SCIREA Journal of Physics
ISSN: 2706-8862
http://www.scirea.org/journal/Physics
May 10, 2020
Volume 5, Issue 1, February 2020

Gravitational Waves Do Not Carry Energy-Momentum

Zhaoyan Wu
Center for Theoretical Physics, Jilin University, China
Email address: zhaoyanwu@jlu.edu.cn

Abstract

From a geometric perspective, we proved the following conclusions, that are against mainstream scholars' viewpoint. (i) Vanishing of the covariant divergence of matter energy- momentum flux density in entire spacetime is a conservation law of matter energy-momentum. It reads the net increase of matter energy-momentum in any infinitesimal neighborhood of spacetime is zero. Hence, introducing gravitational energy-momentum does not save, but destroys the law of energy-momentum conservation. (ii) Interaction or force in physics always means exchange of energy-momentum. The spacetime metric field (gravitational field in general relativity) does not exchange energy-momentum with all mass points and matter fields. Therefore, the metric field of spacetime does not carry energy-momentum, it's not a force field, and gravity is not a natural force. The spacetime metric field is the geometrical aspect of moving matter 4-dimensional continuum. It is not a matter field itself.

Keywords: gravitational energy-momentum, energy-momentum conservation in general relativity

1. Introduction

100 years after A. Einstein founded his theory of general relativity (GR) [1,2], which predicted the existence of gravitational waves, the first successful direct detection of gravitational wave GW150914 was eventually reported by LIGO [3]. And the Nobel prize in physics 2017 was then awarded to this great contribution. Long before LIGO, ever since 1960's, enormous efforts have been made to detect gravitational waves. Most of them have followed the strategy proposed by R. Feynman at Chapel Hill conference 1957 [4]. For Feynman and his followers, to detect gravitational waves means to detect the energy carried by the passing gravitational waves. This is different from LIGO's strategy, which is to perform a pure geometrical measurement directly on the metric field of spacetime. It is worth noting that all the experiments detecting the energy carried by passing gravitational waves have failed so far, while over the past few years more successful direct geometrical measurements of gravitational waves GW151226, GW170104, GW170814, GW170817, etc. have been reported by LIGO and Virgo [5,6,7,8]. Is this contrast just an accidental coincident?

While LIGO's successful measurements of spacetime metric ripple might be the greatest experimental breakthrough in the history of GR; there are still some long existing fundamental theoretical issues in GR, the most beautiful theory in physics, such as (i) whether or not matter energy-momentum is conserved, (ii) whether or not gravitational field carries energy-momentum, (iii) is pseudotensor and non-localizability property of gravitational energy-momentum the inevitable consequence of equivalence principle, etc. The purpose of the present paper is to show, (i) The root cause of all these issues is, vanishing of covariant divergence of matter energy-momentum flux density was misread by Einstein and mainstream scholars as a law of non-conservation of matter energy-momentum; (ii) Since interaction (4-force) in physics always means energy-momentum exchange, and spacetime metric field (gravitational field in GR) does not exchange energy-momentum with matter particles and matter fields. therefore gravitational field is not a force field, and gravity is not a natural force. The spacetime metric field is the geometric aspect of 4-dimensional moving matter continuum, It is not a matter field itself.

The whole argument of this paper will be a deduction based on three starting points: (i) sound facts from modern geometry, (ii) variational principle of dynamics, and (iii) Einstein-Hilbert action of GR. It contains no phenomenological assumption, no approximation, no revision of motion equations of Einstein's GR. The present paper is a companion piece of an earlier work of mine [9].

2. How Was Gravitational Energy-Momentum Introduced into GR?

When Einstein tried to establish his new theory of gravity, to ensure energy-momentum conservation was his top priority. However, from his field equations

$$R^{\lambda\mu}(x) - \frac{1}{2}Rg^{\lambda\mu}(x) = \frac{8\pi G}{c^4}T^{\lambda\mu}(x), \quad \forall\lambda,\mu = 0,1,2,3 \tag{1}$$

by using contracted Bianchi identity, he obtained

$$\nabla_{\lambda} T^{\lambda\mu}(x)|_p = 0, \quad \forall\mu = 0,1,2,3; \, p \in M \tag{2}$$

that is, the covariant divergence of matter energy-momentum flux density vanishes everywhere in spacetime M. Multiplying it by $\sqrt{-|g(x)|}$

$$\frac{\partial}{\partial x^\lambda}\left[\sqrt{-|g(x)|}T^{\lambda\mu}(x)\right] + \sqrt{-|g(x)|}\Gamma^{\mu}_{\lambda\sigma}(x)T^{\lambda\sigma}(x) = 0 \tag{3}$$

and integrating eqn.(3) over 4-dimensional spacetime domain $\Omega(\subset M)$, he got by using Gaussian theorem (Note that the integrand is not a 4-form field, so the integral highly depends on coordinate system!)

$$\int_{\Omega} d^4x\sqrt{-|g(x)|}\,\nabla_{\lambda} T^{\lambda\mu}(x) =$$

$$\int_{\partial\Omega} ds_{\lambda}(x)\sqrt{-|g(x)|}\,T^{\lambda\mu}(x) + \int_{\Omega} d^4x\,\sqrt{-|g(x)|}\Gamma^{\mu}_{\lambda\sigma}(x)T^{\lambda\sigma}(x) = 0,$$

$$\forall\mu = 0,1,2,3 \tag{4}$$

When the boundary of Ω is composed of a past spacelike hyper-surface Σ, a future spacelike hyper-surface Σ' and a timelike hyper-surface Γ which links the boundaries of Σ and Σ', eqn.(4) can be written as

$$[\int_{\Sigma'} + \int_{\Sigma} + \int_{\Gamma}]ds_{\lambda}(x)\sqrt{-|g(x)|}\,T^{\lambda\mu}(x) =$$

$$-\int_{\Omega} d^4x\,\sqrt{-|g(x)|}\Gamma^{\mu}_{\lambda\sigma}(x)T^{\lambda\sigma}(x),\ \forall\mu = 0,1,2,3 \tag{5}$$

Einstein read this as the matter energy-momentum distributed on Σ' minus the matter energy-momentum distributed on Σ and the matter energy-momentum flowing in through Γ equals the integral at left hand side, which is not zero in general. Therefore, he decided that the vanishing of the covariant divergence of matter energy-momentum flux density is a law of non-conservation of matter energy-momentum. In order to save the law of energy-momentum conservation, Einstein rewrite the second term of eqn.(3) as

$$\sqrt{-|g(x)|}\,\Gamma^{\mu}_{\lambda\sigma}(x)\,T^{\lambda\sigma}(x) = \frac{\partial}{\partial x^{\lambda}}\left[\sqrt{-|g(x)|}\,t^{\lambda\mu}(x)\right] \tag{6}$$

where $t^{\lambda\mu}(x)$ is a function of metric field and it's first order derivatives [10]. Substituting eqn.(6)

into eqn.(5), he got

$$[\int_{\Sigma'} + \int_{\Sigma} + \int_{\Gamma}] ds_\lambda(x)\sqrt{-|g(x)|}\, T^{\lambda\mu}(x) =$$

$$-[\int_{\Sigma'} + \int_{\Sigma} + \int_{\Gamma}] ds_\lambda(x)\sqrt{-|g(x)|}\, t^{\lambda\mu}(x), \quad \forall\mu = 0, 1, 2, 3 \qquad (7)$$

Now he was happy. He read the above equations as "the matter energy-momentum distributed on Σ' minus the matter energy-momentum distributed on Σ and the matter energy-momentum flowing in through Γ" plus "the gravitational energy-momentum distributed on Σ' minus the gravitational energy-momentum distributed on Σ and the gravitational energy-momentum flowing in through Γ" is zero, or, the net increase of matter energy-momentum in Ω plus the net increase of gravitational energy-momentum in Ω is zero. And $t^{\lambda\mu}(x)$ was read as the flux density of gravitational energy-momentum, $\sqrt{-|g(x)|}\Gamma^{\mu}_{\lambda\sigma}(x)T^{\lambda\sigma}(x)$ was read as the amount of matter energy-momentum which changes into gravitational energy-momentum per unit 4-dimensional spacetime volume. These viewpoints are still the leading scholars' viewpoints [11,12,13]. But I will show, taking $\nabla_\lambda T^{\lambda\mu}(x)|_p = 0$ as a law of non-conservation of matter energy-momentum and introducing gravitational energy-momentum to save the law of conservation of energy-momentum is a fatal mistake.

