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Statistically Convergent and Cesáro Summable Fuzzy Real-valued Triple Sequences

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Abstract:

In this article, the notion of different types of statistically convergent and statistically null fuzzy real-valued sequences having multiplicity greater than two is introduced. Some algebraic and topological properties such as solid, monotone, symmetric, convergence free, sequence algebra etc. of these spaces are studied. Also fuzzy real-valued Cesáro summable triple sequence space is introduced. A relation between strongly p-Cesáro summability and bounded statistically convergent triple sequences has been established.

Key words and phrases: Fuzzy real-valued triple sequence, solid space, symmetric space, convergence free, sequence algebra, density, statistical convergence, statistical Cauchy, Cesáro summable, strong Cesáro summability.

1. Introduction

Fuzzy set theory, compared to other mathematical theories, is perhaps the most easily adaptable theory to practice. The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [32] in 1965 and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets. In fact the fuzzy set theory has become an area of active area of research in science and engineering for the last 46 years. Fuzzy set theory is a powerful hand set for modelling uncertainty and vagueness in various problems arising in the field of science and engineering. It extends the scope and results of classical mathematical analysis by applying fuzzy logic to conventional mathematical objects, such as functions, sequences and series etc. While studying fuzzy topological spaces, many situations are faced, where we need to deal with convergence of fuzzy numbers.

As a generalization of ordinary convergence for sequences of real numbers, the notion of statistical convergence was first introduced by Fast [7]. After then it was studied by many researchers like Salat [21], Fridy [8], Connor [2], Maddox [12], Fridy and Orhan [9], Kwon [11], Savas [22], Tripathy [26], Savas and Mursaleen [24]. Different classes of statistically convergent sequences were introduced and investigated by Tripathy and Sen [30], Tripathy and Dutta [27] etc. Móricz [13] extended statistical convergence from single to multiple real sequences. Nuray and Savas [18] first defined the concepts of statistical convergence and statistically Cauchy for sequences of fuzzy numbers.

Agnew [1] studied the summability theory of multiple sequences and obtained certain theorems which have already been proved for double sequences by the author himself. The different types of notions of triple sequences was introduced and investigated at the initial stage by Sahiner *et. al*. [19], Some more works on statistical triple sequences are found on Sahiner and Tripathy [20]. Savas and Esi [23], Esi [6], Kumar *et al*. [10], Tripathy and Goswami [29], Dutta *et al.* [4] etc.

A fuzzy real number on R is a mapping $X : R \rightarrow L(=[0,1])$ associating each real number $t \in R$ with its grade of membership $X(t)$. Every real number r can be expressed as a fuzzy real number $\bar{r}$ as follows:

$$\bar{r}(t) = \begin{cases} 1 & if \quad t = r \\ 0 & otherwise \end{cases}$$

The α-level set of a fuzzy real number X, $0 < \alpha \leq 1$, denoted by $[X]^{\alpha}$ is defined as

$$[X]^{\alpha} = \{t \in R : X(t) \geq \alpha\}.$$

A fuzzy real number X is called convex if $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$. If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called normal. A fuzzy real number X is said to be upper semi-continuous if for each $\varepsilon > 0$, $X^{-1}[0, a+\varepsilon))$, for all $a \in L$ is open in the usual topology of R. The set of all upper semi continuous, normal, convex fuzzy number is denoted by $R(L)$. The additive identity and multiplicative identity in $R(L)$ are denoted by $\bar{0}$ and $\bar{1}$ respectively.

Let $C(R^n) = \{A \subset R^n$: A is compact and convex$\}$. The space $C(R^n)$ has a linear structure induced by the operations $A + B = \{a + b : a \in A, b \in B\}$ and $\lambda A = \{\lambda a : a \in A\}$ for $A, B \in C(R^n)$ and $\lambda \in R$. Then $\delta_{\infty}(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} \|a - b\|, \{\sup_{b \in B} \inf_{a \in A} \|a - b\|\}.$

Then clearly $(C(R^n), \delta_{\infty})$ is a complete metric space. Let $X : C(R^n) \to L$.

