SCIREA Journal of Physics
ISSN: 2706-8862
http://www.scirea.org/journal/Physics
December 8, 2020
Volume 5, Issue 6, December 2020

Trigonometric derivations of the fine structure constant, the proton charge radius, the proton to electron mass ratio, the electron charge to mass ratio and the classical expressions for the Compton wavelengths

Emmanuel Saucedo-Flores
DIP-CUCEI Universidad de Guadalajara, José Guadalupe Zuno No. 48, 45101 Zapopan, Jalisco, México
Email: emmmauel.saucedo@cucei.udg.mx

Abstract

Over 20 physical definitions of the Sommerfeld or fine-structure constant α are derived using sine functions plots of a few H atom-related parameters; the presence of the particle-wave duality in most of the derived expressions is indicated. Five expressions for the electron charge to mass ratio are also established. An approximation for the arithmetic value of the proton charge radius is proposed. Also, several analytical expressions for the proton to electron mass ratio are provided. An expression using proton parameters to calculate the exact 2018 CODATA recommended value of Planck constant h is obtained. hc, h/c, $h/q_P/c$ and Compton wavelengths equations having the *quantum* = *classical* configuration were established.

Keywords: Fine-structure constant, proton charge radius, proton to electron mass ratio, electron charge to mass ratio, geometrical mean of a_0 and r_e

Introduction.

The fundamental Sommerfeld or fine-structure constant, SC for short, has received a lot of well-deserved denotations like mysterious, magic, divine and so on, [1]. Its main fundamental physical interpretation is that it describes the interaction intensity of the electric and magnetic fields created by the movement of charged particles and those inherent to photons. It is this involved particle-wave nature of α which this writing is mainly intended to address.

The SC in terms of a_0 and λ_∞

Figure 1 a) shows the cycles in 2π rad for functions defined by sine($x\lambda_\infty/2P_0$) and sine($x\lambda_\infty/\lambda_\infty$)

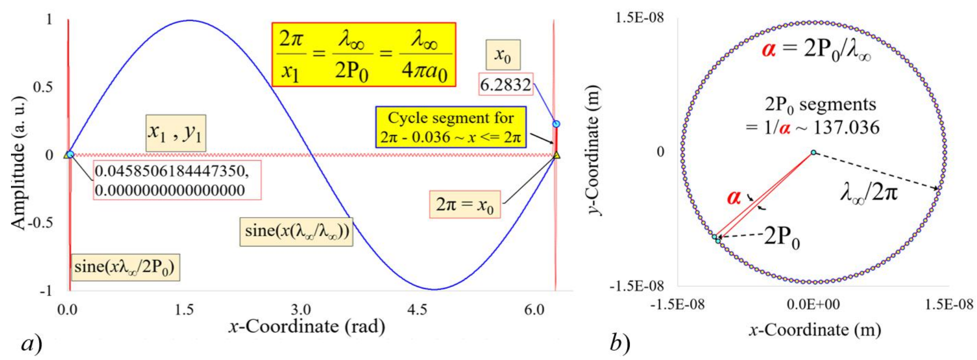

Figure 1. a) One cycle of the Rydberg wavelength, λ_∞, given by sine(x) versus the cycles of 2 times the H atom perimeter, $2P_0$, plotted as sine($x\lambda_\infty/2P_0$) whose inner cycle amplitude is reduced for clarity. b) Circular representation of x_1/x_0.

where λ_∞ (m) is the Rydberg wavelength unit, P_0 (m) is the perimeter of the H atom and x (rad) is the angular coordinate. a_0 (m) stands for the Bohr radius. The x-coordinate value at the first cycle of each function is indicated, namely $x_0 = 2\pi$ and $x_1 \sim 0.04585062$, their quotient gives the cycle ratio between the two functions. x_1 digits were fine-tuned in a spreadsheet cell until a y_1 function amplitude lower than | (4×10^{-17}) | was obtained.

The ratio of the two coordinates states

$$\frac{x_0}{x_1}=\frac{2\pi}{0.0458506184447348}=137.035999083\ \underline{696}=\frac{\lambda_\infty}{4\pi a_0}=137.035999083\ \underline{816} \quad (1)$$

Last value in (1), FSC_inverse, was met using data for a_0 and λ_∞ as internationally recommended by the 2018 CODATA [2] Bohr r NIST and $\lambda_\infty = 1/R_\infty$ - with R_∞ (m^{-1}) being the Rydberg constant R∞ NIST -. It provides

$$\frac{1}{4\pi a_0/\lambda_\infty}=\frac{1}{2P_0/\lambda_\infty}=\frac{1}{0.007\ 297\ 352\ \ 569\ \ \underline{2936}}\equiv\frac{1}{\alpha} \quad (2)$$

where α is the fine-structure constant and its calculated value gives a 12-digit match to the arithmetic value provided in [2] alpha definition; non-matching digits are underlined.

The α physical interpretation from (2), see Figure 1 b), corresponds to an angle given by the ratio of two arc lengths, one associated to a *particle* and the other to a *wave*, and is calculated as follows

$$\alpha=\frac{4\pi a_0}{\lambda_\infty}=\frac{2P_0}{\lambda_\infty}\equiv 4\pi\beta=\frac{2\ \text{x the } \textit{electron orbit length}\ P_0\ (\text{m})}{\text{emitted } \textit{photon wavelength} \text{ for } n=\infty\rightarrow 1\ e\text{-transition (m)}} \quad (3)$$

where the definition $\beta = a_0/\lambda_\infty$ was introduced. This particle-wave duality α-interpretation is kept in terms of an energy ratio by converting (3) as follows

$$\alpha=\frac{hc}{e}\frac{e}{hc}\frac{2P_0}{\lambda_\infty}=\frac{Ry}{E_{2P_0}}=\frac{\textit{energy}\text{ lost by the } \textit{electron} \text{ for } n=\infty\rightarrow 1\ e\text{-transition (eV)}}{\textit{energy} \text{ of a } 2P_0 \text{ wavelength } \textit{photon} \text{ (eV)}} \quad (4)$$

where h (J s) is Planck constant, c (cm s^{-1}) is the speed of the electro-magnetic radiation in vacuum and e (C) is the electron charge.