In fact, for an infinitesimal spacelike (timelike) hyper-surface element $\Delta\Sigma\,(\Delta\Gamma)$, $T^{\lambda\mu}(x)\sqrt{-|g(x)|}ds_\lambda(x)|_{\Delta\Sigma}$ ($T^{\lambda\mu}(x)\sqrt{-|g(x)|}ds_\lambda(x)|_{\Delta\Gamma}$) is really the μ-component in coordinate system $\{x\}$ of matter energy-momentum distributed on $\Delta\Sigma$ (flowing through $\Delta\Gamma$), but for a finite or an infinite spacelike (timelike) hyper-surface Σ (Γ), taking $\int_\Sigma ds_\lambda(x)\sqrt{-|g(x)|}\, T^{\lambda\mu}(x)$ ($\int_\Gamma ds_\lambda(x)\sqrt{-|g(x)|}\, T^{\lambda\mu}(x)$) as the μ-component in coordinate system $\{x\}$ of the matter energy-momentum distributed on Σ (flowing through Γ) is totally wrong! Because the spacetime M in GR is curved, the sum of tangent vectors at different points is meaningless (See [9] or the next section for readers' convenience). There is no flat coordinate system in M, and in a curvilinear coordinate system, the sum of μ-components of vectors at different points does not equal the μ-component of their sum vector, which does not exist at all. (Even in a flat spacetime, when using a curvilinear coordinate system, the sum of μ-components of vectors at different points does not equal the μ-component of their sum vector, which does exist.)

Before we explore further the conservation of matter energy-momentum and gravitational energy-momentum issues, let me present some sound facts from modern geometry in the next section.

3. Some Sound Facts from Modern Geometry

(1) In LIGO's report [3], they talked about "black hole mass" repeatedly. But a black hole is a large celestial body in an extremely curved spacetime region, and in a curved spacetime the energy-momentum of a large celestial body is meaningless. Geometry tells us, even for a system consisting of only two uncharged mass points, its energy-momentum cannot be defined. (See [13], or for readers' convenience, see the following) Let us call these mass points A and B, and denote their world lines by α and β respectively. The energy-momentum 4-vector of a particle is its rest mass times the tangent vector to its world line with the proper time as parameter. Suppose α and β intersect some spacelike hyper-surface Σ at $p =: \alpha(0)$ and $q =: \beta(0)$ respectively ($p \neq q$). And let us consider the system's sum energy-momentum on Σ. The particles' energy-momentum 4-vectors on Σ, P_{Ap} and P_{Bq}, belong to different tangent spaces T_p and T_q, and no one can add up vectors belonging to different vector spaces. In order to add them up, one has to parallelly transport them to a same point, say $r \in M$. The transported vectors P_{Ar} and P_{Br} now belong to the same tangent space T_r, so one can add them up to get a sum vector $S_r = P_{Ar} + P_{Br} \in T_r$. But the spacetime M is curved, parallel transportation of a vector depends on path. To avoid the ambiguity, one might suggest parallelly transporting P_{Ap}, P_{Bq} to r along geodesics. But, even so, when one chooses a different point $s \in M$, and parallelly transport P_{Ap} and P_{Bq} along geodesics to s, one gets a sum vector there, $S_s =: P_{As} + P_{Bs} \in T_s$. However, when parallelly transporting S_s along the geodesic to r, the resulting vector is not S_r in general. Therefore one cannot define the sum of vectors at different points in a curved spacetime, hence the energy-momentum 4-vector of a black hole is an invalid concept.

In general, the sum of (r,s)-tensors of the same kind at different points in curved spacetime cannot be defined, if $r + s > 0$.

One might ask, "In a flat spacetime, vectors of the same kind at different points belong to different tangent spaces too, why can you add them up?"

In fact, when spacetime M is flat, let $\Phi =: \cup_{p \in M} T_p$, and define a binary relation $\sim$ in Φ, such that, For any $p,q \in M$, and $U \in T_p, V \in T_q$, we say $U \sim V$, if and only if parallelly transporting U from p to q will result in V. Because M is flat, parallelly transporting of vectors does not depend on path, therefore $\sim$ is not ambiguous. It is easy to see that $\sim$ is an equivalence relation and each equivalence class contains one and only one representative in every tangent space of M. Denote the equivalence class containing U by $\underline{U}$ and the quotient set by $\underline{\Phi}$. Because parallel transportation keeps linear relation unchanged, we can define addition and multiplication with real numbers in $\underline{\Phi}$ such that, for any $p,q \in M$, and $U \in T_p$, $V \in T_q$, denote by V' *the only vector in* $T_p \cap \underline{V}$, and let $\underline{U} + \underline{V} =: \underline{U+V'}$ and $\alpha\underline{U} = \underline{\alpha U}$, $\forall \alpha \in \mathfrak{R}$. It is easy to see, these definitions are independent of the choice of p,q. With these induced operations, $\underline{\Phi}$ is a real vector space. Because parallel transportation in generalized Riemannian spaces also keeps scalar product, we can define scalar product in $\underline{\Phi}$ such that, for any $p,q \in M$, and $U \in T_p$, $V \in T_q$, denote by V' *the only vector in* $T_p \cap \underline{V}$, and let $\underline{U} \cdot \underline{V} =: U \cdot V'$. It is easy to see, this definition is independent of the choice of p,q too. And now $\underline{\Phi}$ is a real scalar product space. For any $p \in M$, map $U(\in T_p) \mapsto \underline{U}(\in \underline{\Phi})$ is an isometric isomorphism of T_p onto $\underline{\Phi}$. This $\underline{\Phi}$ is the so called scalar product space of free vectors. In the case of a flat spacetime M, when people talk about the sum vector $U + V$ of two tangent vectors $U(\in T_p)$ and $U(\in T_q)$ $(p \neq q)$, they actually mean $\underline{U} + \underline{V} = \underline{U+V'}$ in the scalar product space of free vectors $\underline{\Phi}$. It is clear, when spacetime M is curved, we can not talk about free vectors.

Similarly, scalars of the same kind (such as electrical charges) at different points in curved spacetime can be added up, because parallel transportation of scalar does not depend on path.