The linear structure of $C(R^n)$ induces the addition $X + Y$ and scalar multiplication $\lambda X, \lambda \in R$ in terms of α-level sets by, $[X + Y]^{\alpha} = [X]^{\alpha} + [Y]^{\alpha}$ and $[\lambda X]^{\alpha} = \lambda [X]^{\alpha}$; for each $0 < \alpha \leq 1$,

Now we define, for each $0 \leq p \leq \infty$, $d_p = (\int_0^1 \delta_{\alpha}(X^{\alpha}, Y^{\alpha})^p \, d\alpha)$, and $d_{\infty} = \sup_{0 < \alpha \leq 1} \delta_{\alpha}(X^{\alpha}, Y^{\alpha})$.

Clearly $d_{\infty}(X, Y) = \lim_{p \to \infty} d_p(X, Y)$, with $d_p \leq d_q$ if $p \leq q$.

Throughout the paper, d will denote d_p with $1 \leq p \leq \infty$.

Applying the notion of fuzzy real numbers, a fuzzy real-valued sequence was introduced and it was studied by Nanda [14], Tripathy and Nanda [31], Das and Choudhury [3], Subrahmanyam [25] etc. Recent works on fuzzy triple sequences are found in Nath and Roy ([15], [16], [17]).

2. Preliminaries and background

Throughout the article, c, c_0, ℓ_{∞} denote the spaces of convergent, null and bounded sequences respectively.

A triple sequence can be defined as a function $x : N \times N \times N \to R(C)$ where N, R and C denote the sets of natural numbers, real numbers and complex numbers respectively.

The notion of statistical convergence for triple sequences depends on the density of the subsets of $N \times N \times N$. The notion of the asymptotic density for subsets of $N \times N \times N$ as follows:

A subset E of $N \times N \times N$ is said to have density or asymptotic density $\rho(E)$, if

$\delta_3(E) = \lim_{p,q,r\to\infty} \sum_{n=1}^{p}\sum_{k=1}^{q}\sum_{l=1}^{r} \chi_E(n,k,l)$ exists, where χ_E is the characteristic function of E.

Obviously $\delta_3(E^c) = \rho(N \times N \times N - E) = 1 - \delta_3(E)$.

The concept of statistical convergence for triple sequences was first introduced by Sahiner *et al*. [3] who have given main definition of statistical convergence and statistical Cauchy for triple sequences $x = \langle x_{nkl} \rangle$ as follows:

A real triple sequence $x = \langle x_{nkl} \rangle$ is said to be statistically convergent to the number L, if for each $\varepsilon > 0$, $\delta_3(\{(n,k,l) \in N \times N \times N : |x_{nkl} - L| \geq \varepsilon\}) = 0$.

A real triple sequence $x = \langle x_{nkl} \rangle$ is said to be statistically Cauchy, if for each $\varepsilon > 0$, there exists $p = p(\varepsilon)$, $q = q(\varepsilon)$ and $r = r(\varepsilon) \in N$ such that

$\delta_3(\{(n,k,l) \in N \times N \times N : |x_{nkl} - X_{pqr}| \geq \varepsilon\}) = 0.$

A fuzzy real-valued triple sequence $X = \langle X_{nkl} \rangle$ is a triple infinite array of fuzzy real numbers X_{nlk} for all $n, k, l \in N$ and is denoted by $\langle X_{nkl} \rangle$ where $X_{nkl} \in R(L)$.

A fuzzy real-valued triple sequence $X = \langle X_{nkl} \rangle$ is said to be convergent in Pringshiem's sense to a fuzzy real number X_0, if for each $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon), k_0 = k_0(\varepsilon)$ and $l_0 = l_0(\varepsilon) \in N$ such that $d(X_{nkl}, X_0) < \varepsilon$ for all $n \geq n_0, k \geq k_0$ and $l \geq l_0$.