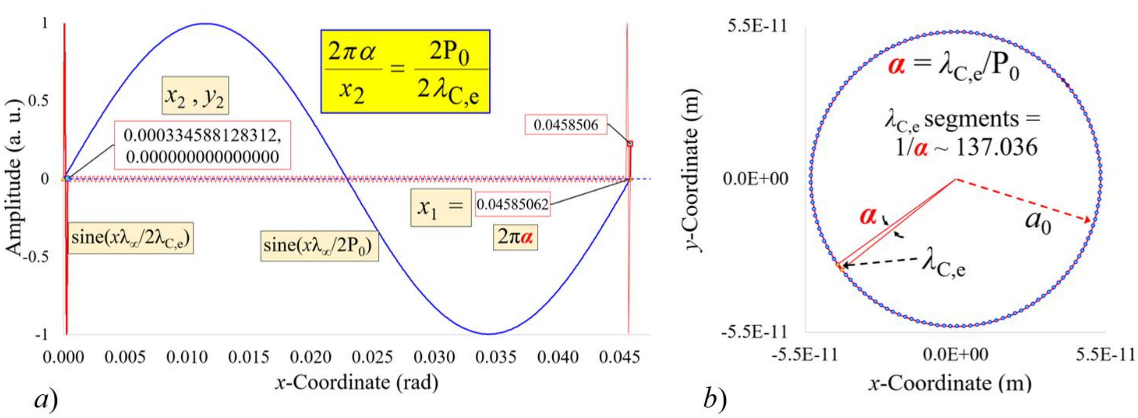

Figure 2. a) One cycle of $2P_0$, given by sine($x\lambda_\infty/2P_0$) versus one cycle of 2 times the Compton wavelength, $2\lambda_{C,e}$, plotted as sine($x\lambda_\infty/2\lambda_{C,e}$). b) Circular representation of x_2/x_1.

The SC in terms of a_0 and $\lambda_{C,e}$, in terms of v_1 and c and four variations.

Figure 2 shows the first cycle for functions defined by sine($x\lambda_\infty/2P_0$) and sine($x\lambda_\infty/2\lambda_{C,e}$) where $\lambda_{C,e} = h/m_e c$ (m) is the Compton wavelength for the electron rest mass m_e (kg). The x-coordinate values for the first cycle of each function are indicated, namely, $2\pi\alpha$ and x_2, whose quotient gives the cycle ratio between the two functions. Coordinate point (x_2, y_2) was determined similarly as (x_1, y_1) above and the following ratio is established

$$\frac{x_1}{x_2} = \frac{2\pi\alpha}{0.000334588128311681} = 137.035\ 999\ 083\ 696 = \frac{1}{\alpha} \tag{5}$$

Also, the argument of the function sine($x\lambda_\infty/2\lambda_{C,e}$) evaluated at x_2, using the 2018 CODATA $\lambda_{C,e}$ value [2] Compton e NIST, gives

$$x_2 = 2\pi\frac{2\lambda_{C,e}}{\lambda_\infty} = 2\pi(5.32514 \text{ x } 10^{-5}) = 2\pi\alpha^2 \tag{6}$$

which provides

$$\alpha^2 \equiv \frac{2\lambda_{C,e}}{\lambda_\infty} = 2\frac{\textit{electron mass} \text{ related wavelength of a } \textit{photon} \text{ (m)}}{\text{Rydberg unit of } \textit{photon} \text{ wavelength (m)}} \tag{7}$$

(7), as done in (4), can be converted into the following energy ratio

$$\alpha^2 \equiv \frac{2Ry}{E_e} = 2\frac{\text{energy of emitted } \textit{photon} \text{ for } n=\infty\rightarrow 1 \ e\text{-transition (eV)}}{\textit{electron energy-mass} \text{ equivalent (eV)}} \tag{8}$$

where $E_e = m_e c^2/e$ (eV) is the electron invariant mass energy equivalent.

(7) and (8) don't show the particle-wave duality, however this feature is unveiled using (3) in (7) to obtain, see plot at Figure 2b,

$$\alpha = \frac{\lambda_{C,e}}{2\pi a_0} \equiv \frac{\lambda_{C,e}}{P_0} = \frac{\textit{wavelength} \text{ of a } \textit{photon} \text{ with energy } E_e \text{ (m)}}{\textit{electron} \text{ ground } \textit{orbit} \text{ length (m)}} \tag{9}$$

This cycle ratio gives another two H atom related lengths ratio α physical interpretation but this time with the particle-wave duality nature inverted in the quotient as compared to (3). Its conversion into an energy ratio keeps this feature and is given as

$$\alpha = \frac{E_{P_0}}{E_\mathrm{e}} = \frac{\text{energy of a } \mathrm{P}_0 \ \textit{wavelength photon} \ (\mathrm{J})}{\textit{electron}-\textit{mass} \ \text{energy equivalent} \ (\mathrm{J})} \tag{10}$$