(2) In a curved spacetime M, even the change of energy-momentum 4-vector of an uncharged mass point is not self-evident; unfortunately nobody has given it a definition in the literature of GR as far as I know. (See [9] or the following) Suppose the particle's world line is $\gamma: \Delta \longrightarrow M$, where $\Delta =: [\tau_i, \tau_f]$, τ_i (τ_f) is the proper time when the particle is created (annihilated), or $\Delta =$

$:[\tau_i, +\infty),(-\infty,\tau_f]$, $(-\infty, +\infty)$, if its life span is infinite. Its energy-momentum 4-vectors at proper times τ_1, $\tau_2 \in \Delta(\tau_1 \neq \tau_2)$, belong to different tangent spaces. We cannot subtract one from the other. Denote by $P(\tau)(\in T_{\gamma(\tau)})$ the particle's energy-momentum 4-vector at proper time $\tau \in \Delta$, and for τ_0, $\tau \in \Delta$, denote by $\widetilde{P}_{\tau_0}(\tau)$ the vector obtained by parallelly transporting $P(\tau_0)$ along the world line from $\gamma(\tau_0)$ to $\gamma(\tau)$. The change of a particle's energy-momentum 4-vector during proper time interval $[\tau_1,\tau_2] \in \Delta$ is a vector field defined only on its world line

$$\delta_{\tau_1,\tau_2}P{:}\Delta \longrightarrow \bigcup\nolimits_{\tau\in\Delta} T_{\gamma(\tau)} \tag{8}$$

such that

$$\delta_{\tau_1,\tau_2}P(\tau) =: \widetilde{P}_{\tau_2}(\tau) - \widetilde{P}_{\tau_1}(\tau) \in T_{\gamma(\tau)}, \forall \tau_1,\tau_2,\tau \in \Delta \tag{9}$$

It is easy to check,

$$\delta_{\tau_1,\tau_2}P + \delta_{\tau_2,\tau_3}P = \delta_{\tau_1,\tau_3}P \qquad 10)$$

That is, the change of a particle's energy-momentum 4-vector during $[\tau_1,\tau_2]$ plus the change during $[\tau_2,\tau_3]$, equals the change during $[\tau_1,\tau_3]$. This is exactly what we expect, but it is not trivial. For, should we define the change of a particle's energy-momentum 4-vector during proper time interval $[\tau_1,\tau_2] \in \Delta$ as $\delta_{\tau_1,\tau_2}P(\tau) =: \overline{P}_{\tau_2}(\tau) - \overline{P}_{\tau_1}(\tau) \in T_{\gamma(\tau)}$, where $\overline{P}_{\tau_0}(\tau)$ is the vector obtained by parallelly transporting $P(\tau_0)$ along the geodesic from $\gamma(\tau_0)$ to $\gamma(\tau)$, the above self-consistency (10) would fail. Therefore, if we wish to talk about the change of a particle's energy-momentum 4-vector in curved spacetime, (8)+(9) is the only reasonable definition. It is worth noting, this definition does not depend on coordinates. (We will double check this definition after we explore the meaning of conservation of matter energy-momentum in GR)

According to this definition, the change of a freely falling mass point's energy-momentum is a vanishing tangent field defined on its world line, which is a timelike geodesics. In other words, a freely falling mass point's energy- momentum does not change, hence the freely falling particle does not exchange energy-momentum with gravitational field and matter fields. This is against leading scholar's viewpoint, say, Sir Herman's. In [11], he wrote, "Every hydroelectric power station depends entirely on this non-conservation of tangible energy, because in it gravitational energy (whatever this term may mean in general relativity) is turned successively into the kinetic energy of water, then of the machinery, and finally into electromagnetic energy".

Yes, energy-momentum conservation is the corner stone of physics, however the notion of energy evolves with people's notion of physical realty. The gravity position potential energy and Coulomb's position potential energy correspond to Newton's notion of physical realty: the world is composed of mass points interacting with each other through instantaneous action at a distance and moving in absolute space and absolute time according to Newton's laws of motion. The founding of Maxwell's electrodynamics and Einstein's SR declared that force field takes place of instantaneous action at a distance in physics forever. Correspondingly electrical field energy takes place of Coulomb's position potential energy. The density of electrical field energy is (Einstein's mass-energy equivalence removes the arbitrary additive constant in field energy)

$$\varepsilon_e(\vec{r}) = \frac{|\vec{E}(\vec{r})|^2}{8\pi}$$

Due to the similarity between Newton's universal law of gravity and Coulomb's law, the density of gravitational field energy should be

$$\varepsilon_g(\vec{r}) = -\frac{|\vec{g}(\vec{r})|^2}{8\pi G}$$

which is negative and without lower bound. This is not allowed physically. The above argument already shows, gravity cannot be treated as a force field in the framework of SR like Coulomb force.

Obviously, the gravitational energy in hydroelectric power station case talked about by Sir Bondi, is not a concept in GR (even not a concept in SR), but the position potential energy of mass points interacting with conservative instantaneous action at a distance in Newton's absolute space and absolute time, which had stepped down the stage of physics long long ago.

(3) In the spacetime of SR, the Minkowski space M, which is a pseudo-Euclidean space, any ordered pair of points $(p,q) \in M \times M$, determines a displacement 4-vector $\overrightarrow{pq}$ or $q - p$. We define a particle's 4-velocity as follows. Let the particle's world line is $\gamma{:}\Delta \longrightarrow \mathrm{M}$. Then its 4-velocity at proper time $\tau \in \Delta$ is

$$\vec{v}(r) = \lim \frac{\overrightarrow{\gamma(\tau)\gamma(\tau')}}{\tau' - \tau} = \lim_{\tau' - \tau \to 0} \frac{\gamma(\tau') - \gamma(\tau)}{\tau' - \tau} \qquad 11)$$

But in GR, the spacetime is no longer flat, the concept of finite displacement vector loses its meaning, and we cannot define a particle's 4-velocity by using its finite displacement vector as eqn. (11). However, we can still talk about a particle's 4-velocity in GR. Please remember, the tangent vectors at a point p of a smooth manifold are defined as certain equivalence classes of 1-parameter smooth curves passing through p. A particle's 4-velocity vector at p is the equivalence class containing its world line with the proper time as parameter.

Briefly speaking, in a curved spacetime, the differences between corresponding coordinates of two finitely separated points do not transform like a vector's components under general coordinate transformations; while the differences between corresponding coordinates of two points close to each other do (neglecting higher order infinitesimal). Therefore, in a curved spacetime, the finite displacement vector loses its meaning, but we can still talk about infinitesimal displacement vector and 4-velocity.

Similarly, in GR, we cannot talk about the amount of matter energy-momentum distributed on a finite spacelike hyper-surface, cannot talk about the amount of matter energy-momentum flowing through a finite timelike hyper-surface, but the flux density T of matter energy-momentum is still a well-defined measurable tensor field. (Here we are talking about matter energy-momentum, it has nothing to do with the equivalence principle.) $\mathrm{T}^{\alpha\beta}(\mathrm{x})\sqrt{-|\mathrm{g}(\mathrm{x})|}\mathrm{ds}_{\alpha}(\mathrm{x})|_{\Delta\Sigma}$ is the β-component (in coordinate system $\{x\}$) of matter energy-momentum located on infinitesimal spacelike hyper-surface element $\Delta\Sigma$. $T^{\alpha\beta}(x)\sqrt{-|g(x)|}ds_{\alpha}(x)|_{\Delta\Gamma}$ is the β-component (in coordinate system $\{x\}$) of matter energy-momentum flowing through infinitesimal timelike hyper-surface element $\Delta\Gamma$. And $\nabla_{\alpha}\mathrm{T}^{\alpha\beta}(\mathrm{x})\sqrt{-|\mathrm{g}(\mathrm{x})|}\mathrm{d}^4\mathrm{x}|_{\Delta\Omega}$ is the β-component (in coordinate system $\{x\}$) of matter energy-momentum net increase in infinitesimal 4-dimensional spacetime volume element $\Delta\Omega$.

