SCIREA Journal of Physics

http://www.scirea.org/journal/Physics

December 26, 2016

Volume 1, Issue 2, December 2016

Convergence of Approximate Solution of Nonlinear Volterra-Fredholm Integral Equations

Monireh Nosrati Sahlan*, Hamid Reza Marasi

Department of Mathematics and Computer Science, Technical Faculty, University of Bonab, Box 55517-61167, Bonab, Iran

Email: nosrati@bonabu.ac.ir (corresponding author), hamidreza.marasi@gmail.com

Tel: +98 413 774 5000, Fax: +98 413 772 1066

Abstract

In this study, an effective technique upon compactly supported semi orthogonal cubic B-spline wavelets for solving nonlinear Volterra-Fredholm integral equations is proposed. Properties of B-spline wavelets and function approximation by them are first presented and the exponential convergence rate of the approximation, $O(2^{-4j})$, is proved. For solving the nonlinear Volterra-Fredholm integral equation, the unknown function of problem is approximated by cubic B-spline wavelets. Then Properties of these functions are used to reduce nonlinear mixed integral equation to some algebraic system. For solving the mentioned system, Galerkin and collocation methods are applied. In the both methods, Cubic B-spline wavelets are used as testing and weighting functions. Convergence and error analysis of the method is described through some proved theorems. Because of having vanishing moments, compact support and semi orthogonality properties of these wavelets, operational matrices of the Galerkin and collocation methods are very sparse. In fact the entries with significant magnitude are in the diagonal of operational matrices, and other entries are very small and hence can be set to zero without significantly affecting the solution. Because of having low

memory requirement, high speed and accuracy of the method, the presented procedure is more practical with respect to many of other methods for solving this class of integral equations. The method is computationally attractive and applications are demonstrated through illustrative examples. As is shown in the reported tables of examples, compare the error of three methods, we can find that the presented method get better approximate solution.

Keywords: Fredholm-Volterra-Hammerstein integral equations, collocation method, Galerkin method, Cubic B-spline wavelets, error analysis

INTRODUCTION

Various problems in physics, mechanics and biology arise to a nonlinear mixed type Volterra–Fredholm integral equation. Such equations also appears in modeling of the spatio-temporal development of an epidemic, theory of parabolic initial boundary value problems, population dynamics, and Fourier problems, see; [1],[2] and [3].

Several numerical methods for solving Volterra-Fredholm integral equations are presented. Brunner [4], Guoqiang [5] and Kumar [6] applied different kinds of collocation method for numerical solution of nonlinear Volterra integral equations. A variation of Nystrom's method for Hammerstein equations is presented by Lardy [7]. Yalcinbas [8] used Taylor polynomial method for approximating the solution of integral equation (1). Linear case of equation (1) is solved with continuous time collocation method by Kauthen [9]. Hacia [10, 11] used projection methods for solving linear Fredholm-Volterra integral equations. Methods based on Adomian decomposition series for approximation the solution of equation had been presented by Maleknejad et al. [12] and Wazwaz [13]. Other numerical methods for solving this class of integral equations such as homotopy perturbation method, Chebyshev collocation and Legendre wavelets methods are discussed by Yildirim [14], Banifatemi et al. [15], Hadizadeh and Asgari [16] and Tricomi [17]. In the present study cubic B-spline wavelets are applied to numerical solution of the second kind nonlinear Fredholm-Volterra-Hammerstein integral equation of the form

$$y(x) = f(x) + \int_0^1 K_1(x,t)G_1(t,y(t))dt + \int_0^x K_2(x,t)G_2\big(t,y(t)\big)dt, \quad 0 \le t, x \le 1, \tag{1}$$

where f, K_1 and K_2 are known L^2 functions, with $G_1(t, y(t))$ and $G_2(t, y(t))$ nonlinear in y, the unknown function that to be determined. Our method consists of reducing the given nonlinear Volterra-Fredholm integral equation to a set of algebraic equations by expanding the unknown function by B-spline wavelets with unknown coefficients. Galerkin and collocation methods are utilized to evaluate the unknown coefficients. The use of semi orthogonal compactly supported spline wavelets is justified by their interesting properties. Among them, the following can be explicitly cited [18], they satisfy all the properties on a bounded interval that are verified by the usual wavelets on the real line, but they do not present the difficulties related to the boundary conditions, when applying such wavelets for problems in finite bounded domains, unlike most of the continuous orthogonal wavelets. Also, the semi orthogonal compactly supported spline wavelets have closed form expressions. In [19], the two categories of wavelets, orthogonal and semi orthogonal are compared, and it is shown that semi orthogonal wavelets are best suited for integral equation applications. Among conventional numerical methods for solving integral equations, the collocation method receives more favorable attention from engineering applications due to lower computational cost generating the coefficient matrix of the corresponding discrete equations. But because of semi orthogonality, compact support and having vanishing moment's properties of these wavelets, the operational matrix corresponding to Galerkin method is very sparse. Thus applying the method presented in this paper determines a strong reduction in the computation time and memory requirement in inverting the matrix.

B-spline scaling and wavelet functions

The general theory and basic concepts of the wavelet theory and MRA is given by Chui [20, 21], Mallat [22] and Daubechies [23]. Wavelets and scaling functions are defined on the entire real line so that they could be outside of the integration domain. This behavior may be required an explicit enforcement of the boundary conditions. In order to avoid this occurrence semi orthogonal compactly supported spline wavelets, constructed for the bounded interval $[0,1]$, have been taken into account in this paper. These wavelets satisfy all the properties verified by the usual wavelets on the real line.

