Analytic mechanics structure of quantum mechanics
and relativity space-time

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Abstract

The least action principle is the basic principle of the Newtonian mechanics and geometrical optics. Various forms of analytical mechanics are based on the principle of least action., including Lagrange mechanics, Hamiltonian mechanics and Hertz mechanics based on the principle of minimum curvature. Max Planck believed that this principle is not limited to a particular coordinate system, it can directly applied on the dynamics and the thermodynamics process without losing its universality. Quantum mechanics starts from action quantization, and its various forms include the mathematical structure of analytical mechanics. Quantum mechanics offers a combination of Newtonian and Einsteinian features. Schrodinger believed that quantum wave function has curvature characteristics.

Keywords: The principle of least action, The principle of least curvature, Non-Euclidean line element.
1. From the least action principle to Schrodinger’s wave mechanics

We can construct a table to show up history development from classical mechanics to quantum mechanics.

<table>
<thead>
<tr>
<th>Time</th>
<th>Physicists or other learners</th>
<th>Articles or works</th>
<th>Principles and Applications</th>
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<tbody>
<tr>
<td>1620</td>
<td>W.R.Snell,1591~1626</td>
<td>Rene Descartes: ‘Refraction optics’(1637)</td>
<td>n₁sinɑ=n₂sinβ, refraction of light</td>
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**Fund Project:**

work principle:
In the balance of all forces, regardless of their mode of action, the sum of the positive energy is always equal to the sum of the negative energy according to the distinction between positive and negative energy.

Later in the early 19th century, Coriolis expressed it as:
Total works of active forces on the virtual displacement is zero under the ideal complete constraints.

Virtual displacement principle:
In terms of the whole object, its internal reaction offset each other, and therefore there is no any contribution for motion, but in fact another set of forces convey movement to the system and make an effective force (Inertia force) statically to be equal to the external forces.[2]

The least action principle:
‘Why should light pass along the shortest distance or the shortest one of two routes? The chosen path should be a real advantage: the least action route.’

‘It is necessary to explain the meaning of action. When an object moves from one point to another point, it must be accompanied by an action. The action depends on the product of the body mass and its movement distance rather than rely on both respectively.’

And De Maupertuis proved that not only when light pass through different media save this action as far as possible, and in a straight propagation, reflection also save the action as much as possible. In latter both cases, the
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<tr>
<td>1744</td>
<td>Leonhard Euler (1707–1783) published his <em>Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes</em> (1744). L. Euler regarded mvds as a particle’s action, he put forward Euler’s principle by variational method: ‘For a particle with a given energy, comparing its true orbit motion between the two fixed points with its adjacent changed orbit with the same energy, there is an infinitesimal difference of two levels between the integrals of the two: ∫vds, that is δ∫vds = 0. He found that real orbit is the orbit whose integral is a stationary value, but this does not guarantee the least action, it may also be the most or neither the most nor the least.’</td>
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<td>1788</td>
<td>J.L. Lagrange (1736–1813) published his <em>Analytic mechanics</em> in 1788, and celebrated a 100 anniversary of Newton’s ‘Mathematical principles of natural philosophy’ (1st version, 1687). W.R. Hamilton (1805–1865) regarded this work as ‘a mathematical poet’. Lagrange started from the least action principle: ∆∫₀²Tdt = 0. And took into account the true path and the variational path have the same energy: ∆[∑(Tᵢ + Vᵢ)] = 0. Finally he obtained an universal equation——d’Alembert-Lagrange principle: ∑ᵢ(fᵢ – maᵢ) · δxᵢ = 0. This equation shows that for the particle system with ideal constraints, the sum of element works of all active force and inertia force acting on a system at any moment in any virtual displacements of the system is zero. In fact, this equation is a combination of</td>
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d’Alembert principle and virtual displacement principle. In a certain condition, we can not only derive the theorem of momentum, theorem of moment of momentum and energy theorem from d’Alembert-Lagrange principle, but also derive further the first and the second Lagrange equation from it, and it is directly used to solve dynamics problems. Jacobi praised it as ‘the mother of all analytical mechanics’ [1,p85-86].

