are to few-body systems, either at the nucleon or the quark level. A variety of nuclear properties have been calculated with Monte Carlo methods, including the binding energies, the expectation values of other operators (besides \hat{H}), electromagnetic form factors, and asymptotic properties of the wave function for three- and four-body nuclei. One can also

study the Hamiltonian itself, in particular the effects of the three-nucleon interaction in light nuclei. Also in this technique we employ random sampling.

The variational Monte Carlo (VMC) method is used to evaluate expressions like [31]:

$$\begin{aligned} E_T &= \frac{\int \Psi_T^*(\mathbf{R}) \hat{H} \Psi_T(\mathbf{R}) d\mathbf{R}}{\int \Psi_T^*(\mathbf{R}) \Psi_T(\mathbf{R}) d\mathbf{R}} = \frac{\int \Psi_T^*(\mathbf{R}) \Psi_T(\mathbf{R}) d\mathbf{R} [\Psi_T^{-1}(\mathbf{R}) \hat{H} \Psi_T(\mathbf{R})]}{\int \Psi_T^*(\mathbf{R}) \Psi_T(\mathbf{R}) d\mathbf{R}}, \\ &= \frac{\int d\mathbf{R} \Psi_T^*(\mathbf{R}) \Psi_T(\mathbf{R}) E_L(\mathbf{R})}{\int d\mathbf{R} \Psi_T^*(\mathbf{R}) \Psi_T(\mathbf{R})} \geq E_0, \end{aligned} \quad (3.4)$$

E_T is an energy expectation value for a system with Hamiltonian operator \hat{H} ; $\Psi_T(\mathbf{R}) \equiv \Psi_T(\mathbf{R}, a)$ is a trial wave function depending on all the particle coordinates possibly including spin, which are represented by the vector \mathbf{R} , a is a set of variational parameters, $E_L(\mathbf{R}) = \Psi_T^{-1}(\mathbf{R}) \hat{H} \Psi_T(\mathbf{R})$ is the "local energy" of the trial wave function and E_0 is the exact ground-state energy. As the quality of the trial wave function improves, E_T becomes closer to E_0 .

The essence of VMC is the sampling of a distribution proportional to $|\Psi_T(\mathbf{R})|^2$ where $\Psi_T(\mathbf{R})$ is a given trial wave function, a function of the 3N-dimensional coordinates \mathbf{R} . The expectation values of non differential operators may be sampled as [32]:

$$\langle \hat{O} \rangle = \frac{\int \hat{O}(\mathbf{R}) |\Psi_T(\mathbf{R})|^2 d^3N\mathbf{R}}{\int |\Psi_T(\mathbf{R})|^2 d^3N\mathbf{R}} \approx \frac{1}{N} \sum_{i=1}^N \hat{O}(\mathbf{R}_i). \quad (3.5)$$

Differential operators are only slightly more difficult, since we can write:

$$\langle \hat{O} \rangle_{VMC} = \frac{\langle \Psi_T | \hat{O} | \Psi_T \rangle}{\langle \Psi_T | \Psi_T \rangle} = \frac{\int [\hat{O} \Psi_T(\mathbf{R}) / \Psi_T(\mathbf{R})] |\Psi_T(\mathbf{R})|^2 d^3N\mathbf{R}}{\int |\Psi_T(\mathbf{R})|^2 d^3N\mathbf{R}} \approx \frac{1}{N} \sum_{i=1}^N \hat{O} \Psi_T(\mathbf{R}_i) / \Psi_T(\mathbf{R}_i). \quad (3.6)$$

3.2 The Variational Wave Function

A suitably parameterized variational wave function Ψ_V which depends upon several variational parameters ($\alpha_1, \alpha_2, \dots, \alpha_n$) is used to calculate an upper bound to the ground-state energy [26]:

$$E_V = \frac{\langle \Psi_V | H | \Psi_V \rangle}{\langle \Psi_V | \Psi_V \rangle} \geq E_0. \quad (3.7)$$

The parameters in Ψ_V can be varied to minimize E_V and the lowest value is taken as the approximate ground-state energy, then the best Ψ_V can be used to evaluate other operators.

The variational wave function has the general form [31,33]:

$$\Psi_V = (S \prod_{i < j} F_{ij}) \Phi, \quad (3.8)$$

where S is a symmetrizing operator (since F_{ij} do not commute), F_{ij} are pair correlation operators and $\Phi = \Phi(JMTM_T)$ is the initial uncorrelated state which is antisymmetrized

product of single-particle spin-isospin states with no spatial dependence [26]. For the ${}^3\text{H}$ and ${}^4\text{He}$ nuclei \emptyset is simply taken to be [31]

$$\emptyset ({}^3H) = A|p \uparrow n \downarrow n \uparrow\rangle. \quad (3.9)$$

$$\emptyset ({}^4He) = A|p \downarrow p \uparrow n \downarrow n \uparrow\rangle. \quad (3.10)$$

The antisymmetrizing operator A in (3.9) and (3.10) takes the form:

$$A = (1 + e_{12} + e_{23} + e_{31} + e_{12}e_{23} + e_{23}e_{12}), \quad (3.11)$$

where e_{ij} are spin-isospin exchange operators

$$e_{ij} = -\frac{1}{4}[1 + \sigma_i \cdot \sigma_j + \tau_i \cdot \tau_j + (\sigma_i \cdot \sigma_j)(\tau_i \cdot \tau_j)]. \quad (3.12)$$

The pair correlation operators are taken to be:

$$F_{ij} = \sum_{p=1,8} f^p(r_{ij}) O_{ij}^p = f_{ij}^c(r_{ij})(1 + \sum_{p=2,8} u_{ij}^p(r_{ij}) O_{ij}^p), \quad (3.13)$$

where f_{ij}^c are the central pair correlation functions.