From (9), it follows that

$$\alpha = \frac{\lambda_{C,\mathrm{e}}}{\mathrm{P}_0} = \frac{\hbar}{a_0 m_\mathrm{e} c} = \frac{\text{angular quantum of } \textit{electromagnetic} \text{ action } (\mathrm{J\cdot s})}{\text{orbital moment of } \textit{electron} \text{ momentum limit } (\mathrm{J\cdot s})} \tag{11}$$

and also

$$\alpha = \frac{[\hbar / a_0 m_\mathrm{e}]}{c} \equiv \frac{v_1}{c} = \frac{\textit{electron} \text{ speed at ground state } (\mathrm{m\cdot s^{-1}})}{\text{speed of } \textit{photons} \text{ in vacuum } (\mathrm{m\cdot s^{-1}})} \tag{12}$$

which gives α a two-speed ratio physical interpretation upon defining $v_1 = h/2\pi a_0 m_\mathrm{e}$ (m s^{-1}) which corresponds to the orbital electron speed in the H atom ground state. This v_1 expression is inserted in the a_0 definition in [2][Bohr r NIST] along with the one originally used by Sommerfeld for v_1 which is as follows

$$v_1 = \left[\frac{1}{4\pi\varepsilon_0}\right]\frac{e^2}{\hbar} \equiv k_\mathrm{e}\frac{e^2}{\hbar} \ (\mathrm{m\cdot s^{-1}}) \tag{13}$$

where $\mathcal{E}_0$ (F/m) is the vacuum dielectric permittivity and k_e (m J C^{-2}) is the Coulomb, or electric force, constant.

Using (13) in (12) gives

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} \tag{14}$$

This expression also has embedded an α physical interpretation as the square of a ratio of two charges given by

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} \equiv \frac{e^2}{\mathrm{q}_\mathrm{P}^2} = \frac{(\textit{classical} \text{ charge})^2 \ (\mathrm{C}^2)}{(\textit{quantum} \text{ Planck charge})^2 \ (\mathrm{C}^2)} \tag{15}$$

where the Planck charge constant, q_P (C), was introduced.

Also, (15) can be rearranged into

$$\alpha = \frac{e^2}{\hbar c}\frac{1}{4\pi\varepsilon_0} \equiv \frac{e^2 k_e}{\hbar c} = \frac{\text{moment of } \textit{electrostatic} \text{ energy (m}\cdot\text{J/rad)}}{\text{moment of } \textit{electromagnetic radiation} \text{ energy (m}\cdot\text{J/rad)}} \tag{16}$$

which neatly gives the α ratio a particle-wave duality nature. Additionally, (16) can be written as

$$\alpha = \frac{e^2 k_e / c}{\hbar} = \frac{\text{(unit of) } \textit{electric field} \text{ action (J s/rad)}}{\textit{quantum of electromagnetic} \text{ action (J s/rad)}} \tag{17}$$

Additionally, (15) and (12) provide

$$\alpha^2 = \frac{e^2 v_1 \mu_0}{4\pi\hbar} = \frac{1}{4\pi}\frac{\text{(unit of) } \textit{magnetic field} \text{ action (J}\cdot\text{s/rad)}}{\textit{quantum of electromagnetic} \text{ action (J}\cdot\text{s/rad)}} \tag{18}$$

where the classical expression for the speed of light in vacuum

$$c^2 = \frac{1}{\mu_0\varepsilon_0} \quad (\text{m}^2\cdot\text{s}^{-2}) \quad \Rightarrow \quad c\mu_0 = \frac{1}{c\varepsilon_0} \quad (\text{J}\cdot\text{s}\cdot\text{C}^{-2})$$

was used. μ_0 (H/m or N/A^2) stands for the vacuum permeability mag_constant.

The SC in terms of r_e and a_0

Figure 3 plots the first cycle for functions defined by sine($x\lambda_\infty/2\lambda_{C,e}$) and sine($x\lambda_\infty/2r_e$) where r_e (m) is the classical electron radius whose numeric value, [1] re_NIST , was used to determine x_3 previously to the figure. From the coordinate cycle ratio x_2/x_3 provided in Figure 3, the following particle-wave duality physical interpretation for α is derived

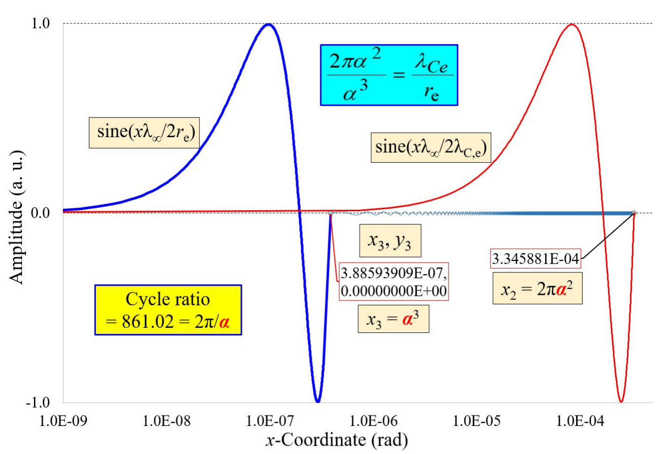

Figure 3. One cycle for $2\lambda_{C,e}$ given by sine($x\lambda_\infty/2\lambda_{C,e}$) versus one cycle for 2 times the electron classical radius, $2r_e$, plotted as sine($x\lambda_\infty/2r_e$).

$$\alpha = \frac{2\pi r_{\text{e}}}{\lambda_{C,\text{e}}} = \frac{\text{P}_{\text{e}}}{\lambda_{C,\text{e}}} = \frac{\textit{electron perimeter } (\text{m})}{\textit{wavelength} \text{ of an } E_{\text{e}} \text{ energy } \textit{photon } (\text{m})} \tag{19}$$

where P_e (m) is the electron perimeter and E_e (eV) is the energy equivalent for the electron rest mass.