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<td>1826</td>
<td>‘Theory of light system’ (W.R.Hamilton, 1826)</td>
<td>E.L. Malus (1775~1812) found a theorem: If the light emits from a point source (or the surface of the vertical light), after reflection and refraction any times, we can always construct a surface perpendicular to all lights. [1,p93-94] Hamilton considered a fixed point light source, and found that the surface perpendicular to all lights is a constant action surface; then he considered a particle motion from a point or motion of some particles emitting vertically from a surface in a definite velocity, and found that the trajectories of these particle are always orthogonal to some constant action surface. So Hamilton analyzed deeply the Fermat principle in optics and occurrence and development of the Maupetuis principle of particles motion. Based on optical closely reasoning, he obtained the characteristic function suitable for mechanics system: [ W=\int_0^t \delta T \text{d}t ] ( T ) is the kinetic energy of the system. [1,p102] For the particle’s movement in the force field that does not change with time, Hamilton introduced a known quantity ( S ) as the ‘main function’.</td>
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According to Hamilton-Jacobi equation:
\[ \frac{\partial S}{\partial t} + H(t, q, \frac{\partial S}{\partial q}) = 0, \]
Hamilton got an equation [1,p102]:
\[ S(q,p,t) = -Et + w(q,p) \]
Hamilton’s principle means: For any possible movements in a mechanical system at any given time, the real movement is whose Hamiltonian function (the main function) quantity takes extreme value [1,p103].

Hamilton introduced a new function H (it is actually the total energy of the system), and got the canonical equation of system movement in 1835. By introducing the generalized coordinates and generalized momentum, Newtonian motion equation of complex systems would transform into a set of simple and symmetrical equation.

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<td>1837</td>
<td>M.H.Jacobi</td>
<td>Eliminated Lagrange time variable in the least action principle by energy integral, and got the least action principle with characteristics of geometry. Changed the least action principle of Lagrange into path integral form between two points in 3n configuration space. And the potential function V, or the force field was classed as a metric of ( g_{ik} ), made the particles’ motion in a force field in flat space (Euclidean space) transformed into free motion in a non-flat space (generalized non-Euclidean space). As a result, the motion law of n-body system in 3d flat space is represented as a geodesic in a curved surface. Hertz mechanics and Einstein’s general relativity are based on this idea.</td>
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<tr>
<td>1883-1884</td>
<td>H.L.F.Helmholtz,1821-1894</td>
<td>Letter to J.J.Thomson (1856-1940), quoting from L.Koenigsberger, Hermann. Helmholtz found that as long as we give generalized force, generalized speed and generalized acceleration in different physical significance, the abstract relationship can</td>
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<td>1894</td>
<td>H.R. Hertz (1857-1894)</td>
<td>‘Mechanics principle’ (1894)</td>
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<tr>
<td>1913</td>
<td>Albert Einstein (1879-1955)</td>
<td>‘The outline of general relativity and the theory of gravity’ (A.Einstein, 1913)</td>
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it will move along the geodesic line:
\[ \delta [ds^2] = 0, \]
Among it, \( ds^2 = g_{\mu \nu} dx^\mu dx^\nu \), \( g_{\mu \nu} \) is a metric tensor to represent the gravitational field. So as long as metric tensor \( g_{\mu \nu} \) of space-time enters into geodesic equation, we can obtain the equation of motion of particles in the gravitational field:
\[ R_{\mu \nu} = k [T_{\mu \nu} - g_{\mu \nu} T/2] \]
\( R_{\mu \nu} \) is Riemann curvature tensor, \( T_{\mu \nu} \) is material energy-momentum tensor.