In general, we cannot talk about the sum of $(r.s)$-tensors of the same kind at different spacetime points in GR, when $r+s>0$, hence cannot talk about the amount of some kind of $(r.s)$-tensor distributed on (flowing through) a finite or an infinite spacelike (timelike) hyper-surface, but its flux density is still a well-defined measurable $(1+\mathrm{r},s)$-tensor field.

(4) Both mathematics and physics pursuit objective truths. That is why a good concept in linear algebra does not depend on the choice of basis of the vector space. That is why all the good concepts in modern geometry, such as tangent vectors in differential manifold, connection of a vector bundle, etc., do not depend on coordinates. And that is why all the good physical concepts should be independent of reference coordinate systems, ways of observing physical processes do not depend on reference coordinate systems and physical laws should be covariant (the principle of general relativity).

In modern geometry, the integrand of an integral over an l-dimensional manifold is an l-form field (l-differential form), otherwise, the integral would highly depend on coordinates. However, in the literature of GR, we can see integrals like $\int_{\Omega}\mathrm{d}^4\mathrm{x}\sqrt{-|\mathrm{g}(\mathrm{x})|}\nabla_{\lambda}\mathrm{T}^{\lambda\mu}(\mathrm{x})$, $\int_{\partial\Omega}\mathrm{ds}_{\lambda}(\mathrm{x})\sqrt{-|\mathrm{g}(\mathrm{x})|}[\mathrm{T}^{\lambda\mu}(\mathrm{x})+\mathrm{t}^{\lambda\mu}(\mathrm{x})]$ pretty often. They are read as common Riemann integrals. Reading them correctly helps clearing confusions and understanding physics correctly. Let me illustrate this idea with Einstein's continuity equations for total energy-momentum conservation

$$\int_{\partial\Omega}ds_{\lambda}(x)\sqrt{-|g(x)|}[T^{\lambda\mu}(x)+t^{\lambda\mu}(x)]=0,\ \forall_{\mu}=0,1,2,3 \qquad 12)$$

Usually the integrand in eqn. (12) is not taken as a 3-form field, the above integral highly depends on coordinate system. Now let us consider it in a reverse way. Assume

$ds_\lambda(x)\sqrt{-|g(x)|}[T^{\lambda\mu}(x)+t^{\lambda\mu}(x)]$ is a 3-form field, then $T^{\lambda\mu}(x)+t^{\lambda\mu}(x)$ must be the λ-component in coordinate system $\{x\}$ of some vector field J,

$$T^{\lambda\mu}(x)+t^{\lambda\mu}(x)=J^\lambda(x), \forall 0\le\lambda\le 3 \qquad 13)$$

So, this vector field J, must be decided by specified coordinate system $\{x\}$ and specified superscript μ. In an arbitrary coordinate system $\{y\}$, let

$$J^\alpha(y)=:\frac{\partial y^\alpha}{\partial x^\lambda}J^\lambda(x)=\frac{\partial y^\alpha}{\partial x^\lambda}[T^{\lambda\mu}(x)+t^{\lambda\mu}(x)],\ \forall 0\le\alpha\le 3 \qquad 14)$$

Then this J is a vector field and its components in specific coordinate system $\{x\}$ is (13), and $ds_\lambda(y)\sqrt{-|g(y)|}\ J^\lambda(y)=:\epsilon_{\lambda\alpha\beta\gamma}\sqrt{-|g(y)|}\ J^\lambda(y)\ dy^\alpha\otimes dy^\beta\otimes dy^\gamma$ is a 3-form field, the integral $\int_{\partial\Omega}ds_\lambda(y)\sqrt{-|g(y)|}J^\lambda(y)$ is independent of coordinate system, hence

$$\int_{\partial\Omega}ds_\lambda(y)\sqrt{-|g(y)|}J^\lambda(y)=\int_{\partial\Omega}ds_\lambda(x)\sqrt{-|g(x)|}T^{\lambda\mu}(x)+t^{\lambda\mu}(x)]=0, \qquad 15)$$

Vector field J is the flux density vector field of some scalar decided by specified coordinate system $\{x\}$ and specified superscript μ, $S[\{x\},\mu]$. Equation (15) is a continuity equation for conservation of scalar $S[\{x\},\mu]$. Note that, in general, $S[\{x\},\mu]\neq S[\{z\},\mu]$, $S[\{x\},\mu]\neq S[\{x\},\nu](\mu\neq\nu)$, we must not take different physical, geometric objects as one and the same.

More generally, we have the following proposition.

Proposition 1: In a curved spacetime M, any integral equation of the following form in a specified coordinate system {ξ}

$$\int_{\partial\Omega} ds_\alpha(\xi)[\sqrt{-|g(\xi)|}F^\alpha(\xi)] = 0, \quad \forall\, 4-dimentioanl\ domain \quad \Omega \subset \mathrm{M} \qquad 16)$$

where $\mathrm{F}^0(\xi)$,$\mathrm{F}^1(\xi)$,$\mathrm{F}^2(\xi)$,$\mathrm{F}^3(\xi)$, are four arbitrary functions of ξ^0,ξ^1,ξ^2,ξ^3, is a continuity equation for conservation of some scalar decided by $\{\xi\}$ and $\mathrm{F}^0(\xi)$,$\mathrm{F}^1(\xi)$,$\mathrm{F}^2(\xi)$,$\mathrm{F}^3(\xi)$.

Proof: Define a vector field J, such that in any coordinate system $\{x\}$,

$$J^\alpha(x) =: \frac{\partial x^\alpha}{\partial \xi^\mu} F^\mu(\xi), \forall 0 \le \alpha \le 3 \qquad 17)$$

It is easy to verify that for any coordinate systems $\{x\}$, and $\{y\}$

$$J^\alpha(x) =: \frac{\partial x^\alpha}{\partial y^\rho} J^\rho(y), in\ particular\ \ J^\alpha(\xi) = F^\alpha(\xi), \forall 0 \le \alpha \le 3 \qquad 18)$$

So, J really is a vector field, and

$$F^\alpha(\xi)\sqrt{-|g(\xi)|}ds_\alpha(\xi) = J^\alpha(x)\sqrt{-|g(x)|}ds_\alpha(x) = J^\alpha(y)\sqrt{-|g(y)|}ds_\alpha(y) \qquad 19)$$

is a 3-form field. From eqn. (16), we have

$$\int_{\partial\Omega} ds_\alpha(\xi)[\sqrt{-|g(\xi)|}F^\alpha(\xi)] = \int_{\partial\Omega} ds_\alpha(x)\sqrt{-|g(x)|}J^\alpha(x) = 0 \qquad 20)$$

This is a continuity equation for the conservation of a scalar $S[\{\xi\},F]$, decided by the specified coordinate system $\{\xi\}$ and $\mathrm{F}^0(\xi)$,$\mathrm{F}^1(\xi)$,$\mathrm{F}^2(\xi)$,$\mathrm{F}^3(\xi)$, and vector field J is the flux density of

$S[\{\xi\},F]$.