Definition 1: Let m and n be two positive integers and

$$a = \mathrm{x}_{-m+1} = \cdots = \mathrm{x}_0 < \mathrm{x}_1 < \cdots < x_n = \mathrm{x}_{n+1} = \cdots =< x_{n+m-1} = b,$$

be an equally spaced knots sequence. The functions

$$B_{m,j,X}(x) = \frac{x - x_j}{x_{j+m-1} - x_j} B_{m-1,j,X}(x) + \frac{x_{j+m} - x}{x_{j+m} - x_{j+1}} B_{m-1,j,X}(x),$$

$$j = -m + 1, \cdots, n - 1,$$

and

$$B_{1,j,X}(x) = \begin{cases} 1, & x_j \le x < x_{j+1}, \\ 0, & O.W., \end{cases}$$

are called cardinal B-spline functions of order $m \ge 2$ for the knot sequence $X = \{x_i\}_{-m+1}^{n+m-1}$, and Supp $B_{m,j,X}(x) = [x_j, x_{j+m}] \cap [a, b]$.

For the sake of simplicity, suppose $[a, b] = [0, n]$ and $x_k = k, k = 0, \cdots, n$. The $B_{m,j,X} = B_m(x - j), j = 0, \cdots, n - m$, are interior B-spline functions, while the remaining $B_{m,j,X}, j = -m + 1, \cdots, -1$ and $j = n - m + 1, \cdots, n - 1$ are boundary B-spline functions, for the bounded interval $[0, n]$. Since the boundary B-spline functions at 0 are symmetric reflections of those at n, it is sufficient to construct only the first half functions by simply replacing x with $n - x$.

By considering the interval $[a, b] = [0,1]$, at any level $j \in Z^+$, the discretization step is 2^{-j}, and this generates $n = 2^j$ number of segments in $[0,1]$ with knot sequence

$$X^{(j)} = \begin{cases} x_{-m+1} = \cdots = x_0 = 0, \\ x_k = \dfrac{k}{2^j}, \quad k = 1, \cdots, n - 1, \\ x_n = \cdots = x_{n+m-1} = 1, \end{cases}$$

Let j_0 be the level for which $2^j \ge 2m - 1$, for each level $j \ge j_0$ the scaling functions of order m can be defined as

$$\varphi_{m,k}^{(j)}(x) = \begin{cases} B_{m,j_0,k}(2^{j-j_0}x), & k = -m + 1, \cdots, -1, \\ B_{m,j_0,2^j-m-k}(1 - 2^{j-j_0}x), & k = 2^j - m - 1, \cdots, 2^j - 1, \\ B_{m,j_0,0}(2^{j-j_0}x - 2^{j_0}k), & k = 0, \cdots, 2^j - m, \end{cases}$$

And the two-scale relation for the m -order semi orthogonal compactly supported B-wavelet functions are defined as

$$\psi_{m,j,i-m}(x) = \sum_{k=i}^{2i+2m-2} q_{i,k} B_{m,j,k-m}, \quad i = 1, \cdots, m - 1, \tag{2}$$

$$\psi_{m,j,i-m}(x) = \sum_{k=2i-m}^{2i+2m-2} q_{i,k} B_{m,j,k-m}, \quad i = m, \cdots, n-m+1, \tag{3}$$

$$\psi_{m,j,i-m}(x) = \sum_{k=2i-m}^{i+m+n-1} q_{i,k} B_{m,j,k-m}, \quad i = n-m+2, \cdots, n-1, \tag{4}$$

where $q_{i,k} = q_{k-2i}$. Hence, there are $2(m-1)$ boundary wavelets and $n-2m+2$ inner wavelets in the boundary interval $[a,b]$. Finally by considering the level j with $j \geq j_0$, the B-wavelet functions in [0,1] can be expressed as follows

$$\psi_{m,j,i}(x) = \begin{cases} \psi_{m,j_0,i}(2^{j-j_0}x), & i = -m+1, \cdots, -1, \\ \psi_{m,2^j-2m+1-i,i}(1-2^{j-j_0}x), & i = 2^j-2m+2, \cdots, 2^j-m, \\ \psi_{m,j_0,0}(2^{j-j_0}x - 2^{j_0}k), & i = 0, \cdots, 2^j-2m+1, \end{cases} \tag{5}$$

The scaling functions $\varphi_{m,k}^{(j)}(x)$, occupy m segments and the wavelet functions $\psi_{m,j,i}(x)$ occupy $2m-1$ segments. Therefore the condition $2^j \geq 2m-1$, must be satisfied in order to have at least one inner wavelet.

Some of the important properties relevant to the present work are given below.

(1) Vanishing moments: a wavelet $\psi(x)$ is said to have a vanishing moments of order m if

$$\int_{-\infty}^{+\infty} x^p\, \psi(x) dx = 0; \qquad p = 0,1, \dots, m-1.$$

All wavelets must satisfy the above condition for $p = 0$. Cubic B-spline wavelet has four vanishing moments. That is

$$\int_{-\infty}^{+\infty} x^p\, \psi(x) dx = 0; \qquad p = 0,12,3.$$

(2) Semi orthogonality: the wavelets $\psi_{j,k}(x)$ form an semiorthogonal basis if

$$\langle \psi_{j,k}, \psi_{s,i} \rangle = 0; \qquad j \neq s.$$

Cubic B-spline wavelets are semi orthogonal.