Schrodinger noticed the deep thought relation between Einstein and Hertz, he wrote in 1918 an important manuscripts: Hertz mechanics and Einstein's theory of gravity. He said: ‘with the help of a clear link between Hertz mechanics and Einstein gravitational theory, I have to point out the importance of this fact, namely in the two theory,’ force’ is represented as the same mathematical symbols-----the location coordinates of differential mode of flat Riemann three indicator------Christoffel symbols ’ [3]. Just as there is no potential energy in Hertz’s mechanics of the system, the gravitational force is the space-time itself in the general theory of relativity, the degree of curvature of the world line means that the intensity of the gravitational field. As a result, the gravitational field transforms into a sort of geometry. The particle in the gravitational field moves along a geodesic line like a free particle. In these two kinds of theories, the equations of motion make space-time curved, so the world line of the planet is curved. In these two theories, the equations of motion are solved by the path (geodesic), and the only difference lies in the fact that the configuration space of Riemann curvature in Hertz mechanics is generated by the kinetic
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<th>Year</th>
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| 1911 | Arnoud Sommerfeld (1868-1951) | Action quantum (Plank, 1900) + The least action principle → quantum condition of atomic radiation: \[ \int_0^t L \, dt = \hbar \]
|       |                | L is Lagrange function of an electron, t is the duration of the radiation. (A. Sommerfeld, 1911) |
|       |                | Sommerfeld reconstructed Plank’s action quantum hypothesis (1916): \[ \int p \, dq = n_1 \hbar \]
|       |                | Matter wave = Quantum phase wave |
|       |                | Fermat’s principle for phase wave is equivalent to Maupertuis principle for moving objects. The dynamics possible orbits of a moving object are equivalent to possible wavefronts of wave. [4] |
| 1926 | E. Schrodinger (1887-1961) | ‘Quantization as eigenvalue problem’ (E. Schrodinger, 1926) |
|       |                | Hamilton-Jacobi equation |
|       |                | Variational principle |
|       |                | phase wave |
|       |                | → Schrodinger wave equation |
2. Einstein’s non-Euclidean line element theory

In his “Four Lectures on Wave Mechanics”, Schrodinger thought that Hamilton-Maupertuis principle as classical starting point of wave mechanics introduced Heinrich Hertz’s generalized non-Euclidean line element $ds^2=2T(qk ,dqk /dt)dt^2$, when we defined a line element in generalized coordinate $q$ space.\[5,p43\] And finally we obtained wave equation(or amplitude equation) $\nabla^2 \psi +8\pi^2 (E-V)\psi/\hbar^2=0$, where $\nabla^2$ could be regarded as neither a basic Laplacian operator in 3-dimentional space, nor a basic Laplacian operator in high-dimentional space, we should regard it as an extension of Laplacian operator in a line element of generalized coordinate $q$ space.\[5,p21-22\] And all geometry expressions of $q$ space had generalized non-Euclidean line elements’ meaning.\[5,p43\]

According to Schrodinger’s viewpoint, geometrical optics is only coarse approximate optics, along with wave optics line, we should develop a new wave mechanics in $q$ space. Perhaps our classical mechanics is similar to geometrical optics, as an error, it doesn’t conform to reality. Once curvature radius and route length can compare with a wavelength in $q$ space, classical mechanics becomes invalid. So we seek to a new wave mechanics what starts from Hamilton similarity and find answers along with wave optics line.\[5,p45-46\]

Einstein was inspired by the idea of Schrodinger, and hoped he could restore determinism by a model that integrated wave and particle notions in his note sent to Max Born in April or May of 1927. Einstein published a paper titled ‘Does Schrodinger's wave mechanics is to determine the motion of a system or only in the sense of statistics?’ on May 5 1927. In this paper, Einstein obtained an expression for kinetic energy using any solution of the Schrodinger equation, and used this to define $dq/dt$, a velocity component of an individual particle. J.T.Cushing describes in detail how he was able to obtain a unique value for $dq/dt$ in term of the wave-function $\psi$, using a non-Euclidean metric in configuration space in his ‘Quantum mechanics: Historical Contingency and the Copenhagen Hegemony’ published by The University of Chicago Press in 1994.\[6\]

J.T.Cushing wrote:

‘Einstein’s basic idea was that the time-independent Schrodinger equation

\[
(\hbar^2/2m)\nabla^2 \psi+(E-V)\psi=0
\]

can be used to find the kinetic energy $K=E-V$ for any given wave function solution $\psi$ defined on an n-dimensional configuration space. He used the quantum-mechanical
expression for the kinetic energy \[ K = -(\hbar^2/2m)\nabla^2\psi/\psi \]
to define an equivalent kinetic energy in point-particle mechanics as \[ K = -(\hbar^2/2m)\nabla^2\psi/\psi \]
where \( g_{\mu\nu} \) is the metric tensor for the configuration space and \( \dot{q}_\mu \) is the velocity component of the particle. These \( \dot{q}_\mu \) are functions of the configuration-space coordinates (that is, they define a velocity field, the tangents to which are the “flow lines” or possible particle trajectories). Specifically, having set \[ \nabla^2\psi = g^{\mu\nu}\psi_{\mu\nu}, \]
where \( \psi_{\mu\nu} \) (which Einstein termed “the tensor of \( \psi \)-curvature) is the covariant derivative, he then sought a “unit” vector \( A^\mu \)
\[ g_{\mu\nu}A^\mu A^\nu = 1 \]
that would render \( \psi_{\mu\nu}A^\mu A^\nu \equiv \psi_A \)
an extremum. This is the normal curvature of the differential geometry of surfaces. A hermitian quadratic form \( \psi_{\mu\nu}A^\mu A^\nu \equiv \psi_A \) is rendered an extremum by those vectors \( A^\mu \) that are the solution to the eigenvalue problem \( (\psi_{\mu\nu} - \lambda g_{\mu\nu})A^\nu = 0 \).

In terms of these \( A^\mu \) and their eigenvalues \( \lambda_{(\alpha)} \), Einstein was able to give an expression for uniquely assigning the \( \dot{q}_\mu \) in terms of a given \( \psi \). (The details of the recipe need not concern us here.)[7,p139]

Heisenberg wrote a letter to Einstein on 19 May 1927, and expressed a ‘burning interest’ in its subject matter, and Paul Ehrenfest and Max Born had heard about Einstein's work. However, on May 21 1927, Einstein telephoned the Academy of Sciences of the Academy of Sciences to give up this paper because Walther Bothe raised some objections and doubts.[8,p437-438]

The essence of Bothe’s objection is that the (covariant) derivative \( \psi_{\mu\nu} \) for such a product wave function \( \psi = \psi_1 \psi_2 \) is not zero when \( \mu \) is an index for referring to the first subsystem and \( \nu \) one for the second subsystem. That is why the motions of the compound system will not be simply combinations of motions for the subsystems, as Einstein demanded that they be on physical ground.[7,p139]

In fact, Bothe pointed out that the problem seems to be similar to quantum entanglement, it plays a central role in the later discussion of the EPR paradox, J. S. Bell’s inequality about quantum non-locality and quantum information theory research.

Einstein tried to make use of generalized coordinates in 3N configuration space as hidden
variables to determine generalized velocity uniquely in his 1927 paper, but we often get a pair of opposite directions from the resulting velocity, and they are undetermined eigenspeeds. Einstein linked directions uncertainty of generalized velocities with quasi-periodicity of particle fluctuation, this caused that the non-Euclidean line element representation of quantum mechanics is greatly reduced in the realization of determinism ideal. Einstein does not examine the ability of many body wave mechanics to express independence by using an integrated state; he does not consider quantum entangled state that is suggested in the EPR paper, and Schrodinger pointed out it clearly as $\psi$’s non-integrable quantum state which involves in non-separability, non-locality and entanglement. Grommer did some adjustments on Einstein’s non-Euclidean line element formulation of wave function, we can guarantee that the product of the wave function of the two combination systems is exactly the wave function of the whole system.

