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Responses to Frequency Modulation in a Quantum Harmonic Oscillator

Chang Jae Lee

Department of Nanochemistry, Sun Moon University, Asan 31460, Korea

Email: coolcjl@sunmoon.ac.kr

Abstract

A quantum harmonic oscillator with a frequency modulation is an extensively studied model, and has relevance to important physical processes. In this paper we consider response of the frequency modulation in terms of the time dependence of quasi energies and the deviation from the quantum virial theorem. Four modulation types— linear, quadratic, exponential, and sinusoidal — are considered. In all cases the average potential and the average kinetic quasi-energies as well as the deviation from the quantum virial theorem exhibit oscillations in response to frequency modulation. A comparative study of the responses for these modulation types are given.

Keywords: Time-dependent harmonic oscillator, Frequency modulation, quasienergy, quantum virial theorem

I. Introduction

The problem of a time-dependent harmonic oscillator has been extensively discussed in the literature [1]. Such a model has relevance to some processes of physical significance such as atom cooling [2]. In this paper we consider a particle in a one-dimensional harmonic potential with time-dependent frequency. For a conservative system with a potential given as a homogeneous function of degree n in coordinates the quantum virial theorem states that $2 < T > \; = n < V >$ for a bound stationary state [3]. For a static harmonic oscillator the relation reduces to $< T > \; = \; < V >$. Since we are dealing with time-dependent problems, there is no guarantee that the virial theorem be valid for all times. We use the model to study the deviation from the virial theorem as a function of time for various types of frequency modulations.

II. Time-dependent quantum harmonic oscillator

The Hamiltonian of a one-dimensional frequency-modulated harmonic oscillator is given by

$$H = \; -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\, m\, \omega(t)^2\, x^2, \qquad (1)$$

and the wave function satisfies the time-dependent Schrödinger equation (TDSE)

$$i\hbar\,\frac{\partial}{\partial t}\psi(x,t) = H(t)\psi\,(x,t). \qquad (2)$$

Given the initial wave function $\psi(x,t_0)$, the wave function at time t may be obtained using the Feynman propagator as

$$\psi(x,t) = \int_{-\infty}^{+\infty} dx' K(x,x';t,t_0)\psi(x',t_0), \qquad (3)$$

or from the evolution operator as

$$\psi(x,t) = U(t,t_0)\psi(x,t_0). \qquad (4)$$

Formally, the wave function $\psi(x,t)$ for general harmonic oscillators with arbitrary time-dependent frequencies can be calculated following Lewis and Riesenfeld [4, 5].

$$\psi_n(x,t) = \; e^{i\alpha_n(t)}\left[\frac{i}{2\hbar}\left(\frac{\dot{\rho}}{\rho} + \frac{i}{\rho^2}\right)x^2\right] H_n\left(\frac{x}{\sqrt{\hbar}\rho}\right), \qquad (5)$$

where $H_n(x)$ are the Hermite polynomials, ρ satisfies the Milne-Pinney equation [6, 7]

$$\ddot{\rho} + \omega^2(t)\rho = \frac{1}{\rho^3}, \quad (6)$$

and the phase is obtained by [8]

$$\alpha_n(t) = -\left(n + \frac{1}{2}\right)\int_{t_0}^{t} \frac{1}{\rho^2} dt'. \quad (7)$$

In general, Eqs. (6) and (7) still need to be solved numerically. Thus, instead of going through Eqs. (5)-(7), we choose to integrate numerically Eq. (2) directly. There are several efficient numerical methods available for solving TDSE [9]. In this paper we choose the 4th-order Runge-Kutta method.

Once $\psi(x,t)$ is obtained, the quasi energies

$$\langle E(t)\rangle = \int \psi(x,t')^* H(x,t')\psi(x,t')\, dt', \quad (8a)$$

$$\langle V(t)\rangle = \int \psi(x,t')^* V(x,t')\psi(x,t')\, dt', \quad (8b)$$

$$\langle T(t)\rangle = \int \psi(x,t')^* \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\psi(x,t')\, dt' \quad (8c)$$

are computed.

III. Case studies

For simplicity, in all calculations in this paper we use the units $m = \hbar = \omega(0) = 1$, assume the initial wave function $\psi(x,0)$ is in the ground state of a simple static harmonic oscillator, and consider the following four cases of frequency modulation. Thus, initially $\langle E(0)\rangle = 1/2$, $\langle V(0)\rangle = \langle T(0)\rangle = 1/4$.