The general form of Ψ_V is [26]

$$\Psi_V = \{S \prod_{i<j} f_{ij}^c(r_{ij}) [1 + \sum_{p=2,6} (\prod_{k \neq i,j} f_{ijk}^p) u_{ij}^p(r_{ij}) O_{ij}^p]\} \emptyset, \quad (3.14)$$

where f_{ijk}^p are the three-body correlations which represent the effect of the other particles on the u^p ,

$$f_{ijk}^p = 1 - t_1 \left(\frac{r_{ij}}{R_{ijk}} \right)^{t_2} \exp(-t_3 R_{ijk}). \quad (3.15)$$

Also, $R_{ijk} = r_{ij} + r_{jk} + r_{ki}$, t_1 and t_2 are variational parameters.

4. The Urbana Model Three Nucleon Interactions

Explicit treatments of the pions and deltas degrees of freedom, or implicit treatment via an effective three-body force (3BF), are different ways to approach the same physics. In approach-1, the nucleus is assumed to be consisting of nucleons only. As nucleon-nucleon (N-N) forces one uses one of the standard two-body force (2BF) potentials (Reid Soft Core, Paris, Bonn, ...), supplemented by a phenomenological or microscopically derived 3BF such as the Tucson-Melbourne (TM) force. This approach is taken by the Los Alamos, Urbana, Tohoku groups [34-36].

The second approach referred to as "3-body force" is based on an explicit treatment of the non-nucleonic degrees of freedom in the ground state wave function. It can be included microscopically by allowing in the wave function for pions, deltas, and pairwise interactions with these additional constituents. This approach is taken by the Hannover group [37].

The Urbana model (UVII) three nucleon interaction is written as a sum of long-range two pion exchange and intermediate-range repulsive terms [26]

$$V_{ijk} = V_{ijk}^{FM} + V_{ijk}^R = V_{ijk}^{2\pi} + V_{ijk}^R. \quad (4.1)$$

$$V_{ijk}^{FM} = \sum_{cyc} -0.0333 \left(\{\tau_i \cdot \tau_j, \tau_j \cdot \tau_k\} \{x_{ij}, x_{ik}\} + \frac{1}{4} [\tau_i \cdot \tau_j, \tau_j \cdot \tau_k] [x_{ij}, x_{ik}] \right),$$

$$x_{ij} = T(r_{ij})S_{ij} + \sigma_i \cdot \sigma_j Y(r_{ij}),$$

$$V_{ijk}^R = \sum_{cyc} 0.0038 T^2(r_{ij})T^2(r_{jk}). \quad (4.2)$$

The $T(r)$ and $Y(r)$ are radial functions associated with the tensor and Yukawa parts of the one pion-exchange interaction:

$$Y(r) = \frac{e^{-\mu r}}{\mu r} (1 - e^{-br^2}),$$

$$T(r) = \left[1 + \frac{3}{\mu r} + \frac{3}{(\mu r)^2} \right] (1 - e^{-br^2}). \quad (4.3)$$

Here the pion mass $\mu = 0.7 fm^{-1}$, and $b = 2 fm^{-2}$. The $V_{ijk}^{2\pi}$ is the familiar Fujita-Miyazawa two-pion exchange operator, and is attractive. The V_{ijk}^R is repulsive, and its strength U_0 is 0.0038 in model VII instead of 0.003 in model V.

The Coulomb interaction is taken as [38]

$$V_c(r_{ij}) = \frac{e^2}{4r_{ij}} (1 + \tau_{3,i})(1 + \tau_{3,j}) \left[1 - \frac{1}{48} e^{-x} (48 + 33x + 9x^2 + x^3) \right], \quad (4.4)$$

and $x = \sqrt{12}r_{ij}/R_{cp}$, where R_{cp} is the rms charge radius of the proton.

5. Results and Discussions

The methods of calculating the one-, two-, three- and four-particle FPC in the TISM are given in [2,39]. Also, a recurrence relations for the two-particle orbital FPC and tables of these coefficients for $3 \leq A \leq 6$ and $N \leq 3$ are given by Vanagas [2]. General and direct method for calculating the two-particle orbital FPC and tables of these coefficients for $A = 6$ and

$2 \leq N \leq 4$ are given by Doma and Machabeli [40]. Finally, we have calculated in the present paper the necessary two- and three-particle orbital FPC for nuclei with $A = 3$ and 4.

The ground-state wave function of each nucleus is expanded in series in terms of the basis functions of the TISM with even number of quanta of excitations N . Accordingly, each one of these basis functions is then expanded in terms of the two-particle total fractional parentage coefficients which are products of orbital and spin-isospin coefficients in order to calculate the two-particle operators in the Hamiltonian. The ground-state of triton has total angular momentum $J = \frac{1}{2}$, isotopic spin $T = \frac{1}{2}$ and even parity, i.e. $(J^\pi, T) = \left(\frac{1}{2}^+, \frac{1}{2}\right)$. In Table-1 we present the used TISM basis functions for the ground-state of triton corresponding to number of quanta of excitations $N = 0, 2, 4, 6, 8, 10, 12, 14, 16, 18$ and 20. The basis functions which produce in the final calculations weights $\leq 10^{-6}$ are eliminated and then the resulting nuclear wave functions are renormalized to unity.