A fuzzy real-valued triple sequence $X = \langle X_{nkl} \rangle$ is said to be a Cauchy sequence, if for each $\varepsilon > 0$, there exists $p = p(\varepsilon)$, $q = q(\varepsilon)$ and $r = r(\varepsilon) \in N$ such that

$$\delta_3(\{(n,k,l) \in N \times N \times N : d(X_{nkl}, X_{pqr}) \geq \varepsilon\}) = 0.$$

A fuzzy real-valued triple sequence $X = \langle X_{nkl} \rangle$ is said to be bounded if $\sup_{n,k,l} d(X_{nkl}, \bar{0}) < \infty$.

A fuzzy real-valued triple sequence $X = \langle X_{nkl} \rangle$ is said to be *statistically convergent* in Pringsheim's sense to the fuzzy real number X_0, if for all $\varepsilon > 0$, $\delta_3(\{(n,k,l) \in N \times N \times N : d(X_{nkl}, X_0) \geq \varepsilon\}) = 0.$ We write $stat_3 - \lim_{n,k,l \to \infty} X_{nkl} = X_0.$

A fuzzy real-valued triple sequence $X = \langle X_{nkl} \rangle$ is said to be statistically null if it is statistically convergent to zero.

A fuzzy real-valued triple sequence $X = \langle X_{nkl} \rangle$ is said to be statistically bounded if there exists a real number μ such that $\delta_3(\{(n,k,l) \in N \times N \times N : \bar{d}(X_{nkl}, \bar{0}) > \mu\}) = 0.$.

A fuzzy real-valued triple sequence $X = \langle X_{nkl} \rangle$ is said to be statistically Cauchy if for every $\varepsilon > 0,$ there exists $p = p(\varepsilon),\ q = q(\varepsilon)$ and $r = r(\varepsilon) \in N$ such that

$$\delta_3(\{(n,k,l) \in N \times N \times N : d(X_{nkl}, X_{pqr}) \geq \varepsilon\}) = 0.$$

A fuzzy real-valued triple sequence $X = \langle X_{nkl} \rangle$ is said to be statistically regularly convergent if it convergent in Pringsheim's sense and in addition the following statistical limits holds:

$stat_3 - \lim_{n \to \infty} X_{nkl} = L_{kl} (k, l \in N);$

$stat_3 - \lim_{k \to \infty} X_{nkl} = L_{nl} (n, l \in N);$

and $stat_3 - \lim_{l \to \infty} X_{nkl} = L_{nl} (n, k \in N);$

Let $\langle X_{nkl} \rangle$ and $\langle Y_{nkl} \rangle$ be two triple sequences of fuzzy real numbers, then we say that $X_{nkl} = Y_{nkl}$ for almost all *n*, *k* and *l* (in short *a. a. n, k* and *l*) if $\delta_3(\{(n,k,l) \in N \times N \times N : X_{nkl} \neq, Y_{nkl}\}) = 0.$

A fuzzy real-valued triple sequence space E^F is said to be solid if $\langle Y_{nkl} \rangle \in E^F$ whenever $\langle X_{nkl} \rangle \in E^F$ and $d(Y_{nkl}, \bar{0}) \leq d(X_{nkl}, \bar{0})$ for all $n, k, l \in N.$

A fuzzy real-valued triple sequence space E^F is said to be monotone if E^F contains the canonical pre-image of all its step spaces.

A fuzzy real-valued triple sequence space E^F is said to be symmetric if $\langle X_{\pi(nkl)} \rangle \in E^F$, whenever $\langle X_{nkl} \rangle \in E^F$ where π is a permutation on $N \times N \times N.$

A fuzzy real-valued triple sequence space E^F is said to be sequence algebra if $\langle X_{nkl} \otimes Y_{nkl} \rangle \in E^F$, whenever $\langle X_{nkl} \rangle, \langle Y_{nkl} \rangle \in E^F$.

A fuzzy real-valued triple sequence space E^F is said to be convergence free if $\langle Y_{nkl} \rangle \in E^F$ whenever $\langle X_{nkl} \rangle \in E^F$. and $X_{nkl} = \bar{0}$ implies $Y_{nkl} = \bar{0}$.