(a) Linear modulation [10]: $\omega(t)^2 = s\,t$

In this modulation the frequency squared is linearly increasing, and we consider two cases with the rates $s = 1$ and 2. Figure 1 shows the variation of quasi energies $\langle E(t)\rangle$, $\langle V(t)\rangle$ and $\langle T(t)\rangle$ as a function of time for each rate. Because the energies of a (time-independent) harmonic oscillator is proportional to ω, it is reasonable to expect $\langle E(t)\rangle$ to increase as $\omega(t)$ increases. However, the figure shows that $\langle E(t)\rangle$ does not grow as $\sim\sqrt{t}$, and there is a slight oscillation. The oscillation is much more significant for $\langle V(t)\rangle$ and $\langle T(t)\rangle$ and both the amplitude and frequency of the oscillation increase as $\langle E(t)\rangle$ gets larger. As stated in the Introduction, for a static harmonic oscillator $\langle V\rangle - \langle T\rangle = 0$. Thus, the deviation from the virial theorem is characterized by $\delta = \langle V(t)\rangle - \langle T(t)\rangle$, and it is shown in Fig. 2. The figure shows that the deviation is also oscillatory with both the amplitude and the frequency commensurating with the quasi energy $\langle E(t)\rangle$. If we regard these quasi energies as 'responses' to up-chirping of the frequency, these energies response by exhibiting up-chirped oscillatory behavior. It also shows that at certain stroboscopic times the virial theorem is satisfied.

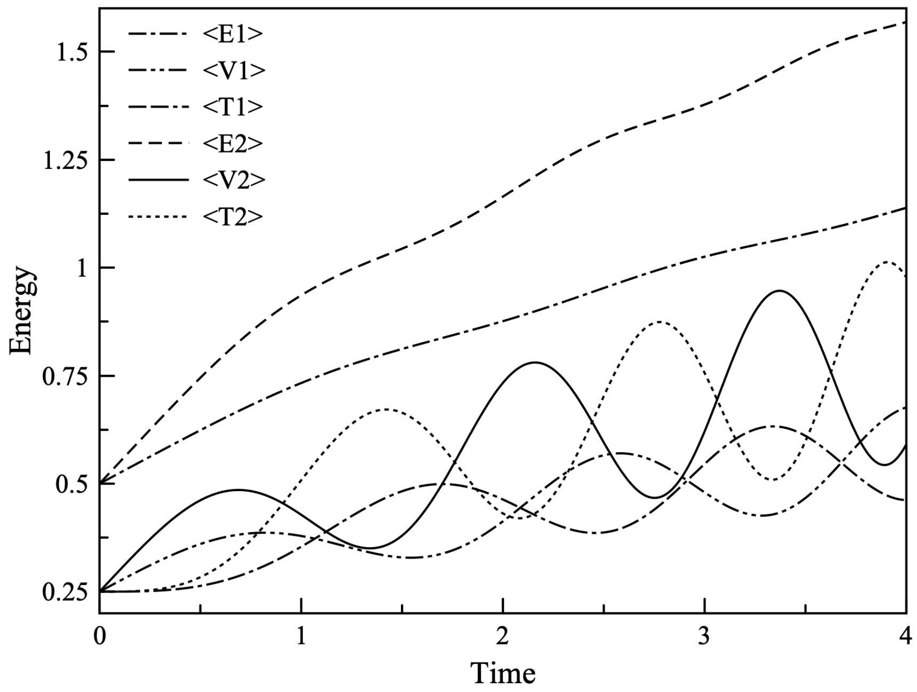

Figure 1. Quasi energies as a function of time. $\langle Ek\rangle$, $(k = 1, 2)$, etc. correspond to modulation rates $s = 1, 2$.