Cubic B-spline scaling and wavelet functions on [0, 1]

Cubic B-spline scaling function $\varphi_4(x)$ is given by [24]

$$\varphi_4(x) = \frac{1}{6}\sum_{k=0}^{4}\binom{4}{k}(-1)^k\,(x-k)_+^3, \tag{6}$$

where

$$(x)_+^n = \begin{cases} x^n, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

And its two-scale dilation equation defined as

$$\varphi_4(x) = \sum_{k=0}^{4}\frac{1}{8}\binom{4}{k}\,\varphi_4(2x-k).$$

In this section, the scaling functions used in this work, for $j = 3$ and $m = 4$, are reported.

Boundary scalings

Left boundary cubic B-spline scaling functions are constructed by the following formula

$$\varphi_{3,k}(x) = \varphi_4(8x-k).\chi_{[0,1]}(x), \qquad k = -3,-2,-1, \tag{7}$$

and for other levels of j, we have

$$\varphi_{j,k}(x) = \varphi_4\left(2^j x-k\right).\chi_{[0,1]}(x), \qquad k = -3,-2,-1, \qquad j = 4,5,\cdots, \tag{8}$$

left and right boundary scaling functions are symmetric with respect to 0, so right boundary scalings are constructed by

$$\varphi_{3,5}(x) = \varphi_{3,-1}(1-x), \tag{9}$$

$$\varphi_{3,6}(x) = \varphi_{3,-2}(1-x), \tag{10}$$

$$\varphi_{3,7}(x) = \varphi_{3,-3}(1-x), \tag{11}$$

and for other levels of j, we have

$$\varphi_{j,2^j-k-3}(x) = \varphi_{3,k}\left(2^j x-k\right), \qquad k = -3,-2,-1, \qquad j = 4,5,\cdots. \tag{12}$$

Inner scalings

Inner cubic B-spline scaling functions are constructed by the following formula

$$\varphi_{3,k}(x) = \varphi_4(8x-k).\chi_{[0,1]}(x), \qquad k = 0,\cdots,4, \tag{13}$$

and for other levels of j, we have

$$\varphi_{j,k}(x) = \varphi_4\left(2^j x-k\right).\chi_{[0,1]}(x), \qquad k = 0,\cdots,2^j-4, \qquad j = 4,5,\cdots. \tag{14}$$

Fig. 1 is helpful to get a geometric understanding boundary and inner cubic B-spline scaling functions.

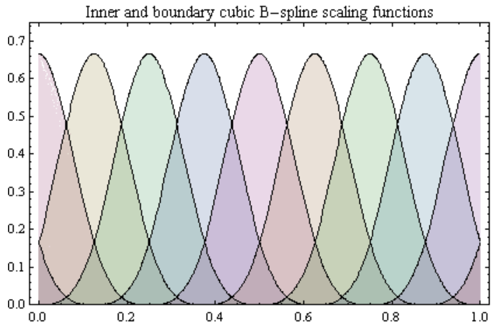

Figure 1: Cubic B-spline Inner and boundary scaling functions

Two scale dilation equation for cubic B-spline wavelet is given by

$$\psi_4(x) = \sum_{k=0}^{10} \frac{(-1)}{8} \sum_{l=0}^{4} \binom{4}{l} \varphi_8(k-l+1)\, \varphi_4(2x-k). \tag{15}$$

Similarly, cubic B-spline inner and boundary wavelet functions are constructed by equations (2)-(5). Figure 2 shows cubic B-spline inner and boundary wavelet functions.

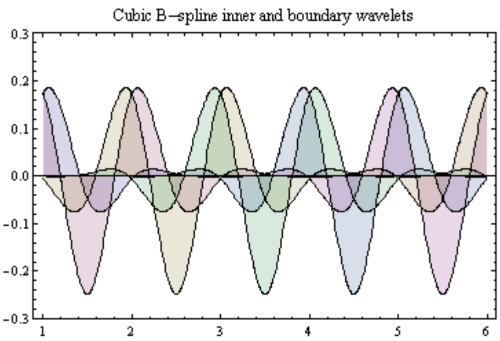

Figure 2: Cubic B-spline inner and boundary wavelets

Function approximation

A function $f(x)$ defined over $[0,1]$ may be approximated by cubic B-spline wavelets as

$$f(x) = \sum_{i=-3}^{2^{j_0}-1} c_{j_0,i} \Phi_{j_0,i}(x) + \sum_{j=j_0}^{\infty} \sum_{k=-3}^{2^k-4} d_{j,k}\, \psi_{j,k}(x), \tag{16}$$

where $\Phi_{j_0,i}$ and $\psi_{j,k}$ are scaling and wavelets functions, respectively. If the infinite series in equation (16) is truncated, then it can be written as

$$f(x) = \sum_{i=-3}^{2^{j_0}-1} c_{j_0,i}\Phi_{j_0,i}(x) + \sum_{j=j_0}^{j_u}\sum_{k=-3}^{2^j-4} d_{j,k}\,\psi_{j,k}(x) = C^T\Psi \tag{17}$$