The duality nature is kept after converting (19) into an energy ratio as follows

$$\alpha = \frac{E_{\text{e}}}{E_{\text{P}_{\text{e}}}} = \frac{\textit{electron} \text{ energy - mass equivalent (eV)}}{\text{energy of a } \text{P}_{\text{e}} \textit{ wavelenth photon } (\text{eV})} \tag{20}$$

Solving for $\lambda_{C,e}$ in (9) and using it in (19) gives

$$\alpha^2 = \frac{r_{\text{e}}}{a_0} = \frac{\text{electron radius (m)}}{\text{H atom radius (m)}} \tag{21}$$

Replacing $\lambda_{C,e}$ in (19) gives

$$\alpha = \frac{r_{\text{e}} m_{\text{e}} c}{\hbar} = \frac{\text{spin related } \textit{electron} \text{ angular momentum } (\text{J} \cdot \text{s/rad})}{\textit{quantum of electromagnetic} \text{ action } (\text{J s/rad})} \tag{22}$$

which also is a particle-wave duality ratio.

Replacing one α factor in (21) with (22) establishes

$$\alpha = \frac{\hbar}{a_0 m_{\text{e}} c} = \frac{\textit{quantum of electromagnetic} \text{ action } (\text{J s/rad})}{\text{orbit related } \textit{electron} \text{ angular momentum } (\text{J} \cdot \text{s/rad})} \tag{23}$$

This expression is used at the NIST web page to define a_0[Bohr r NIST]. Solving (21) for r_e and substituting in (19) gives upon rearranging

$$\alpha = \frac{\lambda_{\text{C,e}}}{2\pi a_0} = \frac{\textit{wavelength} \text{ of an } E_{\text{e}} \text{ energy } \textit{photon } (\text{m})}{\textit{H atom perimeter } (\text{m})} \tag{24}$$

which gives (23) also. The physical interpretation for the ratios in expressions (19) and (24) are presented in Figure 4.

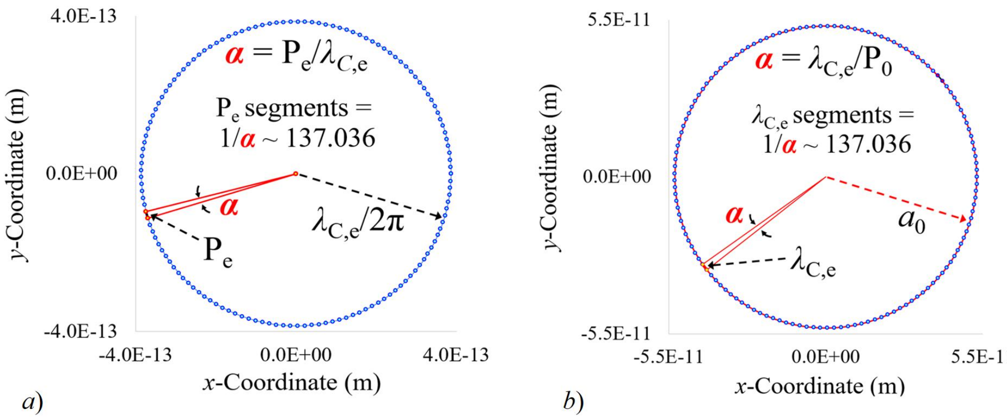

Figure 4. Trigonometric representation of expressions (19), a), and (24), b).

The electron charge to mass ratio.

Solving for $2\pi r_e$ in (22), (19) can be transformed into

$$\alpha = \frac{\lambda_\infty \alpha^3}{2h/m_e c}\left\{\frac{ec}{ec}\right\} = \frac{\alpha^3 c^2}{2hc/e\lambda_\infty}\frac{m_e}{e} = \frac{\alpha(\alpha c)^2}{2Ry}\frac{m_e}{e} \tag{25}$$

so that the electron charge to mass ratio electron e/m, introducing a negative sign, is given by

$$\frac{e}{m_e} = -\alpha^2\frac{c^2}{2Ry} = -\frac{r_e c^2}{2a_0 Ry} = -\frac{v_1^2}{2Ry} = -\frac{2\mu_B}{\hbar} = -\frac{c^2}{E_e} \quad (\mathrm{C\cdot kg^{-1}}) \tag{26}$$

where $\mu_B = eh/4\pi m_e$ ($\mathrm{JT^{-1}}$) is the Bohr magneton for the electron Bohr_μ_e.

From (26) it follows that the first orbit electron speed can also be expressed as

$$v_1 = 2\sqrt{\frac{\mu_B \mathrm{Ry}}{\hbar}} \quad (\mathrm{m\cdot s^{-1}}) \tag{27}$$

The proton charge radius

Figure 5 depicts the first cycle for functions sine($x\lambda_\infty/2\lambda_{C,p}$) and sine($x\lambda_\infty/2r_{pc}$) where $\lambda_{C,p}$ (m) is the Compton wavelength for the proton ($h/m_p c$) where m_p (kg) is the proton rest mass and r_{pc} (m) is the proton charge radius. Their respective first cycle x-coordinate were determined in a preliminary plot and using the numeric values given in [1] Compton for p and rp NIST. Although with this latter parameter a slight modification will be introduced shortly. μ represents the proton to electron mass ratio μ NIST.