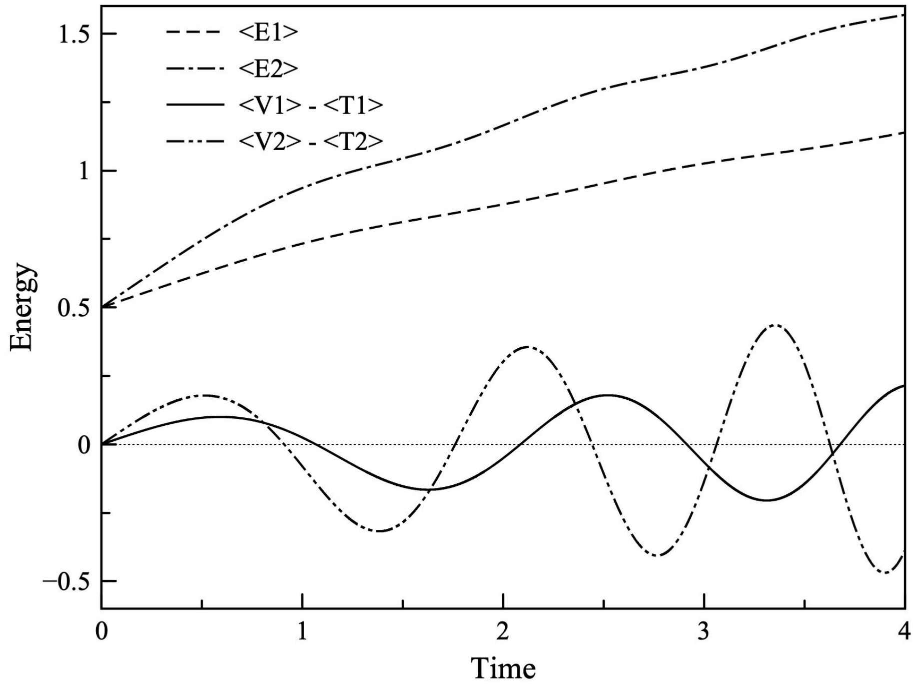

Figure 2. Quasi energies and the deviation from the virial theorem for $s = 1, 2$. A dotted horizontal line is added at Energy = 0 for convenience.

(b) Quadratic modulation: $\omega(t)^2 = s\,t^2, s = 1$

The quasi energies for the quadratically modulated frequency are shown in Fig. 3. In this case the energies increase more rapidly than the linear case. There are oscillations in energies and the deviation δ as in the linear case, but the variations in amplitude and period are different.

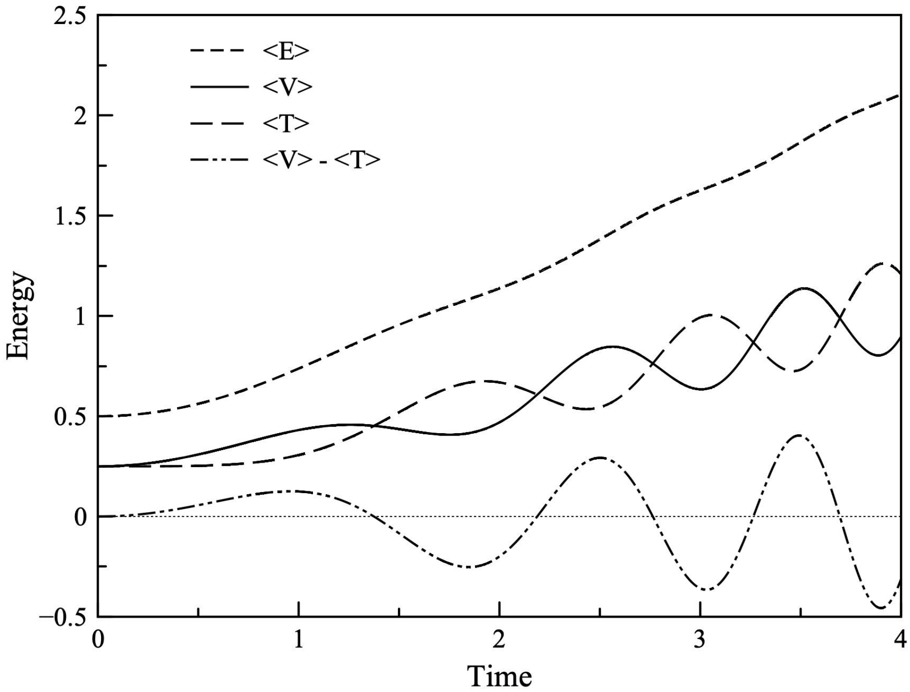

Figure 3. Same as in Fig. 2 for quadratically modulated frequency.