The energy eigenvalues which result from the diagonalization of the Hamiltonian matrices for triton showed two accepted values for the energy of the state $\left(\frac{1}{2}^+, \frac{1}{2}\right)$, the lowest one belongs to the ground-state, and hence the negative value of the binding energy, and the highest belongs to the first-excited state energy: E^* , which is not yet assigned experimentally.

In Table-2 we present the triton ground-state wave function (Ψ), after eliminating the bases with weights $\leq 10^{-6}$ and then renormalizing Ψ , binding energy (B.E.), in MeV, root mean-square radius (R), in fm, and first excited-state energy (E^*), in MeV, as functions of the oscillator parameter $\hbar\omega$. The corresponding experimental values are given in this table. The improved values arised from using the Skyrme III three-body interaction together with Pot-I are also given in this table.

It is seen from Table-2 that the calculated values of the binding energy and the root mean-square radius of ${}^3\text{H}$ are in good agreement with the corresponding experimental values and that the inclusion of the three-body interaction together with the two-body interaction improved the results as expected.

Table-1 TISM-basis functions for the triton nucleus with $0 \leq N \leq 20$. The basis functions which produce in the final calculations weights $\leq 10^{-6}$ are eliminated.

Ψ_i i	N	$\{\rho\}$	(ν)	$[f]$	L	S
1	0	{0}	(0)	[3]	0	1/2
2	2	{2}	(0)	[3]	0	1/2
3	2	{2}	(2)	[21]	0	1/2
4	2	{2}	(2)	[21]	2	3/2
5	2	{11}	(0)*	[1 ³]	1	1/2
6	4	{4}	(0)	[3]	0	1/2
7	4	{4}	(2)	[21]	0	1/2
8	4	{4}	(2)	[21]	2	3/2
9	4	{31}	(2)	[21]	1	1/2
10	6	{6}	(0)	[3]	0	1/2
11	6	{6}	(2)	[21]	0	1/2
12	8	{8}	(0)	[3]	0	1/2
13	10	{10}	(0)	[3]	0	1/2
14	12	{12}	(0)	[3]	0	1/2
15	14	{14}	(0)	[3]	0	1/2
16	16	{16}	(0)	[3]	0	1/2
17	18	{18}	(0)	[3]	0	1/2
18	20	{20}	(0)	[3]	0	1/2

The ground state of ${}^4\text{He}$ has total angular momentum quantum number $J = 0$, isotopic spin $T = 0$ and even parity and so is the first-excited state. The Hamiltonian matrices for the ground state of ${}^4\text{He}$ are constructed and diagonalized with respect to the oscillator parameter $\hbar\omega$, which is allowed to vary in a large range of values $8 \leq \hbar\omega \leq 28$ MeV in order to obtain the minimum energy eigenvalues. In Table-3 we present the TISM basis functions for the ground-state of ${}^4\text{He}$ corresponding to number of quanta of excitations $N = 0, 2, 4, 6, 8, 10, 12, 14, 16, 18$ and 20 . The basis functions which produce in the final calculations weights $\leq 10^{-6}$ are eliminated and then the resulting nuclear wave functions are renormalized to unity.

Since the range of values of the oscillator parameter $\hbar\omega$ is large we present only in Table-4 the resulting ground-state nuclear wave functions of ${}^4\text{He}$ by using Pot-I together with the Skyrme III three-body interaction (the improved case), at the values of $\hbar\omega$ which gave the best fit between the calculated ground-state characteristics of ${}^4\text{He}$ and the corresponding experimental values.

Table-2 Ground-State Characteristic of 3H with $0 \leq N \leq 20$ by using Pot-I alone and by adding to it the Skyrme III three-body interaction (the improved case). The basis functions which produce in the final calculations weights $\leq 10^{-6}$ are eliminated and then the resulting nuclear wave functions are renormalized to unity. The wave functions for the improved cases are only given.

Charact.		$\hbar\omega$ (in MeV)							Exp.
		11	12	13	14	15	16	17	
Ψ	Ψ_1	0.8599	0.8795	0.8893	0.8906	0.8982	0.9029	0.9035	
	Ψ_2	0.1961	0.1125	0.0960	0.0691	0.0395	0.0251	0.0498	
	Ψ_3	-0.3507	-0.3415	-0.3222	-0.3222	-0.3126	-0.3026	-0.2919	
	Ψ_4	0.1334	0.1365	0.1373	0.1370	0.1360	0.1340	0.1309	
	Ψ_5	0.0574	0.0596	0.0614	0.0626	0.0633	0.0635	0.0632	
	Ψ_6	0.1812	0.1773	0.1750	0.1735	0.1740	0.1760	0.1791	
	Ψ_7	0.1392	0.1357	0.1321	0.1283	0.1245	0.1204	0.1160	
	Ψ_8	0.1266	0.1271	0.1277	0.1282	0.1291	0.1300	0.1309	
	Ψ_9	0.0241	0.0250	0.0258	0.0262	0.0267	0.0269	0.0269	
	Ψ_{10}	0.0207	0.0192	0.0210	0.0161	0.0200	0.0302	0.0403	
	Ψ_{11}	-0.0284	-0.0265	-0.0247	-0.0228	-0.0213	-0.0197	-0.0181	
	Ψ_{12}	0.0472	0.0456	0.0447	0.0563	0.0449	0.0460	0.0477	
	Ψ_{13}	0.0403	0.0354	0.0313	0.0410	0.0237	0.0203	0.0169	
	Ψ_{14}	0.0388	0.0312	0.0299	0.0334	0.0285	0.0308	0.0288	
	Ψ_{15}	0.0354	0.0247	0.0310	0.0303	0.0305	0.0253	0.0298	
	Ψ_{16}	0.0324	0.0371	0.0364	0.0486	0.0371	0.0331	0.0377	
	Ψ_{17}	0.0228	0.0263	0.0298	0.0325	0.0329	0.0261	0.0341	
	Ψ_{18}	0.0213	0.0221	0.0214	0.0246	0.0216	0.0210	0.0112	
B.E.		7.8180	8.0137	8.1393	8.2173	8.2043	8.1863	8.1182	8.48