By using Gaussian theorem, we have

$$\int_{\Omega} d^4x \frac{\partial}{\partial x^\alpha}[\sqrt{-|g(x)|}J^\alpha(x)] = \int_{\Omega} d^4x[\sqrt{-|g(x)|}\nabla_\alpha J^\alpha(x)] = 0 \quad 21)$$

and due to the arbitrariness of Ω, we have

$$\nabla_\alpha J^\alpha(x) = 0 \quad 22)$$

Eqns. (20), (22) are conservation laws of scalar $S[\{\xi\},F]$. Note that, $\int_{\partial\Omega} \mathrm{ds}_\alpha(\xi)[\sqrt{-|\mathrm{g}(\xi)|}\mathrm{F}^\alpha(\xi)] = 0$ and $\int_{\partial\Omega} \mathrm{ds}_\alpha(\zeta)[\sqrt{-|\mathrm{g}(\zeta)|}\mathrm{F}^\alpha(\zeta)] = 0$ are conservation laws of different scalars $S[\{\xi\},F]$, $S[\{\zeta\},F]$.

We are now in a position to explore the meaning of conservation and non-conservation in curved spacetime.

4. Conservation and Non-Conservation in GR

It is Einstein and mainstream scholar's viewpoint that the perfect mathematical expression for conservation law of a physical quantity is a continuity integral equation.

When the physical quantity is a scalar, say, the electric charge Q, denote by J its flux density vector field, then the continuity equation for charge conservation is

$$\int_{\partial\Omega} ds_\lambda(x)\sqrt{-|g(x)|}J^\lambda(x) = [\int_{\Sigma'} + \int_\Sigma + \int_\Gamma] ds_\lambda(x)\sqrt{-|g(x)|}J^\lambda(x) = 0,$$

$$\forall \Omega \subset M \,\&\, \overline{\Omega}\ is\ compact \qquad 23)$$

It reads, the electric charge on Σ' equals the electric charge on Σ plus the electric charge flowing through Γ, or the net increase of electric charge in Ω is zero. As has been shown in section 3, scalars of the same kind at different points in curved spacetime can be added up. All the terms in eqn. (23) are meaningful. The above reading is good. Due to Gaussian theorem, eqn. (23) is equivalent to

$$\nabla_\lambda J^\lambda(x)|_p = 0, \forall\, p \in M \qquad 24)$$

that is the covariant divergence of J vanishes everywhere in spacetime. Eqn. (24) reads, the net increase of electric charge in any small 4-dimensional spacetime neighborhood is zero.

When the physical quantity is a vector, say, matter energy-momentum P, vanishing of the covariant divergence of P's flux density tensor field T

$$\nabla_\lambda T^{\lambda\mu}(x)|_p = 0, \forall_\mu = 0,1,2,3;\, p \in M \qquad (2)$$

is equivalent to

$$[\int_{\Sigma'} + \int_\Sigma + \int_\Gamma] ds_\lambda(x)\sqrt{-|g(x)|}T^{\lambda\mu}(x) +$$

$$+\int_\Omega d^4x\sqrt{-|g(x)|}\Gamma^\mu_{\lambda\sigma}(x)T^{\lambda\sigma}(x) = 0, \forall_\mu = 0,1,2,3 \qquad (4)$$

which is not the continuity equations for matter energy-momentum conservation accepted by Einstein and mainstream scholars:

$$[\int_{\Sigma'} + \int_{\Sigma} + \int_{\Gamma}] ds_\lambda(x)\sqrt{-|g(x)|}T^{\lambda\mu}(x) = 0, \forall_\mu = 0,1,2,3 \qquad 25)$$

Eqn. (25) is read by them as the matter energy-momentum distributed on Σ' equals the matter energy-momentum distributed on Σ plus the matter energy-momentum flowing in through Γ, or the net increase of matter energy-momentum in Ω is zero. We have pointed out in section 2, and proved in section 3 that, (i) All those concepts like the matter energy-momentum distributed on $\Sigma(\Sigma')$, the matter energy-momentum flowing through Γ,the matter energy-momentum created in Ω, are meaningless in curved spactime. (ii) There is no flat coordinate system in curved spacetime, and in a curvilinear coordinate system, the sum of β-components of vectors at different points is not equal to the β-components of the sum vector which does not exist really.

No one can do things which are not allowed by geometry. Therefore, the only correct math expression for matter energy-momentum conservation in curved spacetime is eqn. (2), $\nabla_\lambda T^{\lambda\mu}(x)|_p = 0, \forall 0 \leq \alpha \leq 3, p \in M$, which reads the net increase of matter energy-momentum in any small 4-dimensional spactime neighborhood is always zero.

We see, Einstein's field equations do not lead to non-conservation of matter energy-momentum, therefore introducing gravitational energy-momentum to save the law of energy-momentum conservation is just like painting the lily, or in Chinese proverb, adding feet to a snake's picture. It does not save but destroys the law of energy-momentum conservation, and causes century long confusions in GR.

Right after Einstein introduced his gravitational energy-momentum flux density $t^{\alpha\beta}(x)$, Bauer immediately pointed out that this $t^{\alpha\beta}(x)$ is not a tensor and is not localizable [14]. Besides, it is not symmetrical. When there is no point-like angular momentum distribution (spin) and no spin-orbit coupling, a non-symmetrical stress tensor is not acceptable. Decades later, Landau and Lifshitz proposed a symmetrical gravitational energy-momentum flux density $t^{\alpha\beta}(x) = t^{\beta\alpha}(x)$, satisfying the following equation [15]

$$\frac{\partial}{\partial x^\lambda}[-|g(x)|(T^{\lambda\mu}(x)+t^{\lambda\mu}(x))]=0,\ \forall_\mu=0,1,2,3 \qquad 26)$$

which is equivalent to

$$\int_{\partial\Omega} ds_\lambda(x)(-|g(x)|)(T^{\lambda\mu}(x)+t^{\lambda\mu}(x))=0,\ \forall_\mu=0,1,2,3 \qquad 27)$$

However, its hyper-surface element $ds_\lambda(x)(-|g(x)|)$ does not have the correct transformation property under general coordinate transformations. Hence it is wrong to take $T^{\lambda\mu}(x)(-|g(x)|)ds_\lambda(x)|_{\Delta\Sigma}$ as the μ-component of matter energy-momentum on small hyper-surface element $\Delta\Sigma$ in coordinate system $\{x\}$, let alone to take $\int_{\partial\Omega} ds_\lambda(x)(-|g(x)|)T^{\lambda\mu}(x)$ as the μ-component of matter energy-momentum 4-vector on

hyper-surface Σ in coordinate system $\{x\}$.

Following Einstein, Tolman, Trautman, Papapetrou, Bergmann and Moller et al. also proposed several gravitational energy-momentum complexes [16,17,18,19,20]. They are all pseudotensors without exception in the following sense.

$$t^{\lambda\mu}(y)\neq\frac{\partial y^\lambda}{\partial x^\alpha}\frac{\partial y^\mu}{\partial x^\beta}t^{\alpha\beta}(x),\ \forall 0\ll\lambda,\mu\ll 3 \qquad 28)$$

One of the direct consequences of pseudotensor character is the non-localizability. Efforts to search for a covariant localizable description of gravitational energy-momentum have never ceased; but all of them failed. Pseudotensor character and non-localizability of gravitational energy-momentum are attributed to the equivalence principle (See, e.g. [21]). Accepting that the non-localizability of gravitational energy-momentum is an inevitable consequence of equivalence principle, some relativists switched to search for the total gravitational energy when spacetime is asymptotically flat at spacelike and null infinity [22],[23],[24]. The proof of the positivity of the

ADM mass and Bondi mass is considered the greatest achievements in classical GR in the last quarter of 20th century [25], [26]. This success inspired the search for quasi-local conserved quantities (quantities which can be associated to extended but finite spacetime domains) [27]. However, finding an appropriate quasi-local notion of energy-momentum has proven to be surprisingly difficult.