where C and Ψ are $2^{j_u+1}+3$ column vectors given by

$$C = \left(c_{j_0,i}, \cdots, c_{j_0,2^{j_0}-1}, d_{j_0,-3}, \cdots, d_{j_0,2^j-4}, \cdots, d_{j_0,2^j-4}\right), \tag{18}$$

$$\Psi(x) = \left(\Phi_{j_0,-3}, \dots, \Phi_{j_0,2^{j_0}-1}, \psi_{j_0,-3}, \dots, \psi_{j_0,2^{j_0}-4}, \dots, \psi_{j_u,2^{j_u}-4}\right), \tag{19}$$

where

$$c_{j_0,i} = \int_0^1 f(x)\,\widetilde{\Phi}_{j_0,i}(x)dx, \qquad i = -3, \cdots, 2^{j_0}-1,$$

$$d_{j,k} = \int_0^1 f(x)\,\tilde{\psi}_{j,k}(x)dx, \qquad j = j_0, \cdots, j_u\,, \qquad k = -3, \cdots, 2^j-4,$$

and $\widetilde{\Phi}_{j_0,i}$ and $\tilde{\psi}_{j,k}$ are dual functions of $\Phi_{j_0,i}, i = -3, \dots, 2^{j_0}-1$ and $\psi_{j,k}, j = j_0, \cdots, j_u, k = -3, \cdots, 2^j-4$, respectively. These can be obtained by linear combinations of $\Phi_{j_0,i}$ and $\psi_{j,k}$. Let

$$\Phi(x) = \left(\Phi_{j_0,-3}, \Phi_{j_0,-2}, \cdots, \Phi_{j_0,2^{j_0}-1}\right), \tag{20}$$

$$\Upsilon(x) = \left(\psi_{j_0,-3}, \cdots, \psi_{j_0,2^{j_0}-4}, \dots, \psi_{j_u,-3} \cdots, \psi_{j_u,2^{j_u}-4}\right), \tag{21}$$

using equations (7)-(15) and (20), we get

$$\int_0^1 \Phi(x)\Phi^T(x)dx = P_1, \tag{23}$$

and from equations (2)-(5) and (21), we have

$$\int_0^1 \Upsilon(x)\Upsilon^T(x)dx = P_2, \tag{24}$$

where P_2 is $(2^{j_u+1}-8) \times (2^{j_u+1}-8)$ matrix. Suppose $\widetilde{\Phi}(x)$ and $\widetilde{\Psi}(x)$ are the dual functions of $\Phi(x)$ and $\Psi(x)$, respectively, given by

$$\widetilde{\Phi}(x) = \left(\widetilde{\Phi}_{j_0,-3}, \widetilde{\Phi}_{j_0,-2}, \cdots, \widetilde{\Phi}_{j_0,2^{j_0}-1}\right), \tag{25}$$

$$\tilde{\Upsilon}(x) = \left(\tilde{\psi}_{j_0,-3}, \cdots, \tilde{\psi}_{j_0,2^{j_0}-4}, \dots, \tilde{\psi}_{j_u,-3} \cdots, \tilde{\psi}_{j_u,2^{j_u}-4}\right), \tag{26}$$

using equations (23)-(26), we get

$$\int_0^1 \tilde{\Phi}(x)\Phi^T(x)dx = I_{2^{j_0}+3}, \qquad \int_0^1 \tilde{\Upsilon}(x)\Upsilon^T(x)dx = I_{2^{j_u+1}-8},$$

$I_{2^{j_0}+3}$ and $I_{2^{j_u+1}-8}$ are identity matrices. So that

$$\tilde{\Phi}(x) = P_1^{-1}\Phi(x), \qquad \tilde{\Psi}(x) = P_2^{-1}\Upsilon(x).$$

Numerical implementation

In this section, we solve the integral equation of the form (1) by using Galerkin and collocation method based on cubic B-spline wavelets. For this purpose, the unknown functions in equation (1) is expanded in term of the selected scaling and wavelet functions as follows

$$y(x) = C^T\Psi(x), \quad G_1(x, y(x)) = A_1^T\Psi(x), \quad G_2\big(x, y(x)\big) = A_2^T\Psi(x), \tag{27}$$

also the known functions in equation (1) can be expanded in term of the selected dual scaling and wavelet functions as follows

$$f(x) = D^T\Psi(x), \quad K_1(x,t) = \tilde{\Psi}^T(t)H_1\tilde{\Psi}(x), \quad K_2(x,t) = \tilde{\Psi}^T(t)H_2\tilde{\Psi}(x), \tag{28}$$

where C, A_1, A_2 and D are $(2^{j_u+1}+3)$- column vectors, as (18), H_1 and H_2 are $(2^{j_u+1}+3)\times(2^{j_u+1}+3)$ square matrices, defined as

$$[H_i]_{mn} = \int_0^1 \left(\int_0^1 K_i(x,t)\,\Psi_m(t)dt\right)\Psi_n(x)dx, \quad i = 1,2,$$

and Ψ_m is the $m-$ element of the column vector Ψ. Using equations (27) and (28), we get

$$\int_0^1 K_1(x,t)\,G_1\big(t, y(t)\big)dt = \int_0^1 A_1^T\Psi(t)\tilde{\Psi}^T(t)H_1\tilde{\Psi}(x)\,dt$$

$$= A_1^T\left(\int_0^1 \Psi(t)\tilde{\Psi}^T(t)dt\right)H_1\tilde{\Psi}(x) = A_1^T H_1\tilde{\Psi}(x),$$