B.E. Improved	8.0213	8.2159	8.3510	8.4311	8.4176	8.3992	8.3293	
R	2.2209	2.1303	2.0520	1.9240	1.9799	2.0265	2.1088	1.75
R Improved	2.0464	1.9654	1.8909	1.7729	1.8244	1.8674	1.9432	
E^*	8.5323	8.6042	8.6897	8.7754	8.8779	8.9843	9.1565	
E^* Improved	8.4645	8.5272	8.5923	8.6654	8.7523	8.8413	9.0623	---

Table-3 The TISM bases of the ground-state of 4He. The basis functions which produce in the final calculations weights $\leq 10^{-6}$ are eliminated.

Ψ_i i	N	$\{\rho\}$	(ν)	$[\lambda]$	L	S
1	0	{0}	(0)	[4]	0	0
2	2	{2}	(0)	[4]	0	0
3	2	{2}	(0)	[22]	0	0
4	2	{2}	(2)	[22]	2	2
5	2	{11}	(1)	[211]	1	1
6	4	{4}	(0)	[4]	0	0
7	4	{4}	(2)	[22]	0	0
8	4	{4}	(2)	[22]	2	2
9	4	{31}	(2)	[31]	1	1
10	6	{6}	(0)	[4]	0	0
11	6	{6}	(2)	[22]	2	2
12	8	{8}	(0)	[4]	0	0
13	8	{8}	(2)	[22]	2	2
14	10	{10}	(0)	[4]	0	0
15	10	{10}	(2)	[22]	2	2
16	12	{12}	(0)	[4]	0	0
17	14	{14}	(0)	[4]	0	0
18	16	{16}	(0)	[4]	0	0
19	18	{18}	(0)	[4]	0	0
20	20	{20}	(0)	[4]	0	0

Table-4 Ground-state nuclear wave function of ${}^4\text{He}$ by using Pot-I together with the Skyrme III three-body interaction. The basis functions which produce in the final calculations weights $\leq 10^{-6}$ are eliminated and then the resulting nuclear wave function is renormalized to unity. The value of the oscillator parameter $\hbar\omega$ is also given.

Nuclear wave function	
i	Ψ_i
1	0.8764
2	0.1528
3	-0.0671
4	0.1723
5	-0.1129
6	0.1784
7	0.1223
8	-0.0920
9	0.1540
10	0.0930
11	-0.0997
12	0.1305
13	0.0971
14	-0.1453
15	-0.1001
16	0.0453
17	0.0184
18	0.0443
19	0.0370
20	0.0282
$\hbar\omega$ (MeV)	17

The energy eigenvalues which result from the diagonalization of the Hamiltonian matrices for ${}^4\text{He}$ showed two accepted values for the energy of the state $(0^+, 0)$, the lowest one belongs to the ground-state and the highest belongs to the first-excited state: E^* . The obtained nuclear wave functions are used to calculate the root mean-square radius of ${}^4\text{He}$. We present in Table-

5 the best values of the ${}^4\text{He}$ binding energy (B.E.), in MeV, root mean-square radius (R), in fm, and first-excited state energy eigenvalue (E^*), in MeV, for Pot-I alone and for the improved case by adding to it the Skyrme III three-body interaction. The corresponding experimental values and previous results by using the Gogny, Pires and De Turreil (GPT)-potential [19] with the TISM bases corresponding to $0 \leq N \leq 10$ [10] are also given in this table. In Table-5 we present also previous results obtained by using large-basis shell model with a multivalued G-matrix effective interaction [41]. The values of the oscillator parameter $\hbar\omega$, which produce the minimum energy eigenvalues in each case, are also given in Table-5.

Table-5 The best values of the ground state characteristics of ${}^4\text{He}$ by using Pot-I alone and by adding to it the Skyrme III three-body interaction. The corresponding experimental values and the values of the oscillator parameter $\hbar\omega$ are also given. Previous results by using the GPT-potential and the G-matrix method are also given.

Characteristic	B. E.	R	E^*	$\hbar\omega$
Case	(MeV)	(fm)	(MeV)	(MeV)
Pot-I alone	27.895	1.534	20.654	17
Pot-I + Skyrme III-Pot	28.286	1.468	20.354	
GPT-PotI, $N \leq 10$ [10]	25.480	1.70	20.70	17
G-matrix [40]	26.459	1.492	21.82	14
Experiment	28.3 [42]	1.46 [27]	20.21 [26]	–

It is seen from Table-5 that the calculated values of the binding energy, the root mean-square radius and the first excited-state energy eigenvalue of ${}^4\text{He}$ are in good agreement with the corresponding experimental values and that the inclusion of the three-body interaction together with the two-body interaction improved the results as expected.