Equivalence principle, which is important part of GR's foundation, has been used by some leading scholars as the shelter for non-localizable gravitational energy-momentum pseudotensors for more than half century long. So, a revisit to the equivalence principle might be helpful in clearing the confusion. Let me quote a typical argument from section 20.4 of an influential book by Misner et al [21]. *"…One can always find in any given locality a frame of reference in which all local 'gravitational fields' (all Christoffel symbols; all $\Gamma^{\alpha}_{\mu\nu}$) disappear. No $\Gamma^{'}s$ means no 'gravitational field' and no local gravitational field means no 'local gravitational energy-momentum.' …Nobody can deny or wants to deny that gravitational forces make a contribution to the mass-eneryy of a gravitationally interacting system. …At issue is not the existence of gravitational energy, but the localizability of gravitational energy. It is not localizable. The equivalence principle forbids.*"

To illustrate why the above argument is wrong, let us consider the case, which has been used by H. Bauer in [14]: When the matter energy-momentum flux density tensor field vanishes everywhere in spacetime: $T|_p = 0, \forall p \in M$, there exists flat solution to Einstein's field equations:

$$g_{\alpha\beta}(x)|_p = \eta_{\alpha\beta}, \forall\, 0 \leq \alpha, \beta \leq 3,\ p \in M \qquad 29)$$

that is, the spacetime is Minkowski space and $\{x\}$ is a Lorentzian (inertial) coordinate system. Choose a locality

$$\Delta\Omega =: \{q \in M: |x^{\alpha}_{q}| < \delta, \forall 0 \leq \alpha \leq 3\} \qquad 30)$$

and define a new coordinate system $\{y\}$, such that $y^{\alpha}_{p} = x^{\alpha}_{p}$, $\forall\, 0 \leq \alpha \leq 3$, $p \notin \Delta\Omega$, and when restricted on $\Delta\Omega$, $\{y\}|_{\Delta\Omega}$ is a curvilinear local coordinate system: $\Gamma^{\alpha}_{\beta\gamma}(y)|_q \neq 0$, $\forall\, q \in \Delta\Omega$. According to the authors of ref. [21], using coordinate system $\{y\}$, there would be gravitational energy-momentum in $\Delta\Omega$ (more accurately, there would be gravitational energy-momentum on spacelike hyper-surface in $\Delta\Omega$). Where does the gravitational energy-momentum come from? The flux density of matter energy-momentum vanishes everywhere, $T^{\alpha\beta}(y)|_q = 0$, $\forall q \in M$, $0 \leq \alpha$, $\beta \leq 3$, that means no "matter energy-momentum" changes into gravitational energy-momentum in any small spacetime neighborhood. Besides, there is no gravitational wave outside $\Delta\Omega$: $g_{\alpha\beta}(y)|_p = \eta_{\alpha\beta}$, $\forall p \notin \Delta\Omega$, hence no gravitational energy-momentum flows into $\Delta\Omega$. The law of conservation of energy-momentum is broken!

The ratio of inertial mass to gravitational mass for an object does not vary from object to object. This is an experimental fact. As a consequence of $m_I = m_G$, gravitational force can not be distinguished from inertial force, hence they are the same thing. This enlightened Einstein to realize that gravity is only the manifestation of spacetime bending. In GR, the spacetime bending is completely described by metric field g. There is no other dynamic variable describing gravity or inertial force in GR. The concepts of gravity and inertial force belong to Newtonian physics. Equivalence principle is only the midwife of GR, it does not belong to GR itself, as Synge has put it [28].

To finish the argument, let us pay a revisit to the fundamental concept of GR, spacetime. It might be helpful for understanding the above conclusion.

5. Spacetime

In Einstein's classical theory of general relativity, spacetime M is a 4-dimensional Lorentzian generalized Riemannian manifold. Geometry tells us, a topological manifold becomes a differential manifold when it is given a differential structure; and it further becomes a generalized Riemannian manifold, when it is further given a metric field g. If g is positive definite or negative definite, we call it a Riemannian manifold. If the signature of g is $(p,1)(p > 0)$ or $(1,q)(q > 0)$, we call it a Lorentzian generalized Riemannian manifold, or simply a Lorentzian manifold.

5.1 Physical realization of the differential structure of spacetime

In physics, a chart (U,u) of spacetime manifold M (U is a non-empty open subset of M) is constructed as follows. Placing infinite point-like (small) clocks such that their world lines cover U and do not intersect with each other within U (that is, for any $q \in U$, there is one and only one point-like clock's world line passing through q). Each point-like clock is given three specific ordered real numbers u^1,u^2,u^3, and the clock's numerical reading increases (or decreases) in its own way. If some event q happens within U, at a point-like clock labeled u^1,u^2,u^3, when the point-like clock's reading is u^0, then $(u^0 u^1,u^2,u^3)$ will be the coordinates of this event (spacetime point) $q(\in U)$.

When we have a collection of charts of M, $\{(U(a),u(a))|a \in A\} = :\mathcal{A}'$ (A is some set of indices) such that $\cup_{\alpha \in A} U(a) = M$, and all its members are c^∞-compatible with each other, then the c^∞-compatible atlas of M, $\mathcal{A}^{'}$, decides a unique smooth differential structure of M, $\mathcal{A}$, the collection of all charts of M that are c^∞-compatible with all members of $\mathcal{A}^{'}$.

5.2 Physical measurement of spacetime metric field

Every concept in physics should be measurable. The Lorentzian signatured metric field g of spacetime manifold M is no exception. For any spacetime point $p \in M$, and its neighborhood coordinate chart (X, x), we can measure all the components of g at p, $\{g_{\alpha\beta}(x)|_p | 0 \leq \alpha,\beta \leq 3\}$, by using several standard clocks, light pulse signal emitters and receivers.

Let us measure the components of metric field at $p \in M$, $g_{\alpha\beta}(x)$. Place a standard clock at the point-like clock labeled x^1,x^2,x^3 before the latter's reading reaches x^0, Write down the standard clocks' numerical reading τ, when the point-like clock's reading is x^0. And write down their numerical readings $\tau + \Delta\tau$, $x^0 + \Delta x^0$ again shortly.