Remark 2.1. *If a sequence space E^F is solid, then it is monotone.*

Throughout the article $(w^F)_3, (\ell_\infty^F)_3, (\bar{c}^F)_3^R, (\bar{c}_0^F)_3^R, (\bar{c}^F)_3^P, (\bar{c}_0^F)_3^P$ denote the spaces of all, bounded, statistically regularly convergent, statistically regularly null, statistically convergent in Pringsheim's sense and statistically null in Pringsheim's sense triple sequence spaces of fuzzy real numbers respectively.

Further the following bounded fuzzy real-valued triple sequence spaces are introduced:

$(\bar{c}^F)_3^{RB} = (\bar{c}^F)_3^R \cap (\ell_\infty^F)_3$;

$(\bar{c}_0^F)_3^{RB} = (\bar{c}_0^F)_3^R \cap (\ell_\infty^F)_3$;

$(\bar{c}^F)_3^{PB} = (\bar{c}^F)_3^P \cap (\ell_\infty^F)_3$;

$(\bar{c}_0^F)_3^{PB} = (\bar{c}_0^F)_3^P \cap (\ell_\infty^F)_3$.

From the above definitions, it is clear that $(c)_3^P \subset (\bar{c}^F)_3^P, (c_0)_3^P \subset (\bar{c}_0^F)_3^P, (c)_3^{PB} \subset (\bar{c}^F)_3^{PB}$ and $(c_0)_3^{PB} \subset (\bar{c}_0^F)_3^{PB}$ and the inclusions are proper.

Now we give the definitions of Cesáro summability and p-Cesáro summability of fuzzy real-valued triple sequences.

A fuzzy real-valued triple sequence $\langle X_{nkl} \rangle$ is said to be Cesáro summable to a fuzzy real number X_0, if $\lim_{u,v,w\to\infty} d\left(\frac{1}{uvw} (\sum_{n=1}^{u}\sum_{l=1}^{v}\sum_{k=1}^{w} X_{nkl}), X_0 \right) = 0.$

A fuzzy real-valued triple sequence $\langle X_{nkl} \rangle$ is said to be strongly p-Cesáro summable to a fuzzy real number X_0, if $\lim_{u,v,w\to\infty} \left(\frac{1}{uvw} \sum_{n=1}^{u}\sum_{l=1}^{v}\sum_{k=1}^{w} [d(X_{nkl}, X_0)]^p \right) = 0.$

The space of all strongly p-Cesàro summable triple sequences of fuzzy real numbers is denoted by $(w^F)_3^P$.

3. Main Results:

The following result is the **decomposition theorem** for statistically convergent fuzzy real-valued triple sequence space.

Theorem 3.1. *The following statements are equivalent*:

(i) *The triple sequence* $\langle X_{nkl} \rangle$ *of fuzzy real numbers is statistically convergent to* X_0.

(ii) *There exists a triple sequence* $\langle Y_{nkl} \rangle \in (\bar{c}^F)_3^P$ *such that* $X_{nlk} = Y_{nlk}$ *for a.a. n, l and k.*

(iii) There exists a subset $M = \{n_p, k_q, l_r) \in N \times N \times N : p, q, r \in N\}$ *of* $N \times N \times N$ *such that* $\rho(M) = 1$ *and* $\left\langle X_{n_p k_q l_r} \right\rangle \in (\bar{c}^F)_3^P$.

***Proof.* (*i*) ⇒ (*ii*)** Let $stat_3 - \lim_{n,k,l \to \infty} X_{nkl} = X_0$. Then for every $\varepsilon > 0$,

$\delta_3(\{(n,k,l) \in N \times N \times N : d(X_{nkl}, X_0) \geq \varepsilon\}) = 0.$

Now selecting the increasing sequences (T_j), (U_j) and (V_j) of natural numbers such that

if $p > T_j, q > U_j$ and $r > V_j$, then $\frac{1}{pqr}\left\{(n,k,l) \in N \times N \times N : n \leq p; k \leq q; l \leq r : d(X_{nkl}, X_0) \geq \frac{1}{j}\right\} < \frac{1}{j}$,

where $|E|$ denote the cardinality of the set E.