According to the above results, the variations of the binding energy (B.E.), root mean-square radius (R), and first excited–state energy (E^*), with respect to the oscillator parameter $\hbar\omega$ (i.e. the dependence of the obtained results on the used model and its wave function) for each case, with and without improvements arised from using the Skyrme III three-body interaction, are given in Figs. 1-6, for the nuclei ${}^3\text{H}$ and ${}^4\text{He}$. The figures show minima for the values of the root mean-square radius, the ground-stae energy (the negative value of the binding energy) and the first-excited state energy for the nucleus ${}^4\text{He}$, for the two cases, in agreement with the basic property of the used model. For the triton nucleus minima have been obtained only for

the ground-state and the root-mean square radius. Concerning the second $\left(\frac{1}{2}^+, \frac{1}{2}\right)$ state of ${}^3\text{H}$, we would like to point out that the three- and four-nucleon states have only one bound state, and all excited states are in the continuum. The use of a bound-state approach with square-integrable basis function is only meaningful for bound states, and narrow resonances. Contrary to ${}^4\text{He}$, which presents a broad 0^+ resonance, there are no resonances in the $\frac{1}{2}^+$ channel of tritium. The reason for which its energy increases with increasing harmonic oscillator frequency and does not present a clear minimum is that this eigenstate and those above it represent a discretization of the energy continuum and as such the continuously move as the model space size is increased or other parameters are varied.

Moreover, it is seen from Table-2 and Figs.1, 2 and 3, for ${}^3\text{H}$, that the differences arising from using potential-I alone and by adding to it the Skyrme III three-body interaction in the calculations of the binding energy, the root mean-square radius and the first excited-state energy of ${}^3\text{H}$ are 0.2138 MeV, -0.1511 fm and 0.11 MeV, respectively. We can see also that at $\hbar\omega = 14$ MeV the improved calculations arised from using the Skyrme III three-body interaction together with Pot-I gave values close to the expermintal ones, where the differences between them are 0.0489 MeV in the binding energy of ${}^3\text{H}$ and -0.1229 fm in the value of the root mean-square radius of ${}^3\text{H}$.

Also, it is seen from Table-5 and Figs. 4, 5 and 6, for ${}^4\text{He}$, that the differences between using potential-I alone and by adding to it the Skyrme III three-body interaction in the binding energy, the root mean-square radius and the first excited-state energy are 0.5195 MeV, -0.175 fm and -0.293 MeV, respectively. We see also that at $\hbar\omega = 17$ MeV the improved calculations arised from using the Skyrme III three-body interaction together with Pot-I gave values, for ${}^4\text{He}$, close to the expermintal ones where the differences between them are 0.2265 MeV in the binding energy, -0.052 fm in the root mean-square radius and -1.012 MeV in the first excited-state energy.

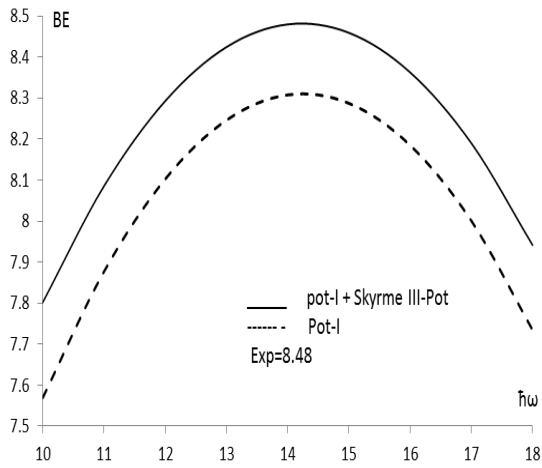


Fig. 1

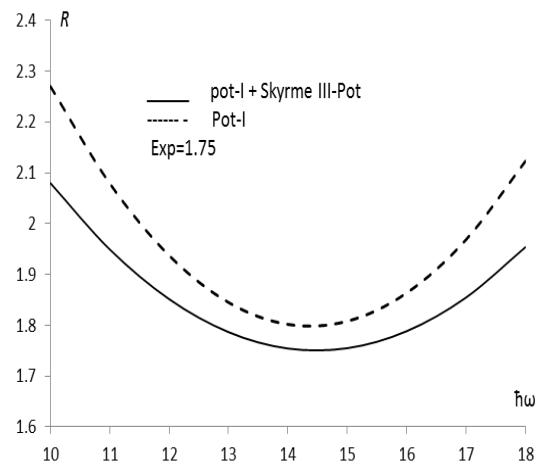


Fig. 2

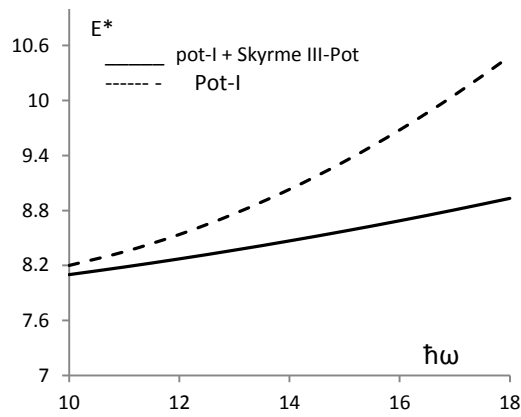


Fig. 3

Figs.1, 2 and 3 Variations of the ${}^3\text{H}$ binding energy (B.E.), root mean-square radius (R) and first excited–state energy (E^*) with the oscillator parameter $\hbar\omega$ for Pot-I alone and for the improved case, arised from ading to it the Skyrme III three-body interaction.