Because a standard clock tells its proper time, we have (neglecting higher order infinitesimals)

$$-c^2(\Delta\tau)^2 = g_{\alpha\beta}(x)\Delta x^\alpha \Delta x^\beta = g_{00}(x)\Delta x^0 \Delta x^0 \qquad 31)$$

hence

$$g_{00}(x) =- c^2(\Delta\tau)^2/(\Delta x^0)^2 \qquad 32)$$

In order to determine the other components, emit a light pulse signal at p. When this signal arrives at point-like clock labeled $x^1 + \Delta x^1, x^2, x^3 (x^1 - \Delta x^1, x^2, x^3)$, write down the clock's numerical reading $x^0 + \xi(x^0 + \xi^{'})$. Because the world lines of light pulse signals are light-like, we have (neglecting higher order infinitesimals)

$$0 = g_{00}(x)\xi^2 + 2g_{01}(x)\xi\Delta x^1 + g_{11}(x)(\Delta x^1)^2 \qquad 33)$$

$$0 = g_{00}(x)\xi^{'2} - 2g_{01}(x)\xi^{'}\Delta x^1 + g_{11}(x)(\Delta x^1)^2 \qquad 34)$$

Subtracting eqn. (34) from eqn. (33) gives

$$g_{01}(x) = g_{00}(x)\frac{\xi^{'} - \xi}{2\Delta x^1} \qquad 35)$$

$$g_{11}(x) =- g_{00}(x)\frac{\xi\xi'}{(\Delta x^1)^2} \qquad 36)$$

Similarly, if when the above mentioned signal arrives at point-like clock labeled $x^1, x^2 + \Delta x^2, x^3$ $(x^1, x^2 - \Delta x^2, x^3)$ $\{x^1, x^2, x^3 + \Delta x^3$ $(x^1, x^2 - \Delta x^2, x^3)\}$ the clock's numerical reading is $x^0 + \eta\ (x^0 + \eta^{'})$ $\{x^0 + \zeta\ (x^0 + \zeta^{'})\}$, then we have

$$g_{02}(x) = g_{00}(x)\frac{\eta' - \eta}{2\Delta x^2} \qquad 37)$$

$$g_{22}(x) = - g_{00}(x)\frac{\eta\eta'}{(\Delta x^2)^2} \qquad 38)$$

and

$$g_{03}(x) = g_{00}(x)\frac{\zeta' - \zeta}{2\Delta x^3} \qquad 39)$$

$$g_{33}(x) = - g_{00}(x)\frac{\zeta\zeta'}{(\Delta x^3)^2} \qquad 40)$$

Let us determine the rest. If when the above-mentioned signal arrives at point-like clock labeled $x^1 + \Delta x^1, x^2 + \Delta x^2, x^3 (x^1, x^2 + \Delta x^2, x^3 + \Delta x^3, x^1 + \Delta x^1, x^2, x^3 + \Delta x^3)$, the clock's numerical reading is $x^0 + \rho(x^0 + \sigma, x^0 + \tau)$, then we have

$$g_{00}(x)\rho^2 + 2g_{01}(x)\rho\Delta x^1 + 2g_{02}(x)\rho\Delta x^2 + g_{11}(x)(\Delta x^1)^2$$

$$+ 2g_{12}(x)\Delta x^1\Delta x^2 + g_{22}(x)(\Delta x^2)^2 = 0 \qquad 41)$$

hence

$$g_{12}(x) = - \frac{g_{00}(x)}{2\Delta x^1\Delta x^2}[\rho^2 + \rho(\xi' - \xi) + \rho(\eta' - \eta) - \xi\xi' - \eta\eta'] \qquad 42)$$

Similarly we have

$$g_{23}(x) =- \frac{g_{00}(x)}{2\Delta x^2 \Delta x^3} [\sigma^2 + \sigma(\eta' - \eta) + \sigma(\zeta' - \zeta) - \eta\eta' - \zeta\zeta'] \qquad 43)$$

$$g_{13}(x) =- \frac{g_{00}(x)}{2\Delta x^3 \Delta x^1} [\tau^2 + \tau(\zeta' - \zeta) \quad + \tau(\xi' - \xi) - \zeta\zeta' - \xi\xi'] \qquad 44)$$

Thus, for any spacetime point $p \in M$, we can determine the spacetime metric at p, $g|_{\mathrm{p}}$, by using the above-mentioned finite measurement results. These measurement results are the least measurement results needed for determining $g|_{\mathrm{p}}$. Note that, eqns. (31) through (44) ($\because$ eqn. (31) $\Leftrightarrow$ eqn. (32), eqn. (33) and eqn. (34) $\Leftrightarrow$ eqn. (35) and eqn. (36), eqn.(41) $\Leftrightarrow$ eqn.(42), $\therefore$ there are 10 independent equations) are inhomogeneous linear equations for the 10 unknowns $\{g_{\alpha\beta}(\mathrm{x}) | 0 \leq \alpha \leq \beta \leq 3\}$.

Whether more similar measurement results would lead to contradiction or not? If they would, the fundamental concept of spacetime in GR would be wrong. I do not believe this could happen, because Einstein's spacetime concept is logically so beautiful, and there have been no experimental observation definitely against GR so far. Nevertheless, people can do more measurements to examine GR's spacetime concept. We see, by doing more than necessary above-mentioned measurements provides a way to prove or disprove GR's spacetime concept.

In geometry, the metric field contains all geometric information of a generalized Riemannian manifold. Particularly, in GR, the metric field g contains all geometric information of the spacetime. There is no dynamic variable other than spacetime metric field describing gravitation. The procedure for determining g discussed above can be taken as the working definition of g or the working definition of space-time. Clearly, this metric field g (the gravitational field in GR) is not a new kind of matter field itself, it is the geometric aspect of the moving matter continuum, it is a pure geometric notion.

6. Appendix

In subsection (2) of section 2, the change of a mass point's energy-momentum was defined by eqns. (8) and (9). According to this definition, a mass point's energy-momentum keeps unchanged if and only if its world line is a timelike geodesics (if and only if it is freely falling). In section 4, we proved the conclusion, $\nabla_\lambda T^{\lambda\mu}(x) = 0$ is the law of matter energy-momentum conservation. In view of the fact that both conclusions are new, here I am going to prove they are not in contradiction to each other for a double check.

Suppose the world line of a mass point is

$$x^\alpha = X^\alpha(\tau),\ \ \tau \in \Delta,\ \ \forall\ \ 0 \le \alpha \le 3 \tag{A1}$$

where τ is its proper time. The contribution of the mass point to the total matter energy-momentum flux density is

$$T_{mp}^{\alpha\beta}(x) = \int_{\tau\in\Delta} d\tau \frac{dX^\alpha(\tau)}{d\tau} \frac{\delta^4(x - X(\tau))}{\sqrt{-|g(x)|}} m \frac{dX^\beta(\tau)}{d\tau} \tag{A2}$$

If both the definition eqns. (8) plus (9) and the conclusion of section 4 are correct, we should have the following proposition.

Proposition 2: $\nabla_\alpha T_m^{\alpha\beta}(x) = 0,\ \ if\ and\ only\ if\ \ x^\alpha = X^\alpha(\tau)$ is a timelike geodesic.