According to Einstein's Non-Euclidean line element orbital theory, if we calculate the velocity of particle motion, and then we often get wrong velocity eigenvalue. There is a formal parallel between Einstein’s expression for the kinetic energy $L$ and Bohm’s expression for the quantum potential $U$; but in the de Broglie-Bohm hidden variable theory, if Einstein regarded a physical quantity as kinetic energy, and Bohm regarded this same physical quantity as the quantum potential, thus it can be concluded that the eigenvalue of the particle velocity is consistent with the predictions of quantum mechanics. Therefore, Einstein’s non-Euclidean line element theory’s physical meaning is ambiguous, it might even contain some imaginary velocity components (negative energy), and the same situation also appears in the other two modified versions of Einstein thought proposed by Grommer. D.I.Blokhintsev also paid attention to that if we assume that the electrons have certain positions and momentums (even though these values may be unknown and are not observable) when we consider electrons in a Helium atom, then the unexcited electrons can appear in the region above the ground state potential energy, and their kinetic energy becomes negative[9,p190-193].

### 3. The dual features of quantum mechanics

On March 25 in 1935, A. Einstein, D.Podolsky and N.Rosen submitted a paper titled ‘Can we think quantum mechanics description of physical reality is complete it?’ to ‘Physics Review’, and soon on June 29 in 1935, Niels Bohr submitted a paper of the same name to ‘Physics Review’. We can find in these two papers that quantum mechanics seems to imply that[9,
We can not only establish the particle's wave function by an experiment on a single particle, but also can calculate the wave function of its partner particle without interfering with its partner. Schrödinger noticed that quantum mechanics’ characteristic nature is located in this kind of quantum entanglement—‘It is exactly this point caused the complete departure from the classical thought course’[9, p247]. Annihilation photon polarization experiments have showed that an experimenter can really manipulate it into any state of an infinite number of possible states under no interference on condition of a distance system. Schrödinger believed that this is not only a theoretical problem, but a serious defect problem in theoretical basis of quantum mechanics. In his view, a possible source of this kind of defect is the ‘time’ concept or variable's special status in the quantum mechanics, especially time role in quantum measurements. Quantum measurement theory maybe generalize non-relativistic quantum theory to outside its legal scope. [9,p257]

Einstein pointed out in ‘Physics and Reality’ in 1936: ‘Absolute time and potential energy play decisive roles in Schrödinger equation, these two concepts have been identified as not allowed in principle by relativity. If people want to get rid of this difficulty, they will have to put this theory to be based on a field and its laws, but not based on mutual forces.’[10] And Pauli pointed out in a footnote of an essay on his ‘handbook of physics papers’ in 1933: ‘Although according to the requirements of Lorentz covariance, there should be an energy-time commutation relations similar to coordinate-momentum commutation relations in quantum mechanics: Et-tE=ħ/i. But we can prove that there is no Hermitian operator t to satisfy the relationship: \( t = \frac{\hbar}{i} \frac{\partial}{\partial E} \), otherwise, it will be in conflict with the fact that there is a discrete spectrum.’[11] Space and time form a four-dimensional world in the special theory of relativity so that in Lorentz transformation, x, y, z, t, in their respective sense is ‘synchronization’, but time t keeps the classic sense still in quantum mechanics in spite of p and q being endowed with classic roles (operators and commutation relations), and there isn’t commutation relationship between t and E in classical dynamics.

So in quantum mechanics, time t usually refers to the Newton's absolute space. In Schrodinger equation, t is an evolution parameter of system as an external time what does not contain the physical events and has nothing to do with the system. As a parameter t, we can’t put uncertainty relation between time and energy in the same base as uncertainty relation between position and momentum. There is a fundamental difference between classical mechanics and relativity mechanics at this level: in Newtonian mechanics, time is an external and absolute
parameter, and action at a distance is possible; but in Einsteinnian mechanics, space and time are not independent and influences may not propagate faster than light. As Thomas Durt said: ‘This difference is manifest at the level of the Cauchy problem: in the Newtonian approach, the evolution in present time of local physical quantities is conditioned by the past history of the whole universe; in special relativity it depends only on the interior and on the surface of the backwards light cone.’[12,p5]