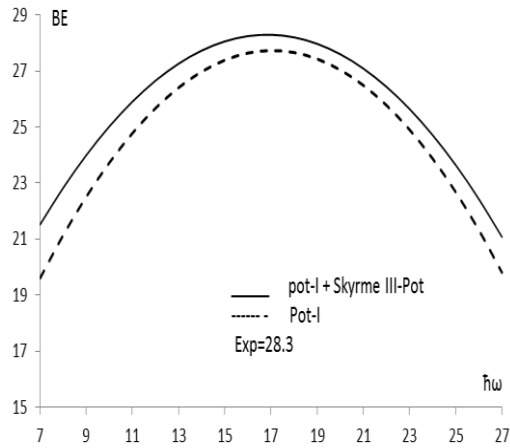


Fig. 4

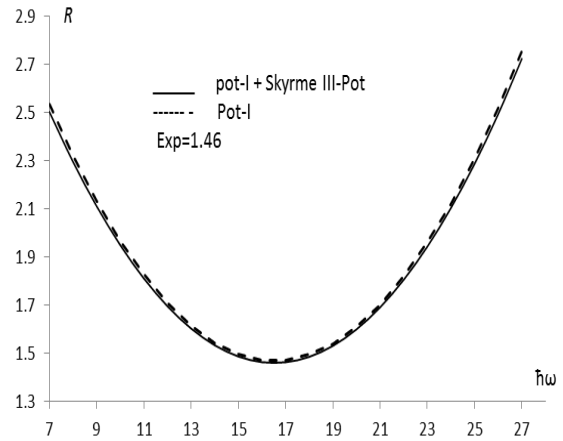


Fig. 5

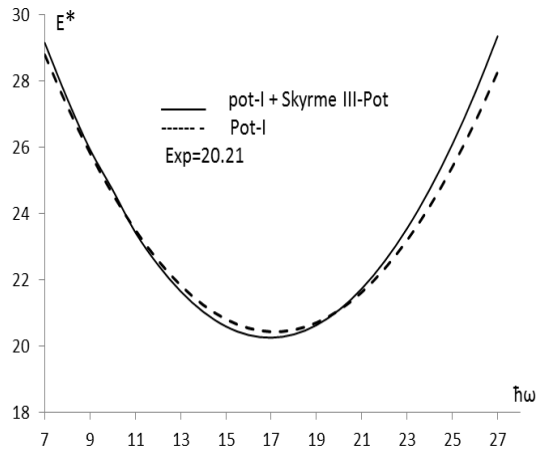


Fig. 6

Figs.4, 5 and 6 Variations of the binding energy (B.E.), root mean-square radius (R) and first excited-state energy (E^*) of ${}^4\text{He}$ with the oscillator parameter $\hbar\omega$ for Pot-I alone and for the improved case, arising from adding to it the Skyrme III three-body interaction.

In the Monte Carlo calculations the role of trial wave function is very important in order to obtain good results for the binding energy and the root mean-square radius of the considered nucleus as well as to accelerate the convergence of the resulting integrals. The using computer program enables us to make good search for such trial wave function. The optimized variational parameters that we have obtained for ${}^3\text{H}$ with the Reid V_8 interaction are listed in Table-6. Once the optimum wave function has been determined, a set of Monte Carlo calculations should be undertaken to determine all of the expectation values. The results obtained with this wave function are summarized in Table-7. Table-7 presents all of the

energy expectation values, as well as point-particle root mean square radii of the neutron and proton density. In this Table, V_{ij} gives the total nucleon-nucleon potential energy, and T_i is the total kinetic energy. The two-body potential is also split into the V_6 contribution and the remaining L.S terms (V_b)

In addition, the contribution of the various pieces of the Urbana model (UVII) three-nucleon interaction (TNI) are given in this Table. The commutator and anti-commutator pieces of the two-pion-exchange TNI are listed as $V3_c$ and $V3_a$ respectively, and the short-ranged repulsive piece is $V3_u$. Each of these terms is a very small fraction of the total potential energy, yet they constitute a significant part of the binding energy of the triton nucleus.

The contribution of the Urbana model (UVII) three-nucleon interaction (TNI) to the expectation value of the binding energy of triton is 0.76 MeV which represents 10% of the binding energy of triton when we use the Reid V_8 two-body interaction only. This increase depends on the exact 2BF and 3BF used. The inclusion of the 3BF improves the Coulomb energy difference ${}^3\text{H}$ - ${}^3\text{He}$. The 3BF leads to a change of the proton point density [43]. This change of the point density due to the 3BF is very desirable one.

Table-6 Triton variational parameters for the VMC method

	1S_0	1P_1	3S_1	3P_1	3D_1	3F_1
$E_{S,T}$	6.00	2.00	12.00	6.00	---	---
η_T					0.026	-0.10
C_x	1.00	1.00	3.00	3.00	2.00	2.00
a_x	0.40	0.40	0.40	0.40	0.40	0.40
R_x	1.00	1.00	2.80	2.80	360	3.60
α_x	1.00	0.92	0.92	0.92	0.92	0.92
	$t1_{ST}$	$t1_{tT}$	$t2_{ST}$	$t2_{tT}$	$t3_{ST}$	$t3_{tT}$
	10.00	10.00	4.00	4.00	0.05	0.05

Table-7 The calculated ground-state energy of triton by using the variational Monte Carlo method with two- and three-body interactions.