$$\int_0^x K_2(x,t)\,G_2\big(t, y(t)\big)dt = \int_0^x A_2^T\Psi(t)\tilde{\Psi}^T(t)H_2\tilde{\Psi}(x)\,dt$$

$$= A_2^T \left(\int_0^x \Psi(t)\widetilde{\Psi}^T(t)dt \right) H_2 \widetilde{\Psi}(x) = A_2^T \Lambda(x) H_2 \widetilde{\Psi}(x),$$

where

$$\Lambda(x) = \int_0^x \Psi(t)\widetilde{\Psi}^T(t)dt.$$

By substituting current expressions in equation (1) and computing the residual function, we get

$$R(x) = C^T\Psi(x) - D^T\widetilde{\Psi}(x) - A_1^T H_1 \widetilde{\Psi}(x) - A_2^T \Lambda(x) H_2 \widetilde{\Psi}(x), \tag{29}$$

equation (29) has $3 \times (2^{j_u+1} + 3)$ unknowns. For solving this system of nonlinear equations, first we apply Galerkin method via cubic B-spline scaling and wavelet functions as weighting functions. For this purpose we put $\langle R(x), \Upsilon(x) \rangle = 0$. That is, equation (29) is multiplied by $\Psi^T(x)$, then is integrated from 0 to 1, so we have

$$C^T P = D^T - A_1^T H_1 - A_2^T \Gamma, \tag{30}$$

in which P is $(2^{j_u+1} + 3) \times (2^{j_u+1} + 3)$ square matrix, given by

$$P = \begin{pmatrix} P_1 & \\ & P_2 \end{pmatrix}$$

and

$$\Gamma = \int_0^1 \Lambda(x) H_2 \widetilde{\Psi}(x)\, \Psi^T(x) dx.$$

Equation (30) is a nonlinear system of algebraic equations with $3 \times (2^{j_u+1} + 3)$ unknowns and $\left(2^{j_u+1} + 3\right)$ equations. For having unique solution we need $2 \times (2^{j_u+1} + 3)$ equations, too. These new equations are generated by collocation method. Now, we are collocate the equations

$$A_1^T \Psi(x) = G_1\big(x, C^T\Psi(x)\big), \qquad A_2^T \Psi(x) = G_2\big(x, C^T\Psi(x)\big),,$$

in the following points

$$x_m = \frac{m}{2^{j_u+1}+3}, \; m = 1,2,\cdots, 2^{j_u+1} + 3.$$

The current equations generate a set of $3 \times (2^{j_u+1} + 3)$ algebraic equations that could be easily solved by one of iterative methods.

Convergence and error estimate

In this section, we found an error bound for the presented method.

Theorem 1: We assume that $f \in C^4[0,1]$ is represented by cubic B-spline wavelets as (18), where ψ_4 has 4 vanishing moments, then

$$\left|d_{j,k}\right| \leq \alpha\beta \frac{2^{-5j}}{4!}, \tag{31}$$

where $\alpha = max\left|f^{(4)}(t)\right|_{0\leq t\leq 1}$ and $\beta = \int_0^1 x^4 \tilde{\psi}_4(x)dx$.

Proof: Taylor expansion of $f \in C^4[0,1]$ in arbitrary $x_0 \in [0,1]$ can be written as

$$f(x) = \sum_{l=0}^{3} \frac{(x-x_0)^l}{l!} f^{(l)}(x_0) + \frac{(x-x_0)^4}{4!} f^{(4)}(\xi), \qquad \xi \in D_f, \tag{32}$$

and $f(x)$ may be represented by cubic B-spline wavelets as equation (17) where

$$d_{j,k} = \int_0^1 f(x)\, \tilde{\psi}_{j,k}(x)dx, \tag{33}$$

with substituting equation (32) in equation (31) we get

$$d_{j,k} = \int_0^1 \sum_{l=0}^{3} \frac{(x-x_0)^l}{l!} f^{(l)}(x_0)\, \tilde{\psi}_{j,k}(x)dx + \int_0^1 \frac{(x-x_0)^4}{4!} f^{(4)}(\xi)\, \tilde{\psi}_{j,k}(x)dx. \tag{34}$$

By putting $x_0 = \frac{k}{2^j}$ and $u = 2^j x - k$ in the first integral of equation (34), we have

$$d_{j,k} = \sum_{l=0}^{3} \frac{f^{(l)}\left(\frac{k}{2^j}\right)}{2^{j(l+1)} l!} \int_0^1 u^l\ \tilde{\psi}_{j,k}(u)du + \int_0^1 \frac{(x-x_0)^4}{4!} f^{(4)}(\xi)\, \tilde{\psi}_{j,k}(x)dx,$$

Now suppose T be the linear transformation that

$$T\psi = \tilde{\psi},$$

thus we get

$$d_{j,k} = \frac{f^{(4)}(\xi)}{4!} 2^{-5j} \int_0^1 x^4\, \tilde{\psi}_{j,k}(x)dx,$$

Because ψ has 4 vanishing moments then,

$$\sum_{l=0}^{3} \frac{f^{(l)}\left(\frac{k}{2^j}\right)}{2^{j(l+1)} l!} T\left(\int_0^1 u^l\, \psi_{\ j,k}(u)du\right) = 0,$$

and

$$\psi_{j,k}(x) = \psi\left(2^j x - k\right),$$

Thus proof is completed.