Curiously, quantum mechanics offers a combination of Newtonian and Einstein features that presents strong analogies with Poincare’s views on relativity. In quantum mechanics, the state of a system undergoes two difference types: ‘When no measurement is performed, the state of the system obeys a Schrodinger-like equation of evolution that is unitary, deterministic and continuous in time. It is possible to express such a law in a relativistically covariant form (this is the case for instance with Dirac’s equation that describes the evolution of the wave-function of one electron in the presence of external potentials). When a measurement is performed, the system is assumed to perform a sudden transition (quantum jump or collapse of the wave-function) that is supposedly instantaneous in time, non-deterministic and non-unitary.’[12,p6] In other words, we can’t define exactly what is the border line between a classical measurement device and a quantum system.

In non-relativistic quantum mechanics, as Peter Holland said: ‘The circumstance that the configuration space function depends on a single evolution parameter t implies that the state of the n particles is specified at a common time, and that there is a nonlocal connection between them brought about by the classical and quantum potentials.’[13,p282] This brought out three features in quantum mechanics: ‘the dependence of each particle orbit on all the others, the response of the whole to localized disturbance, the extension of actions to large interparticle distances------will be referred to as ‘nonlocal connection’.’[13,p282] Nonlocality is a generic feature of quantum many-body systems.

For Einstein, non-locality is a representation of theoretical incomplete description given by \( \psi \)-function. Some scholars assume that quantum entanglement can be understood by the wormhole in general relativity, but such a model also caused serious technical difficulties and conceptual difficulties. These wormhole topologies maybe destroy the law of causality, even if there is no connection between the system, space will appear from multiply connected to simply connected topological change after a measurement is completed.[13,p482-483]
In the relativistic quantum field theory, Lorentz covariance and locality are statistical effects. Quantum effects in a single process can lead to break Lorentz covariance, and field quantization produces non-localization effects. In quantum field theory, EPR correlations are widespread too. For example, Rindler quantization studies the EPR correlation of vacuum fluctuations in space-like areas in Minkowski space-time. [14] Peter Holland pointed out: ‘Dirac(1951) observed that quantum mechanics could reconcile with the preferred spacetime direction picked out by a mechanical ether at each point if the wavefunction ascribed equal probabilities to all ether velocities. Developing this theme, Dirac(1953) pointed out that the notions of absolute simultaneity and absolute time are also compatible with Lorentz covariance in quantum context. While we have not uncovered a mechanical ether of the type envisaged by Dirac, it is clear that absolute time is playing a fundamental role in the causal theory of fields.’[13,p529-530] Thomas Durt thought that physicists can design quantum simulation experiment of Michelson - Morly experiment to reveal ‘quantum ether wind’ in the process of wave packet collapse.[15] Therefore, mathematical equivalence of Lorentz electronic theory and Einstein's special relativity are only effective at the classical level, Lorentz electronic theory shows to be conflict with special relativity at the quantum level but not just the difference space-time views between them. If we reconstruct general relativity according to the Lorentz's train of thought, quantization form of this new theory will no longer be experience equivalent to any kind of quantization form of general relativity.

4. Conclusion

Einstein's non-Euclidean line element theory is an early abortion project that he insisted on within the framework of general relativity in the various attempts to solve the quantum problem. But Einstein's non-Euclidean line element theory understand Schrodinger equation by Hertz’s ‘the least curvature principle’ that is equivalent to Newton mechanics. Although non-Euclidean line element theory is close to the mathematical model of general relativity in form, but it still adhere to the absolute space-time view of Newton mechanics and particle model in essence. Einstein envisioned a type of field theory like general relativity, if it can’t properly handle many body problems associated with quantum non-locality and understand what role time play role in quantum measurement, then quantum mystery is hard to solve. Quantum mechanics offers a combination of Newtonian and Einstein features that presents strong analogies with Poincare’s views on relativity.
Reference