Characteristics	Expectation Value	Statistical Error
$T_i + V_{ij}$	-7.59	0.03
$T_i + V_{ij} + V_{ijk}$	-8.35	0.06
T_i	51.15	0.53
V_{ij}	-58.75	0.53
V_6	59.43	0.53
V_b	.068	0.07
V_{ijk}	-0.75	0.04
$V3_a$	-0.57	0.03
$V3_c$	-0.39	0.01
$V3_c$	0.20	0.006
$\langle r_i^2 \rangle_{\text{proton}}$	1.57	0.001
$\langle r_i^2 \rangle_{\text{neutron}}$	1.64	0.001

In Table-8 we present the ^4He variational parameters. The calculated ground-state energy of helium by using three-body interaction, together with the two-body interaction are given in Table-9.

Table-8 ^4He variational parameters for the VMC method

	1S_0	1P_1	3S_1	3P_1	3D_1	3F_1
$E_{S,T}$	6.00	2.00	12.0	6.00	---	---
η_T					.026	-.010
C_x	1.00	1.00	3.00	3.00	2.00	2.00
a_x	0.40	0.40	0.40	0.40	0.40	0.40
R_x	1.00	1.00	2.80	2.80	3.60	3.60
α_x	1.00	0.80	0.80	0.80	0.80	0.80
	$t1_{ST}$	$t1_{tT}$	$t2_{ST}$	$t2_{tT}$	$t3_{ST}$	$t3_{tT}$
	10.0	10.0	4.00	4.00	0.05	0.05

Table-9 The calculated ground-state energy of helium ${}^4\text{He}$ by using the variational Monte Carlo method with two- and three-body interactions.

Characteristics	Expectation Value	Statistical Error
$T_i + V_{ij}$	-25.30	0.05
$T_i + V_{ij} + V_{ijk}$	-28.296	0.07
T_i	75.025	0.39
V_{ij}	-100.32	0.42
V_6	-102.37	0.39
V_b	1.43	0.04
V_{ijk}	-2.99	0.03
V_{3a}	-2.65	0.03
V_{3c}	-1.74	0.01
V_{3c}	1.41	0.018
$\langle r_i^2 \rangle_{\text{proton}}$	1.92	0.001
$\langle r_i^2 \rangle_{\text{neutron}}$	1.92	0.001

Similar conclusions can be given for the nucleus ${}^4\text{He}$. The contribution of the Urbana model (UVII) three-nucleon interaction (TNI) to the expectation value of the binding energy of helium is 2.996 MeV which represents 9% of the binding energy of helium when we use the modified Reid V_8 two-body interaction only. It is seen from the second row of Table-11 that the inclusion of the three-nucleon interaction improved the calculated ground-state energy of helium as expected.

6. Conclusion

It is well-known that calculations within a three-body translational invariant harmonic oscillator basis and using much more realistic two and three-nucleon forces have been performed (see e.g. [44]) for the three-nucleon system, and for the four-nucleon system (see e.g. [45,46]). These were obtained within the translational invariant form of the no-core shell model, which is equivalent to the TISM with the exception that the antisymmetrization of the wave function is not achieved by means of group theory, but rather by diagonalizing the antisymmetrization operator and retaining as basis states the antisymmetric eigenstates. It is

well-known also that, the three- and four-nucleon systems have been studied by means of numerically exact few-body approaches (such as the Faddeev, Faddeev-Yacubosky and hyperspherical harmonics approaches starting from realistic two and three-nucleon forces, see for example [47-49]). A review can be found e.g., in [50].

However, the calculations presented here does present a certain degree of novelty with respect to the translational invariant no-core shell model. In particular, the direct construction of antisymmetric three- and four-body basis states with the help of group theory is more elegant and may even turn out to be computationally more advantageous.

References

- [1] I. Talmi, **Simple Models of Complex Nuclei**, Harwood Academic publishers (1993).
- [2] V. V. Vanagas, **Algebraic Methods in Nuclear Theory**, Mintis. Vilnius (1971).
- [3] V. Bargman and M. Moshinsky, Nucl. Phys., **18**: 697 (1960); **23**: 177 (1961), S. P. Kramer and Moshinsky, Nucl. Phys., **82**: 241 (1966).
- [4] M. Kretschmar, Z. Phys. **157**: 433 (1960); **158**: 284 (1960).
- [5] S. B. Doma, Bull. Acad. Sci. Georgian SSR, **74**: 585 (1974).
- [6] S. B. Doma, T. I. Kopaleyshvili and I. Z. Machabeli, Yadernaya Physica (Sov. J. Nucl. Phys.) **21**: 720 (1975).
- [7] S. B. Doma and A. M. El-Zebidy, International Journal of Modern Physics E, **14**(2): 189 (2005).
- [8] S. B. Doma, A. M. El-Zebidy and M. A. Abdel-Khalik, J. Phys. G: Nucl. Part. Phys. **34**: 27 (2007).
- [9] S. B. Doma and M. K. Yousef, Indian j. Pure and Appl. Math., **10**(4): 404 (1979); S. B. Doma, Indian j. Pure and Appl. Math., **13**(12): 1455 (1982); Indian J. Pure and Appl. Math., **12**(6): 769 (1981).
- [10] S. B. Doma, Indian J. Pure and Appl. Math., **12**(12): 1468 (1981); Helvetica Physica Acta, **58**: 1072 (1985).
- [11] V. R. Pandharipande, Nucl. Phys. **A446**: 189c (1985).
- [12] M. Viviani, L. Girlanda, A. Kievsky, and L. E. Marcucci, Phys. Rev. Lett. **111**, 172302 (2013).
- [13] D. Lonardonì, S. Gandolfi, and F. Pederiva, Phys. Rev. C **87**, 041303(R) (2013).
- [14] D. Lonardonì, F. Pederiva, and S. Gandolfi, Phys. Rev. C **89**, 014314 (2014).