■

Theorem 2: Consider the pervious theorem, assume that $e_j(x)$ be the error of approximation in V_j, then

$$\left|e_j(x)\right| = O\left(2^{-4j}\right).$$

Proof: By using equations (16) and (17), we get

$$e_j(x) = \sum_{l=j}^{\infty} \sum_{k=-3}^{2^l-4} d_{l,k}\, \psi_{l,k}(x),$$

by putting

$$C_l = max\left\{\left|\psi_{l,k}(x)\right|_{x\in[0,1]}, k = -3, \dots, 2^l - 4\right\},$$

we get

$$\left|d_{l,k}\psi_{l,k}\right| \leq \alpha\beta C_l \frac{2^{-5l}}{4!},$$

and

$$\sum_{k=-3}^{2^l-4} \left|d_{l,k}\psi_{l,k}\right| \leq \alpha\beta C_l \frac{2^{-4l}}{4!},$$

by the current inequality we get

$$\left|e_j(x)\right| = O\left(2^{-4j}\right). \qquad \blacksquare$$

Therefore, order of error depends on the level j. Obviously, for larger level of j, the error of approximation will be smaller.

Theorem 3 [25, 26]: For the $m-th$ order B-spline wavelet the approximation error decreases with the $m-th$ power of the 2^j ,

$$\left\|f - P_j f\right\| \leq C 2^{-jm} \left\|f^{(m)}\right\|.$$

Specifically we can derive the following asymptotic relation [27],

$$lim_{j\to\infty} \left\|f - P_j f\right\| = C_m 2^{-jm} \left\|f^{(m)}\right\|,$$

where the constant C_m is the same for all spline wavelet transforms of a given order m, and is given by

$$C_m = \sqrt{\frac{B_{2m}}{(2m)!}},$$

and B_{2m} is Bernouilli's number of order $2m$.

■

In the above theorem, $\|f\|$ defined as follows

$$\|f\| = \left(\int_0^1 f^2(x)dx \right)^{\frac{1}{2}}.$$

Now we describe the main theorem of this section.

Theorem 4: Assume $K_1,\ K_2 \in L^2$ in rectangle $[0,1] \times [0,1]$ and $G_1,\ G_2 \in C([0,1] \times [0,1])$. If y and y_j are the exact and approximate solution (obtained by m −order B-spline wavelet) of equation (1), respectively, then

$$\|y(x) - y_j(x)\| \leq B2^{-jm}\|y^{(m)}\|.$$

Proof: It is clear that

$$y(x) - y_j(x) = \int_0^1 K_1(x,t)G_1(t, P_j y(t))dt + \int_0^x K_2(x,t)G_2\left(t, P_j y(t)\right)dt, \tag{25}$$

subtracting equation (25) from (1), we get

$$y(x) - y_j(x) = I_1 + I_2,$$

where

$$I_1 = \int_0^1 K_1(x,t)\left(G_1\big(t, y(t)\big) - G_1\left(t, P_j y(t)\right)\right)dt,$$

$$I_2 = \int_0^x K_2(x,t)\left(G_2\big(t, y(t)\big) - G_2\left(t, P_j y(t)\right)\right)dt,$$

then we can write

$$\|y(x) - y_j(x)\| \leq \|I_1\| + \|I_2\|.$$

Now, by Cauchy Schwartz inequality $\|I_1\|$ can be written as

$$\|I_1\| = \left\| \int_0^1 K_1(x,t)\left(G_1\big(t,y(t)\big) - G_1\left(t, P_j y(t)\right)\right) dt \right\|$$

$$\leq M_1 \left(\int_0^1 \left(G_1\big(t,y(t)\big) - G_1\left(t, P_j y(t)\right)\right)^2 dt \right)^{\frac{1}{2}},$$

where

$$M_1 = \sup\{|K_1(x,t)|_{0\leq t\leq 1}, 0 \leq x \leq 1\},$$

on the other hand by the mean value theorem for G_1 we have

$$G_1\big(t,y(t)\big) - G_1\left(t, P_j y(t)\right) \leq A_1 \left|y(t) - P_j y(t)\right|,$$

where

$$A_1 = \sup\left|G_{1_2}(t,s(t))\right|_{0\leq t\leq 1},$$

(G_{1_2} is the derivative of G_1 with respect to the second variable) and

$$s(t) = \max\left\{G_1\big(t,y(t)\big), G_1\left(t, P_j y(t)\right)\right\}_{0\leq t\leq 1},$$

thus

$$\|I_1\| \leq M_1 A_1 C 2^{-jm} \left\|y^{(m)}\right\|,$$

Also for I_2, similarly we can write

$$\|I_2\| = \left\| \int_0^x K_2(x,t)\left(G_2\big(t,y(t)\big) - G_2\left(t, P_j y(t)\right)\right) dt \right\|$$

$$\leq M_2 \left(\int_0^1 \left(G_2\big(t,y(t)\big) - G_2\left(t, P_j y(t)\right)\right)^2 dt \right)^{\frac{1}{2}},$$

where

$$M_2 = \sup\{|K_2(x,t)|_{0\leq t\leq 1}, 0 \leq x \leq 1\}.$$

And

$$G_2\big(t,y(t)\big) - G_2\left(t, P_j y(t)\right) \leq A_2 \left|y(t) - P_j y(t)\right|,$$

where

$$A_2 = \sup\left|G_{2_2}(t, u(t))\right|_{0\le t\le 1},$$

(G_{2_2} is the derivative of G_2 with respect to the second variable) and

$$u(t) = \max\left\{G_2\big(t, y(t)\big), G_2\left(t, P_j y(t)\right)\right\}_{0\le t\le 1},$$

thus

$$\|I_2\| \le M_2 A_2 C 2^{-jm}\left\|y^{(m)}\right\|.$$

Putting $B = (M_1 A_1 + M_2 A_2)C$, proof is completed.