(c) Exponential modulation: $\omega(t)^2 = \exp{(-\, s\, t)}\,, s = 0.1$

In the previous two cases we considered frequency up-chirping, and this time we consider down-chirping, with the frequency decreasing exponentially. In the case of frequency up-chirping the energies diverge as $t \to \infty$, which limits examination of long-time behaviors. However, when $\omega(t) \to 0$ as $t \to \infty$, long-time behaviors can be studied. The computed quasi energies for the exponentially decreasing frequency and the potential shape as a function of time are shown in Fig. 4. We note that as expected $\langle E(t) \rangle$ decays to zero as $\omega(t) \to 0$, but the decay rate is much smaller than the potential shape. Also, we see that $\langle V(t) \rangle \to 0$, so $\langle T(t) \rangle \to \langle E(t) \rangle$ as $\omega(t) \to 0$. The oscillatory behavior of $\langle V(t) \rangle$ and $\langle T(t) \rangle$ is much subdued compared with the linear and the quadratic modulations. Figure 5 shows the deviation δ, and the response is reminiscent of the frequency down chirp. Eventually, $\delta(t) \to 0$, since $\langle V(t) \rangle \to 0$ and $\langle T(t) \rangle \to 0$ as $t \to \infty$.

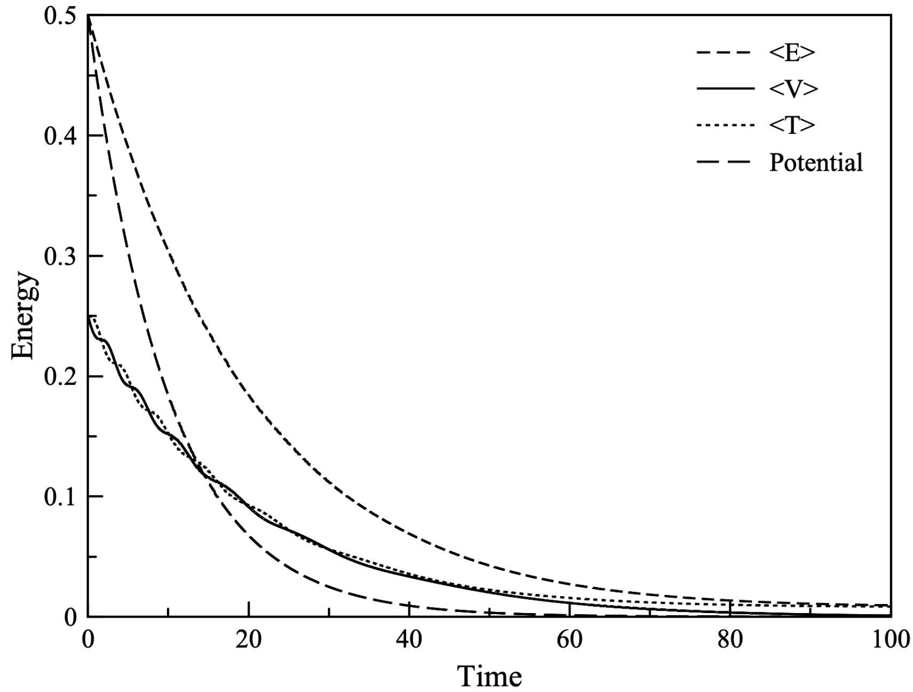

Figure 4. Quasi energies for exponentially decreasing frequency. Also shown is the exponential shape of the potential.

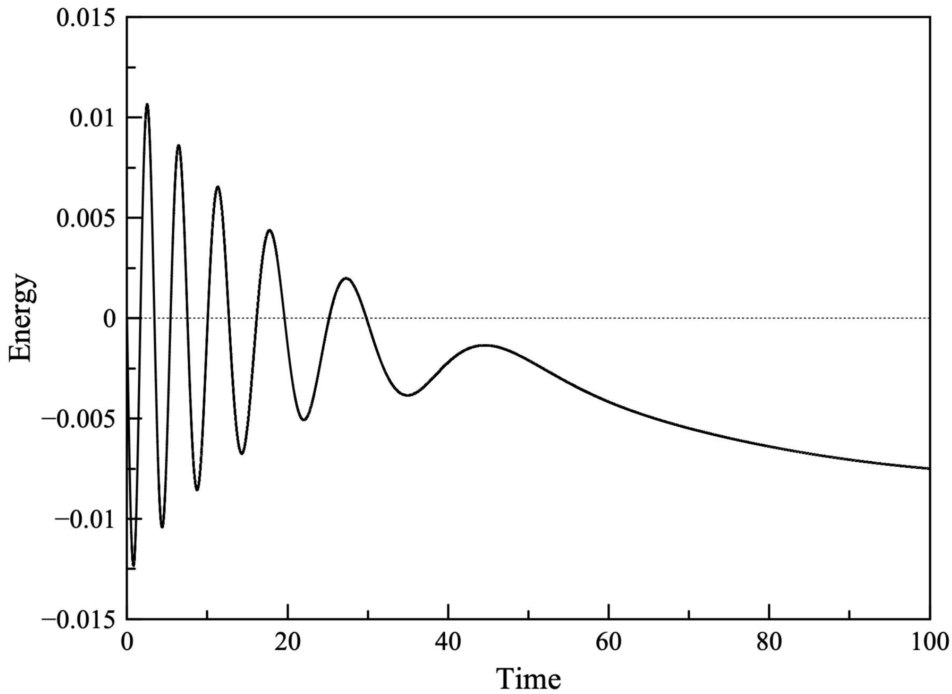

Figure 5. The down-chirp behavior of the deviation δ from the virial theorem.