Using the currently accepted value for the proton radius, the first cycle of function sine($x\lambda_\infty/2r_p$) provides

$$\sin(x\frac{\lambda_\infty}{2r_p}) = \sin(\frac{4\alpha_{r_p}^2}{\mu}\frac{\lambda_\infty}{2r_p}) = 0 \Rightarrow \frac{\lambda_\infty}{r_p}\frac{2\alpha_{rp}^2}{\mu} = 2\pi \tag{28}$$

Using the numeric value of α alpha definition and solving (28) for α_{rp}, their ratio gives the following result

$$\frac{\alpha_{rp}}{\alpha} \equiv \sqrt{\frac{\pi\mu r_p}{\lambda_\infty}} / \alpha = 1.000\ 097\ 684\ 0611 \equiv \frac{1}{\delta^2} \tag{29}$$

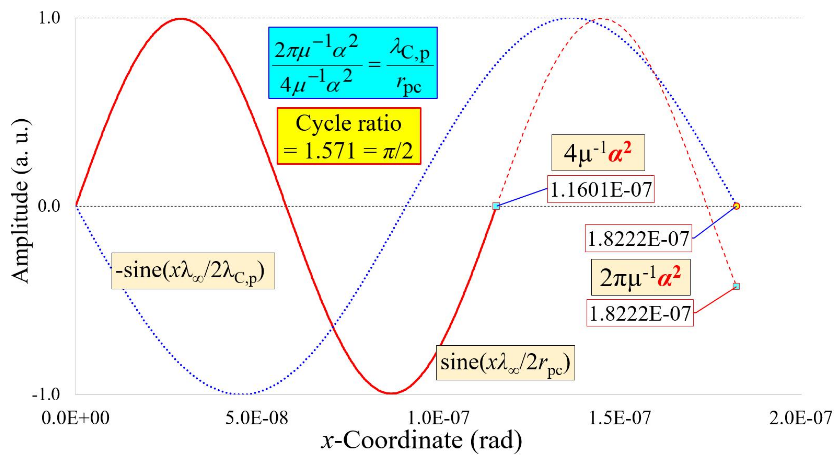

Figure 5. One cycle for 2 times the Compton wavelength for the proton, $2\lambda_{C,p}$, given by sine($x\lambda_\infty/2\lambda_{C,p}$) versus one cycle for 2 times the proton charge radius, $2r_{pc}$, plotted as sine($x\lambda_\infty/2r_{pc}$).

where δ has the value

$$\delta = 0.999\ 804\ 660\ 500\ 573 \tag{30}$$

with which in (29) the following proton charge radius is suggested

$$r_{pc} = r_p\delta = 8.414 \text{ x}10^{-16}\,\delta = \frac{\alpha^2\lambda_\infty}{\pi\mu} = 8.412\ \ 356\ \ 413\ \ 42237\ \text{ x } 10^{-16}\ \ (\text{m}) \tag{31}$$

This value is practically equal to the one proposed in [3]. An alternative derivation of the numeric value in (31) is given in [4].

(31) gives the following particle-wave ratio duality α expression

$$\alpha = \pm\sqrt{\pi\frac{r_{\mathrm{pc}}m_{\mathrm{p}}}{\lambda_\infty m_{\mathrm{e}}}} = \left(\pi\frac{\textit{radius}\ \text{arm moment of proton mass (m}\cdot\text{kg)}}{\text{Ry}\ \textit{wavelength}\ \text{arm moment of electron mass (m}\cdot\text{kg)}}\right)^{1/2} \tag{32}$$

whose numeric value gives a 13-digit match to the one recommended in alpha definition.

(32) equated to (21), provides another wave-particle duality relation given by

$$\pi\frac{\mu}{\rho} \equiv \pi\frac{m_{\mathrm{p}}/m_{\mathrm{e}}}{r_{\mathrm{e}}/r_{\mathrm{pc}}} = \frac{\lambda_\infty}{a_0} = \frac{\textit{photon wavelength}\ \text{for}\ n=\infty\rightarrow 1\ \text{e-transition (m)}}{\textit{H atom radius}\ \text{(m)}} \tag{33}$$

where the definition $\rho = r_{\mathrm{e}}/r_{\mathrm{pc}}$ was made. Equating (33) to (21) and using λ_∞ from (3) provides

$$\alpha = 4\frac{r_{\mathrm{e}}/r_{\mathrm{pc}}}{m_{\mathrm{p}}/m_{\mathrm{e}}} = 4\frac{\rho}{\mu} = 4\frac{\text{the electron to proton radius ratio}}{\text{the proton to electron mass ratio}} \tag{34}$$

This can be tranformed into

$$\alpha = 4\frac{r_{\mathrm{e}}m_{\mathrm{e}}}{r_{\mathrm{pc}}m_{\mathrm{p}}} = 4\frac{\text{spin moment of the}\ \textit{electron}\ \text{mass (m}\cdot\text{kg)}}{\text{spin moment of the}\ \textit{proton}\ \text{mass (m}\cdot\text{kg)}} \tag{35}$$

With (21), (35) becomes

$$\alpha = \frac{1}{4}\frac{r_{\mathrm{pc}}m_{\mathrm{p}}}{a_0 m_{\mathrm{e}}} = \frac{1}{4}\frac{\text{spin moment of}\ \textit{proton}\ \text{mass (m}\cdot\text{kg)}}{\text{orbital moment of}\ \textit{electron}\ \text{mass (m}\cdot\text{kg)}} \tag{36}$$

(35) can be modified as follows

$$\alpha = 4\frac{r_{\mathrm{e}}m_{\mathrm{e}}c}{r_{\mathrm{pc}}m_{\mathrm{p}}c} = 4\frac{\text{spin moment of the}\ \textit{electron}\ \text{momentum (J}\cdot\text{s)}}{\text{spin moment of the}\ \textit{proton}\ \text{momentum (J}\cdot\text{s)}} \tag{37}$$

and similarly (36) gives

$$\alpha = \frac{1}{4}\frac{r_{\mathrm{pc}}m_{\mathrm{p}}c}{a_0 m_{\mathrm{e}}c} = \frac{1}{4}\frac{\text{spin moment of the}\ \textit{proton}\ \text{momentum (J}\cdot\text{s)}}{\text{orbital moment of the}\ \textit{electron}\ \text{momentum (J}\cdot\text{s)}} \tag{38}$$

h **and** $\boldsymbol{\mu}$ **in terms of the proton charge radius.** $\boldsymbol{\lambda_{\mathrm{C,e}}}$ **and** $\boldsymbol{\lambda_{\mathrm{C,p}}}$ **in classical terms.**

From the cycle ratio given in Figure 5 and (32), another wave-particle expression follows as

$$\frac{\lambda_{\mathrm{C,p}}}{0.5\pi r_{\mathrm{pc}}} = 0.999\ 999\ 999\ 999\ 321 \tag{39}$$

where the $\lambda_{C,p}$ value is given in [5]Compton for p.