■

Illustrative examples

In this section, for showing the accuracy and efficiency of the described method we present some examples. In tables the infinity norm of error is defined as follows

$$\|e\|_{L_\infty} = max\{\left|y - y_j\right|, 0 \le t \le 1\},$$

where y and y_j are the exact and approximated solution for $j_u = j$.

Example 1: Consider the nonlinear two-point boundary value problem

$$u''(t) - e^{u(t)} = 0, \qquad 0 \le t \le 1, \qquad u(0) = u(1) = 0,$$

which evidently is of some interest in magneto hydrodynamics [28]. This equation can be reformulated as

$$u(t) = \int_0^1 K(t,s) e^{u(s)} ds,$$

where

$$K(t,s) = \begin{cases} -s(1-t), & s \le t, \\ -t(1-s), & t \le s, \end{cases}$$

The exact solution of this integral equation is

$$u(t) = Ln \frac{c^2}{2Cos^2(\frac{c}{2}(t - \frac{1}{2}))},$$

and c is the root of equation $\frac{c^2}{2} = Cos\left(\frac{c}{4}\right)$. Table 1 represents the error estimates using the method of [6] together with the results obtained for maximum errors by the present method.

Example 2: Consider the equation [29]

$$u(t) = -\frac{1}{30}t^6 + \frac{1}{3}t^4 - t^2 + \frac{5}{3}t - \frac{5}{4} + \int_0^t (t-s)(u(s))^2 ds + \int_0^1 (s+t)u(s)ds,$$

Methods	$\|e\|_{L_\infty}$
Method of [6]	
$N = 17$	5.61×10^{-3}
$N = 33$	1.44×10^{-4}
$N = 65$	3.66×10^{-5}
Presented method $j_0 = 3$	
$j_u = 3$	0.78×10^{-6}
$j_u = 4$	2.36×10^{-8}
$j_u = 5$	3.19×10^{-10}

Table 1: infinity norm of errors of numerical solutions of example 1in different scales of j_u

with the exact solution $u(t) = t^2 - 2$. Table 2 present exact and approximation solution for $u(t)$, obtained by the method in section 4 at the octave level $j_0 = 3$ and at the levels $j_u = 3$, 4 and 5. The results are compared with the result of rationalized Haar functions method [29].

Example 3: Consider the nonlinear Fredholm-Volterra integral equation with the exact solution $u(t) = e^t$.

$$u(t) = \frac{1}{4} + e^t - \frac{1}{3}t^5 - \frac{1}{4}e^4 + \int_0^t t^2 sLn\big(u(s)\big)ds + \int_0^1 e^{s-t}(u(s))^3 ds,$$

		approximate		R.H.F*
x	$j_u = 3$	$j_u = 4$	$j_u = 5$	$k = 16$
0	1.01×10^{-6}	1.18×10^{-8}	0.12×10^{-11}	0.81×10^{-6}
0.2	1.34×10^{-6}	1.96×10^{-8}	0.23×10^{-11}	1.02×10^{-6}
0.4	1.53×10^{-6}	1.85×10^{-8}	0.19×10^{-11}	1.17×10^{-6}
0.6	2.27×10^{-6}	1.63×10^{-8}	0.38×10^{-11}	1.64×10^{-6}

0.8	1.95×10^{-6}	1.57×10^{-8}	0.32×10^{-11}	1.39×10^{-6}
1	2.18×10^{-6}	1.90×10^{-8}	0.44×10^{-11}	1.14×10^{-6}

Table 2: numerical solutions of example 2in different scales of $\mathbf{j_u}$

*R.H.F.: Rationalized Haar Functions

Table 3 present exact and approximation solution for $u(t)$, obtained by the method in section 4 at the octave level $j_0 = 3$ and at the levels $j_u = 3$, 4 and 5. The results are compared with the result of rationalized Haar functions method [29].

		approximate		R.H.F
x	$j_u = 3$	$j_u = 4$	$j_u = 5$	$k = 16$
0	2.27×10^{-6}	3.78×10^{-8}	0.84×10^{-11}	1.63×10^{-6}
0.2	1.92×10^{-6}	4.29×10^{-8}	0.52×10^{-11}	2.58×10^{-6}
0.4	2.04×10^{-6}	5.91×10^{-8}	0.73×10^{-11}	1.95×10^{-6}
0.6	1.85×10^{-6}	3.17×10^{-8}	0.44×10^{-11}	1.57×10^{-6}
0.8	1.97×10^{-6}	4.64×10^{-8}	0.12×10^{-11}	2.13×10^{-6}
1	2.01×10^{-6}	7.12×10^{-8}	0.66×10^{-11}	2.79×10^{-6}

Table 3: numerical solutions of example 3in different scales of $\mathbf{j_u}$

Conclusion

In this paper, we proposed an advanced numerical model in solving nonlinear Fredholm-Volterra-Hammerstein integral equation of the second kind by means of semi orthogonal compactly supported spline wavelets via Galerkin and collocation methods. Based on the consideration reported in tables, the method presented in this paper determines a high accuracy in the solutions of nonlinear Fredholm-Volterra integral equations. The approach can be extended to nonlinear integro-differential equation with little additional work. Further research along these lines is under progress and will be reported in due time.