- [15] A. Cipollone, C. Barbieri, and P. Navrátil, *Phys. Rev. Lett.* **111**, 062501 (2013).
- [16] R. B. Wiringa, R. Schiavilla, Steven C. Pieper, and J. Carlson, *Phys. Rev. C* **89**, 024305 (2014).
- [17] Bruce R. Barrett, Petr Navrátil and James P. Vary, *Progress in Particle and Nuclear Physics*, **69**: 131 (2013).
- [18] C. Forssén, P. Navrátil and S. Quaglioni, *Few-Body Systems* **49**: 11 (2011).
- [19] D. Gogny, P. Pires and R. De Tourreil, *Phys. Lett.*, **32B**: 591 (1970); T. M. Hu and H. S. W. Massey, *Proc. R. Soc.*, **196** : 135 [1949].
- [20] S. B. Doma, N. A. El-Nohy and K. K. Gharib, *Helvetica Physica Acta*, **69**: 90 (1996).
- [21] M. Beiner, H. Flocard, N. Van Giai et al. *Nucl. Phys. A* **238** (1975).
- [22] Reid R. V., *Ann. Phys.*, **50**, 411 [1968].
- [23] R. Schiavilla, V. R. Pandharipande, and R. B. Wiringa, *Nucl. Phys. A* **449**: 219 (1986).
- [24] S. B. Doma, *High Energy Physics and Nuclear Physics*, **26**(9): 941 (2002).
- [25] S. B. Doma, *International Journal of Modern Physics E*, **12**(3): 421 (2003).
- [26] R. B. Wiringa, *Phys. Rev. C* **43**, No. 4: 1585 (1991).
- [27] S. B. Doma and F. N. El-Gammal, *Journal Applied Mathematics & Information Sciences*, **5**(3): 315S (2011).
- [28] S. B. Doma and F. N. El-Gammal, *Acta Physica Polonica A*, **122** (1): 42 (2012).
- [29] S. B. Doma and F. N. El-Gammal, *Journal of Theoretical and Applied Physics*, **6**: 28 (2012).
- [30] R. B. Wiringa and V. R. Pandharipande, *Nucl. Phys. A* **317**: 1 (1979).
- [31] [31] J. A. Carlson and R. B. Wiringa, *Computational Nuclear Physics* **1**: 171, Springer-Verlag, Berlin (1991).
- [32] D. Bressanini and P. J. Reynolds, *J. Chem. Phys.*, Vol. **111**, No. 14 (1999).
- [33] I. E. Lagaris and V. R. Pandharipande, *Nucl. Phys. A* **359**: 349 (1981).
- [34] R. B. Wiringa, J. L. Friar, B. F. Gibson, G. L. Payne and C. R. Chen, *Phys. Lett.* **143B**: 273 (1984)
- [35] C. R. Chen, G. L. Payne, J. L. Friar and B. F. Gibson, preprint LA-UR, 85: 1472
- [36] S. Ishikawa, T. Sasakawa, T. Sawada and T. Ueda, *Phys. Rev. Lett.* **53**: 1877 (1984).
- [37] P. U. Sauer, *Prog. Part. Nucl. Phys.* **16**: 35 (1986).
- [38] N. Auerbach et al., *Rev. Mod. Phys.* **44**: 48 (1972).
- [39] V. G. Neudachin and Yu. F. Smirnov, *Nucleon Association in Light Nuclei*, Nauka, Moscow (1969), in Russian.

- [40] S. B. Doma and I. Z. Machabeli, Proceedings of Tbilisi University, **A9 (157)** (1975).
- [41] D. C. Zheng, J. P. Vary and B. R. Barrett, Physical Review C, **50**: 2841 (1994).
- [42] Samuel S. M. Wong, **Introductory Nuclear Physics**, Wiley-Vch Verlag Gmbl I & Co. KGaA, Weinheim (2004).
- [43] I. Sick, Proc. of the International Symposium, George Washington University, Washington D. C., April 24-26 (1986).
- [44] D. C. J. Marsden, P. Navrátil, S. A. Coon, and B. R. Barrett, Phys. Rev. C **66**, 044007 (2002)
- [45] P. Navrátil, V. G. Gueorguiev, J. P. Vary, W. E. Ormand, and A. Nogga, Phys. Rev. Lett. **99**, 042501 (2007)
- [46] S. Quaglioni and P. Navrátil, Phys Lett. B **652**, 370 (2007).
- [47] A. Nogga, A. Kievsky, H. Kamada, W. Glöckle, L. E. Marcucci, S. Rosati, and M. Viviani, Phys. Rev. C **67**, 034004 (2003)
- [48] A. Nogga, H. Kamada, and W. Glöckle, Phys. Rev. Lett **85**, 944 (2000)
- [49] A Kievsky, S Rosati, M Viviani, L E Marcucci and L Girlanda, J. Phys. G: Nucl. Part. Phys. **35**, 063101 (2008)
- [50] W. Leidemann and G. Orlandini, Progress in Particle and Nuclear Physics **68**, 158 (2013).