(d) Sinusoidal modulation: $\omega(t)^2 = \cos^2(2\pi\nu\, t)$, $\nu = 1/2$

In this type of modulation there are periods of frequency up- and down-chirping.

The responsive quasi energies and the deviation from the virial theorem are shown in Fig. 6. Note that the times at which $\langle V(t)\rangle = 0$ are truly regular at $t_n = (2n+1)/2, (n = 0, 1, 2, \ldots)$ and equal to the times at which $\omega(t_n) = 0$. However, the amplitude variation and the times at which $\langle V(t)\rangle$ has maximal values are not regular. Other quantities $\langle E(t)\rangle$ and δ behave similarly, but $\langle T(t)\rangle$ has a quite distinct behavior.

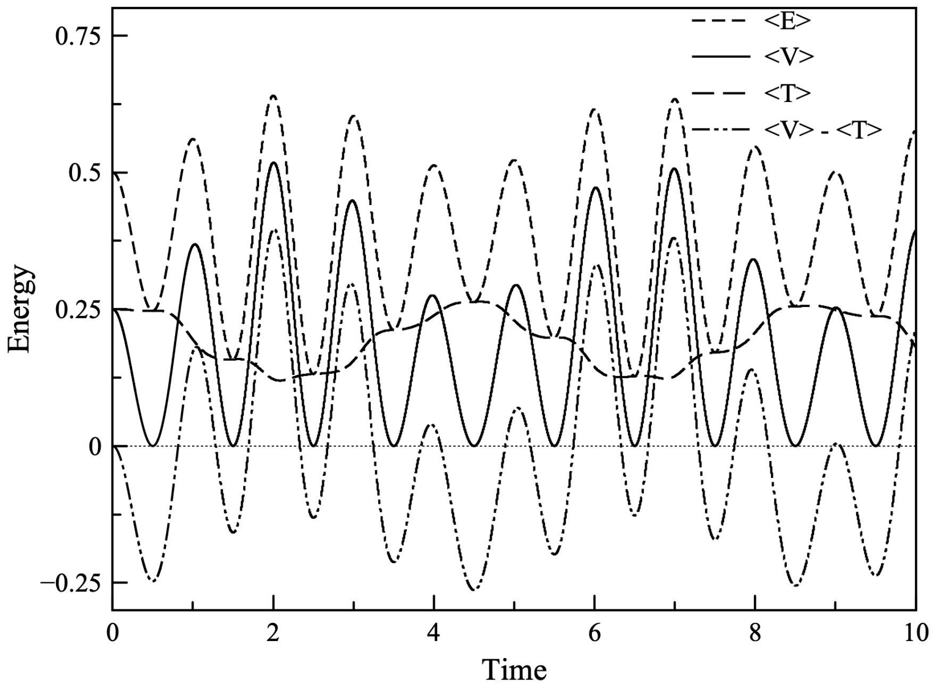

Figure 6. Quasi energies and the deviation for sinusoidally modulated frequency.

IV. Conclusion

In this paper we considered the response of frequency modulation for a quantum harmonic oscillator. The response considered was the variation of quasi energies $\langle E(t)\rangle$, $\langle V(t)\rangle$ and $\langle T(t)\rangle$, and the deviation from the virial theorem $\delta = \langle V(t)\rangle - \langle T(t)\rangle$. Four modulation types were considered: linear and quadratic up-chirping, exponential down-chirping, and sinusoidal chirping. In all cases quasi energies increase(decrease) and as oscillator frequency increases(decreases) as expected, but the rates these responses follow were much smaller than the modulation rates. Especially interesting fact is that there are oscillations in the responses even though the modulations were monotonic functions. The phases of $\langle V(t)\rangle$ and $\langle T(t)\rangle$ oscillations are opposite to each other, and at certain times the responses satisfy the quantum virial theorem. We expect that these informations will be useful when studying atoms trapped in frequency-modulated harmonic potentials.

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