Acknowledgments

The authors are very grateful to the referees for their valuable suggestions for the improvement of this paper.

References

[1] Pathpatte, B.G., On mixed Volterra–Fredholm integral equations. Indian J. Pure. Appl. Math., 17 (1986), pp. 488–496.

[2] Diekmann, O., Thresholds and traveling for the geographical spread of infection. J Math Biol., 6 (1978), pp. 109-130.

[3] Thieme, H.R., A model for the spatial spread of an epidemic. J. Math. Biol., 4 (1977), pp. 337–351.

[4] Brunner, H., Implicitly linear collocation method for nonlinear Volterra equations. J Appl. Num. Math., 9 (1982), pp. 235-247.

[5] Guoqiang, H., Asymptotic error expansion variation of a collocation method for Volterra Hammerstein equations. J Appl. Num. Math., 13 (1993), pp. 357-369.

[6] Kumar, S and I.H. Sloan., A new collocation-type method for Hammerstein integral equations. J Math. Comput., 48 (1987), pp. 123-129.

[7] Lardy, L.J., A variation of Nystrom's method for Hammerstein equations. Journal of Integral Equations, 3 (1982), pp. 123-129.

[8] Yalcinbas, S., Taylor polynomial solution of nonlinear Volterra–Fredholm integral equations. Appl Math Comput, 127 (2002), pp. 195–206.

[9] Kauthen, J.P., Continuous time collocation method for Volterra-Fredholm integral equations. Numer. Math., 56 (1989), pp. 409-424.

[10] Hacia, L., On approximate solution for integral equations of mixed type. ZAMM Z. Angew. Math. Mech., 76 (1996), pp. 415-416.

[11] Hacia, L., Projection methods for integral equations in epidemic. J Math Model Anal, 7 (2) (2002), pp. 229-240.

[12] Maleknejad, K and M, Hadizadeh., A new computational method for Volterra–Fredholm integral equations. J Comput. Appl Math, 37 (1999), pp. 1–8.

[13] Wazwaz, A.M., A reliable treatment for mixed Volterra–Fredholm integral equations. Appl Math Comput, 127 (2002), pp. 405–414.

[14] Yildirim, A., Homotopy perturbation method for the mixed Volterra–Fredholm integral equations. Chaos Solitons Fractals, 42 (2009), 2760–2764.

[15] Banifatemi, E., M, Razzaghi and S, Yousefi., Two-dimensional Legendre wavelets method for the mixed Volterra-Fredholm integral equations. J Vibr Control, 13 (2007), pp. 1667-1675.

[16] Hadizadeh, M and M, Asgari., An effective numerical approximation for the linear class of mixed integral equations. Appl Math Comput, 167 (2005), pp. 1090–1100.

[17] Tricomi, F.G., Integral Equations. Dover, 1982.

[18] Ala, G., M.L. Silvestre., E, Francomano and A, Tortorici., An advanced numerical model in solving thin-wire integral equations by using semi orthogonal compactly supported spline wavelets. IEEE Trans Electromagn Compact, 45 (2003), pp. 218–228.

[19] Nevels, R.D., J.C. Goswami and H, Tehrani., Semi orthogonal versus orthogonal wavelet basis sets for solving integral equations. IEEE Trans Antennas Propagat, 45 (1997), pp. 1332–1339.

[20] Chui, C., An introduction to wavelets. New York, Academic press, 1992.

[21] Chui, C., Wavelets: a mathematical tool for signal analysis. Philadelphia, PA: SIAM, 1997.

[22] Mallat, S.G., A theory for multiresolution signal decomposition: The wavelet representation. IEEE Trans, Pattern Anal Mach Intell, 11 (1989), pp. 674-693.

[23] Daubechies, I., Ten lectures on wavelets. Philadelphia, PA: SIAM, 1992.

[24] Maleknejad, K., K, Nouri and M, Nosrati Sahlan., Convergence of approximate solution of nonlinear Fredholm Hammerstein integral equations. Commun Nonlinear Sci Num Simul, 15(6) (2010), pp. 1432-1443.

[25] Strang G and G, Fix., A Fourier analysis of the finite element variational method, in: Constructive Aspect of Functional Analysis, Edizioni Cremonese, Rome, 1971, pp. 796-830.

[26] Strang G., Wavelets and dilation equations: a brief introduction. SIAM Review, 31 (1989), pp. 614-627.

[27] Unser, M., 1996. Approximation power of biorthogonal wavelet expansions. IEEE Trans Signal Processing, 44(3) (1996), pp. 519-527.0

[28] Kalman, R.E. and R.E. Kalaba., Quasi linearization and Nonlinear Boundary-Value Problems. Elsevier, New York, 1969.

[29] Ordokhani, Y. and M, Razzaghi., Solution of nonlinear Volterra Fredholm Hammerstein integral equations via a collocation method and rationalized Haar functions. Applied Mathematics Letters, 21 (2008), pp. 4-9.