Dividing (39) by 4 gives, see Figure 6,

$$\frac{\lambda_{\mathrm{C,p}}}{2\pi r_{\mathrm{pc}}} = \frac{1}{4} = \frac{\textit{wavelength} \text{ of a } \textit{photon} \text{ with energy } E_{\mathrm{p}} \text{ (m)}}{\textit{proton} \text{ charge radius (m)}} \tag{40}$$

whose convertion into energy provides another wave-particle duality relation given by

$$\frac{E_{\lambda_{\mathrm{Pp}}}}{E_{\mathrm{p}}} = \frac{1}{4} = \frac{\text{Energy of a } \textit{photon wavelength} \text{ of } \mathrm{P_{pc}} \text{ (eV)}}{\textit{proton-mass} \text{ energy equivalent (eV)}} \ . \tag{41}$$

(40) con be transformed into

$$\frac{\hbar}{r_{\mathrm{pc}} m_{\mathrm{p}} c} = \frac{1}{4} = \frac{\textit{quantum of electromagnetic} \text{ action } (\mathrm{J \cdot s/rad})}{\textit{proton radius} \text{ arm moment of its } \textit{mass} \text{ momentum limit } (\mathrm{J \cdot s/rad})} \tag{42}$$

and additionally, multiplying by c/c, to

$$\frac{hc}{r_{\mathrm{pc}} e E_{\mathrm{p}}} = \frac{\pi}{2} = \frac{\text{moment of } \textit{quantum of electromagnetic} \text{ energy } (\mathrm{m \cdot J/cy})}{\textit{proton radius} \text{ arm moment of the } \textit{proton mass}\text{-energy eq. } (\mathrm{m \cdot J/cy})} \tag{43}$$

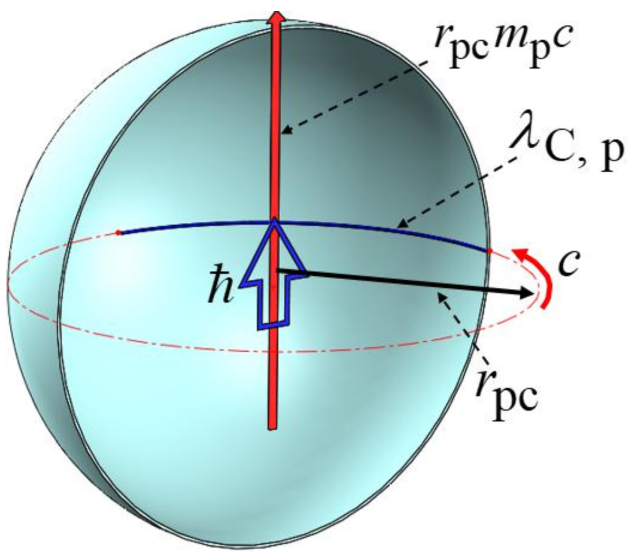

Figure 6. Proton cross section schematics showing the trigonometric representation for (40) and vector representation of (42).

The numeric value of h from (42) is

$$h = \frac{\pi}{2} r_{\text{pc}} m_{\text{p}} c = 6.626\ 070\ 15\ \text{x}\ 10^{-34}\ \ (\text{J} \cdot \text{s/cy})\ \ (\underline{\text{exact}}) \qquad (44)$$

which gives the Planck constant in terms of proton parameters and the speed of the EM radiation in vacuum, expression which can also be interpreted as the proton spin related moment of momentum limit. While for the electron spin-orbit related moment of momentum limit, (11) and (21) give

$$h = \frac{2\pi}{\alpha} r_{\text{e}} m_{\text{e}} c = \pm 2\pi \sqrt{a_0 r_{\text{e}}}\, m_{\text{e}} c = 6.626\ 070\ 1\underline{499}\ \ \underline{8131}\ \text{x}\ 10^{-34}\ \ (\text{J} \cdot \text{s/cy})\ \ (\text{rsu}\ 2.82\ \ \text{x}\ \ 10^{-12}) \qquad (45)$$

where rsu represents the relative standard uncertainty with respect to the h numeric value provided in Planck_h. On regards to the proton to electron mass ratio expressions, (34) gives

$$\mu = \frac{4\rho}{\alpha} = \frac{4 r_{\text{e}}}{\alpha r_{\text{p}}} = 1836.152\ \ 673\ \ 43\ \ \underline{358}\ \ \ (\text{rsu}\ 1.95\ \ \text{x}\ \ 10^{-12}) \qquad (46)$$

Also, (46) and (21) give μ the following comely expression for its ratio of masses in terms of all H radii

$$\mu = 4 \frac{\sqrt{a_0 r_{\text{e}}}}{r_{\text{pc}}} \qquad (47)$$

While (31) provides

$$\mu = \frac{\alpha^2 \lambda_\infty}{\pi r_{\text{pc}}} = 1836.152\ \ \ 673\ \ \ 43\ \ \ \underline{519}\ \ (\text{rsu}\ \ 2.83\ \ \ \text{x}\ \ \ 10^{-12}) \qquad (48)$$