Proof:

$$\nabla_\alpha T_{mp}^{\alpha\beta}(x) = \partial_\alpha T_{mp}^{\alpha\beta}(x) + \Gamma_{\alpha\rho}^{\alpha}(x) T_{mp}^{\alpha\beta}(x) + \Gamma_{\alpha\sigma}^{\beta}(x) T_{mp}^{\alpha\sigma}(x) \tag{A3}$$

$$= \int_{\tau\in\Delta} d\tau \frac{dX^{\alpha}(\tau)}{d\tau} \frac{\partial}{\partial x^{\alpha}} \{ \frac{\delta^4(x - X(\tau))}{\sqrt{-|g(x)|}} \} m \frac{dX^{\beta}(\tau)}{d\tau}$$

$$+ \Gamma^{\alpha}_{\alpha\rho}(x) T^{\alpha\beta}_{m}(x)$$

$$+ \int_{\tau\in\Delta} d\tau \frac{dX^{\alpha}(\tau)}{d\tau} \frac{\delta^4(x - X(\tau))}{\sqrt{-|g(x)|}} m \Gamma^{\beta}_{\alpha\sigma}(x) \frac{dX^{\sigma}(\tau)}{d\tau}$$

$$=: < 1 > + < 2 > + < 3 >$$

$$< 1 > = \int_{\tau\in\Delta} d\tau \frac{dX^{\alpha}(\tau)}{d\tau} \frac{\partial}{\partial x^{\alpha}} \{ \frac{1}{\sqrt{-|g(x)|}} \} \delta^4(x - X(\tau)) m \frac{dX^{\beta}(\tau)}{d\tau}$$

$$+ \int_{\tau\in\Delta} d\tau \frac{dX^{\alpha}(\tau)}{d\tau} \frac{1}{\sqrt{-|g(x)|}} \frac{\partial}{\partial x^{\alpha}} \delta^4(x - X(\tau)) m \frac{dX^{\beta}(\tau)}{d\tau}$$

$$= \int_{\tau\in\Delta} d\tau \frac{dX^{\alpha}(\tau)}{d\tau} \frac{-1}{\sqrt{-|g(x)|}} \Gamma^{\rho}_{\rho\alpha}(x) \delta^4(x - X(\tau)) m \frac{dX^{\beta}(\tau)}{d\tau}$$

$$+ \int_{\tau\in\Delta} d\tau \frac{-1}{\sqrt{-|g(x)|}} \frac{dX^{\alpha}(\tau)}{d\tau} \frac{\partial}{\partial X^{\alpha}(\tau)} \delta^4(x - X(\tau)) m \frac{dX^{\beta}(\tau)}{d\tau} \tag{A4}$$

$$\because \Gamma^{\alpha}_{\beta\gamma}(x) = \frac{1}{2} g^{\alpha\rho}(x) [\partial_{\beta} g_{\rho\gamma}(x) + \partial_{\gamma} g_{\rho\beta}(x) - \partial_{\rho} g_{\beta\gamma}(x)] \tag{A5}$$

$$\therefore \Gamma^{\alpha}_{\alpha\gamma}(x) = \frac{1}{2} g^{\alpha\rho}(x)\partial_{\gamma} g_{\rho\alpha}(x) \tag{A6}$$

$$\frac{\partial}{\partial x^{\gamma}} \frac{1}{\sqrt{-|g(x)|}} = \frac{1}{2} \frac{|g(x)|}{[-|g(x)|]^{3/2}} g^{\alpha\beta}(x)\partial_{\gamma} g_{\beta\alpha}(x) = -\frac{1}{\sqrt{-|g(x)|}} \Gamma^{\alpha}_{\alpha\gamma}(x) \tag{A7}$$

$$< 1 \geq -\Gamma^{\rho}_{\rho\alpha}(x) T^{\alpha\beta}_{m}(x) + \frac{-1}{\sqrt{-|g(x)|}} \int_{\tau\in\Delta} d\tau \frac{d}{d\tau} \delta^{4}(x - X(\tau)) m \frac{dX^{\beta}(\tau)}{d\tau}$$

$$= -< 2 > + \int_{\tau\in\Delta} d\tau \frac{\delta^{4}(x - X(\tau))}{\sqrt{-|g(x)|}} m \frac{d^{2}}{d\tau^{2}} X^{\beta}(\tau) \tag{A8}$$

(The boundary term is zero.)

$$\therefore \nabla T^{\alpha\beta}_{mp}(x) = \int_{\tau\in\Delta} d\tau \frac{\delta^{4}(x - X(\tau))}{\sqrt{-|g(x)|}} m \left[\frac{d^{2}}{d\tau^{2}} X^{\beta}(\tau) + \Gamma^{\beta}_{\alpha\sigma}(x) \frac{d}{d\tau} X^{\alpha}(\tau) \frac{d}{d\tau} X^{\sigma}(\tau) \right] \tag{A9}$$

Eqn. (A9) tells us, If and only if $x^{\alpha} = X^{\alpha}(\tau)$ is a timelike geodesics

$$\frac{d}{d\tau} \frac{dX^{\beta}(\tau)}{d\tau} + \Gamma^{\beta}_{\alpha\sigma}(x) \frac{dX^{\alpha}(\tau)}{d\tau} \frac{dx^{\sigma}(\tau)}{d\tau} = 0 \tag{A10}$$

We have $\nabla_{\alpha} T^{\alpha\beta}_{m}(x) = 0$.

References

[1] A. Einstein, Sitzungsber K. preuss. Akacl. Wiss. 2, 688(1916)

[2] A. Einstein, Sitzungsber K. preuss. Akacl. Wiss. 2, 154(1918)

[3] B.P. Abbott, et. al. (LIGO), Phys. Rev. Lett. 116, 061102 (2016)

[4] R.P. Feynman, et al, Feynman Lectures on Gravitation, Westview Press, Boulder (2002).

[5] B.P. Abbott, et al. Phys. Rev. Lett. 116, 241103 (2016).

[6] B.P. Abbott, et al. Phys. Rev. Lett. 118, 221101 (2017).

[7] B.P. Abbott, et al. Phys. Rev. Lett. 119, 141101 (2017).

[8] B.P. Abbott, et al. Phys. Rev. Lett. 119, 161101 (2017)

[9] Z. Wu, Commun. Theor. Phys. 65 716-730 (2016).

[10] A. Einstein, Berl. Ber. 178 (1915), 448 (1918).

[11] H. Bondi, Proc. R. Soc. Lond. A 427 249-258 (1990).

[12] S. Weinberg, Gravitation and Cosmology (Wiley, New York, 1972).

[13] J.M. Nester, et al, Dynamic geometry and Gravitational Energy, (2004).

[14] H. Bauer, Physikalische Zeitschrift, 19 163 (1918).

[15] L.D. Landau and E.M.Lifshitz, The Classical Theory of Fields, 2nd ed. (Reading, Mass.: Addison-Wesley, 1962).

[16] R.C. Tohnan, Phys. Rev. 35, 875 (1930).

[17] A. Trautman, in Gravitation: An Introduction to Current Research, ed. L. Witten (Wiley, New York, 1962), 169-198.

[18] Papapetrou, Proc.Roy. Irish Acad. A 52, 11-23 (1948).

[19] P.G. Bergmann and R. Thompson, Phys. Rev. 89, 400-407 (1953).

[20] C. Mller, Ann. Phys. 4, 347-371 (1958).

[21] C.W. Misner, K.S. Thorne and J.A. Wheeler, Gravitation (Freeman, San Francisco, 1973).

[22] A. Komar, Physical Rev. 113 (1959) 934.

[23] R. Arnowitt, S. Deser, and C.W. Misner, The Dynamics of General Rela,- tivity, in Gravitation: A Introduction to Current Research, ed. L. Witten, Wiley, New York (1962).

[24] H. Bondi, M.G.J. van der Burg and A.W.K. Metzner, Proc. Roy. Soc. London A 269 {1962} 21.

[25] R. Schoen and S.-T. Yau, Commun. Math. Phys. 79, 231 (1981).

[26] E. Witten, Commun. Math. Phys. 80, 381(1981).

[27] L.B. Szabados, Living Rev. Relativity 7, 4 (2004).

[28] J.L. Synge, Relativity: The General Theory, North-Holland Publishing Company, Amsterdam (1960).