Using (21) and (3) in (48) and rearranging conduces to

$$\mu = \frac{2 Ry E_{\text{p}}}{\alpha^2 E_{\text{e}}^2} \qquad (49)$$

Finally, using c from (12) in (42) and rearranging conduces to

$$\alpha = \frac{r_{\text{pc}} m_{\text{p}} v_1}{4\hbar} = \frac{1}{4} \frac{\textit{proton radius}\ \text{arm moment of}\ \textit{proton mass}\ \text{momentum (J} \cdot \text{s/rad)}}{\textit{quantum of electromagnetic}\ \text{action (J} \cdot \text{s/rad)}} \qquad (50)$$

while doing the same in (43) gives

$$\alpha = \frac{4\hbar}{r_{pc} e E_p / v_1} = 4 \frac{\textit{quantum} \text{ of EM action} \quad (\text{J}\cdot\text{s/cy})}{\textit{proton} \text{ action} \quad (\text{J}\cdot\text{s/cy})} \tag{51}$$

On the other side, from (44) and 45, the following *quantum moment of energy* = *classical moment of energy* equalities for the Einstein product of wavelength, λ_{ph} (m), and energy, E_{ph} (J), can be derived and are given by

$$\lambda_{ph} E_{ph} = hc = \frac{\pi}{2} r_{pc} m_p c^2 \approx 1.986446 \text{ x } 10^{-25} \quad (\text{m}\cdot\text{J/cy}) \tag{52}$$

and

$$hc = \pm 2\pi \sqrt{a_0 r_e}\, m_e c^2 \quad (\text{m}\cdot\text{J/cy}) \tag{53}$$

respectively. Also, combining (16) and (21) conduces to

$$hc = 2\pi \sqrt{a_0 / r_e}\, e^2 k_e \quad (\text{m}\cdot\text{J/cy}) \tag{54}$$

Finally, dividing by c^2, (52) and (53) are converted into

$$\frac{h}{c} = \lambda_{ph} \frac{E_{ph}}{c^2} = \frac{\pi}{2} r_{pc} m_p = \lambda_{C,p} m_p \approx 3.5177 \text{ x } 10^{-43} \text{ (m}\cdot\text{kg/rad) or } \left(\frac{\text{J}\cdot\text{s/cy}}{\text{m}\cdot\text{Hz}}\right) \tag{55}$$

where (40) was used, and

$$\frac{h}{c} = \pm 2\pi \sqrt{a_0 r_e}\, m_e = \lambda_{C,e} m_e \approx 3.5177 \text{ x } 10^{-43} \text{ (m}\cdot\text{kg/rad) or } \left(\frac{\text{J}\cdot\text{s/cy}}{\text{Hz}} \frac{1}{\text{m}}\right) \tag{56}$$

respectively. Note that the moment arm in this expression is given by a radius defined by the geometrical mean of a_0 and r_e, so, it represents a combined spin-orbit effect acting along a circle with perimeter equal to the electron Compton wavelength.

(55) and (56), which are just a modification of (44) and (45), provide *quantum moment of mass* equal to *classical moment of mass* expressions. Also, these two expressions give the following *quantum wave* equal *classical particle* expressions for the Compton wavelengths

$$\lambda_{C,p} = \frac{\pi}{2} r_{pc} \quad \text{(m)} \tag{57}$$

and

$$\lambda_{C,e} = \pm 2\pi \sqrt{a_0 r_e} \quad \text{(m)} \tag{58}$$

respectively. The ratio of the above two parameters gives (47). While their product provides the following magnificent barred *h* physical interpretation equation

$$\hbar = \pm \frac{1}{2} \sqrt{\sqrt{a_0 r_e}\, m_e c \cdot r_{pc} m_p c} = \pm \frac{1}{2} \sqrt[4]{\frac{a_0}{r_e}} \sqrt{r_e r_{pc}} \sqrt{m_e m_p}\, c \quad (\text{J} \cdot \text{s/rad}) \tag{59}$$

which, with an additional retouch gives

$$\hbar = \pm \frac{1}{2\sqrt{\alpha}} \sqrt{r_e r_{pc}} \sqrt{m_e m_p}\, c \quad (\text{J} \cdot \text{s/rad}) \tag{60}$$

Also, multiplying (60) by *c* and rearranging conduces to

$$hc = \pm \pi \sqrt{\frac{r_e m_e c^2 r_{pc} m_p c^2}{\alpha}} = \pm \pi e \sqrt{\sqrt{a_0 r_e}\, E_e \cdot r_{pc} E_p} \quad (\text{m} \cdot \text{J/cy}) \tag{61}$$

which is the geometrical mean of the moments of energy given in (52) and (53).

Now, using (15) in (60) produces the following *quantum* = *classical* moment of mass/charge expression

$$\frac{\hbar}{q_p c} = \pm \frac{1}{2e} \sqrt{r_e r_{pc}} \sqrt{m_e m_p} \cong 1.875 \text{ x } 10^{-25} \quad (\text{m} \cdot \text{kg/rad / C}) \text{ or } (\frac{\text{J} \cdot \text{s/rad}}{\text{A} \cdot \text{m}}) \tag{62}$$

Plots of *α* expressions

Figure 7 gives the SC expressions for each of the H atom length parameters used in the above derivations. Figure 8 shows all the parameter first cycle zero crossing for their sine functions; shifting and/or axis reflextion was used for the sake of clarity.

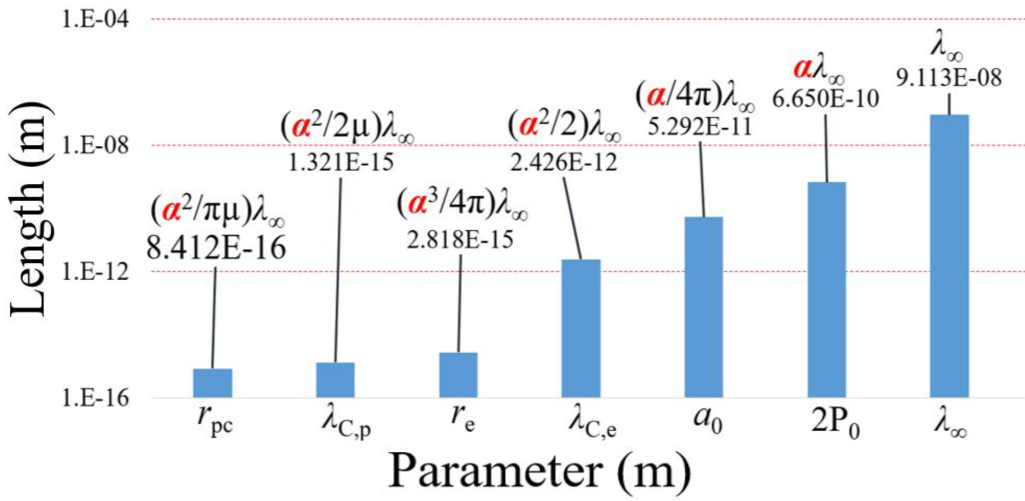

Figure 7. H atom length parameters expressed in terms of the Rydberg wavelength unit and Sommerfeld constant.

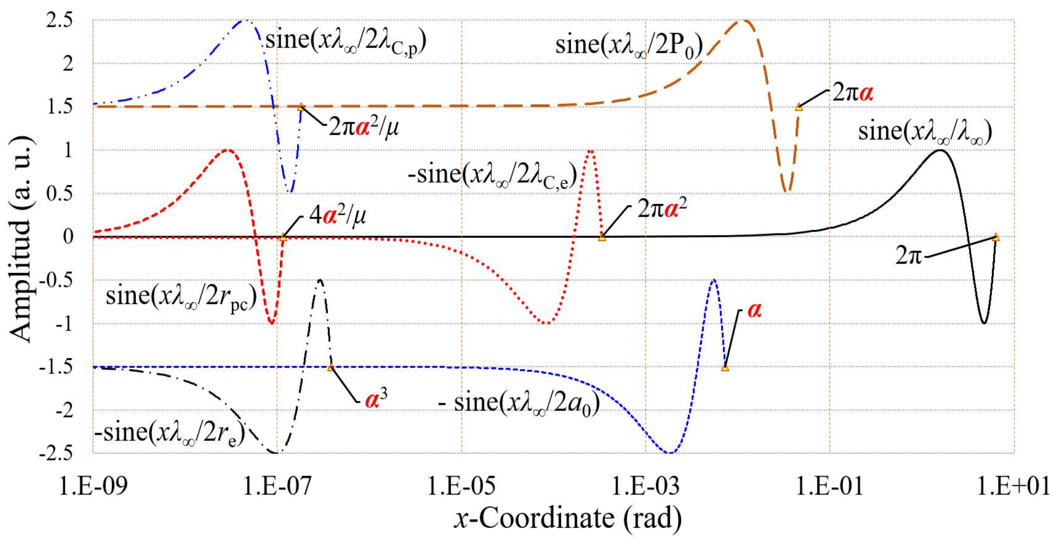

Figure 8. All parameter one-cycle data for sine functions used in above derivations.

Conclusions

Five physical interpretations of the SC were directly established from basic trigonometric plots and over 20 other expressions were subsequently derived. In most cases, a type of wave-particle duality was present in the α ratio definition. Additionally, several expressions for the electron charge to mass ratio, e/m_e, were obtained. An expression for the currently accepted proton charge radius was also derived from a trigonometric plot and a "correcting" factor was calculated based on obtaining an r_{pc} value with which a 13-digit match α calculation was produced. Subsequently, a couple of analytical expression for the proton to electron mass ratio, μ, were derived. Finally, *hc*, *h/c* and Compton wavelengths equations having the *quantum* = *classical* configuration were established.

Acknowledgments

The author appreciates the funding support of the Universidad de Guadalajara during the realization of this manuscript in times of social isolation due the COVID-19 pandemic crisis.

The author declares that there is any potential conflict of interest with respect to the research, authorship and publication of this article.

Data availability

The data that support the findings of this study are available at the National Institute of Standards and Technology (NIST) https://physics.nist.gov/cuu/Constants/index.html .

References

[1] Kragh, H., "Magic Number: A Partial History of the Fine-Structure Constant" Arch. Hist. Exact Sci. 57, 395–431 (2003). https://doi.org/10.1007/s00407-002-0065-7

[2] Mohr, P. J.; Taylor, B. N.; Newell, D. B., (2019), "Fine structure constant". CODATA Internationally recommended 2018 values of the fundamental physical constants. National Institute of Standards and Technology. All NIST values shown in manuscript were last retrieved on 15 October 2020

[3] Backerra, Anna C. M. "Relation between Planck's Constant and Speed of Light, Predicting Proton Radius More Accurately "Applied Physics Research; Vol. 11, No. 5; 2019 ISSN

1916-9639 E-ISSN 1916-9647 Published by Canadian Center of Science and Education doi:10.5539/apr.v11n5p1

[4] Saucedo-Flores, Emmanuel, “The proton charge radius and the proton to electron mass ratio equations” SCIREA Journal of Physics Volume 5, Issue 5, October 2020, PP. 108-117, PDF (231 K) , Pub. Date: October 11, 2020