Define the sequence $\langle Y_{nkl} \rangle$ as follows:

$Y_{nkl} = X_{nkl}$, if $n \leq T_1$ or $k \leq U_1$ or $l \leq V_1$.

Next for all (n,k,l) with $T_j < n \leq T_{j+1}$ or $U_j < k \leq U_{j+1}$, or $V_j < l \leq V_{j+1}$,

let $Y_{nkl} = X_{nkl}$, if $d(X_{nkl}, X_0) < \frac{1}{j}$, otherwise, $Y_{nkl} = X_0$.

We prove that $\langle Y_{nkl} \rangle$ converges to X_0.

Let $\varepsilon > 0$ be given and we choose j such that $\varepsilon < \frac{1}{j}$.

For $n > T_j, k > U_j$ and $l > V_j$, we find that $\bar{d}(Y_{nkl}, X_0) < \varepsilon$.

The fact that $Y_{nkl} = X_{nkl}$, for *a.a. n, k and l.* follows from the following inclusion:

Let $T_j < n \le T_{j+1}, U_j < k \le U_{j+1}$ and $V_j < l \le V_{j+1}$, then

$$\{(n,k,l) \in N \times N \times N : n \le p; k \le q, l \le r \; and \quad X_{nkl} \ne Y_{nkl}\}$$

$$\subseteq \left\{(n,k,l) \in N \times N \times N : n \le p; k \le q, l \le r \; and \quad d(X_{nkl}, X_0) \ge \frac{1}{j}\right\}$$

$$\Rightarrow \frac{1}{pqr}\left|\{(n,k,l) \in N \times N \times N : n \le p; k \le q, l \le r \; and \quad X_{nkl} \ne Y_{nlk}\}\right|$$

$$\le \frac{1}{pqr}\left|\left\{(n,k,l) \in N \times N \times N : n \le p; k \le q, l \le r \; and \quad d(X_{nkl}, X_0) \ge \frac{1}{j}\right\}\right| < \frac{1}{j},$$

$$\Rightarrow \lim_{p,q,r\to\infty} \frac{1}{pqr}\left|\{(n,k,l) \in N \times N \times N : n \le p; k \le q, l \le r \; and \quad X_{nkl} \ne Y_{nkl}\}\right| = 0.$$

(*ii*) ⇒ (*iii*) Let there exists a sequence $\langle Y_{nkl}\rangle \in (\bar{c}^F)_3^P$ *such that* $X_{nkl} = Y_{nkl}$ *for a.a. n, k and l.*

Let $M = \{n,k,l) \in N \times N \times N : X_{nkl} = Y_{nkl}\}$, then $\rho(M) = 1$. Now enumerate M as $M = \{n_p, k_q, l_r) \in N \times N \times N : p,q,r \in N\}$ on neglecting the rows and columns those contain finite number of elements. Clearly $\delta_3(M) = 1$ and $d(X_{n_p k_q l_r}, X_0) = d(Y_{n_p k_q l_r}, X_0) \to 0$ as $p,q,r \to \infty$. This implies $\langle X_{n_p k_q l_r}\rangle \in (\bar{c}^F)_3^P$.

(iii) ⇒ (i) Let there exists a subset $M = \{n_p, l_q, k_r) \in N \times N \times N : p,q,r \in N\}$ of $N \times N \times N$ such that $\rho(M) = 1$ and $\langle X_{n_p l_q k_r}\rangle \in (\bar{c}^F)_3^P$.

Then by definition, we have $d(X_{n_p l_q k_r}, X_0) \to 0$ as $p,q,r \to \infty$.

For any $\varepsilon > 0$, let $A = \left\{(n,k,l) : \bar{d}(X_{nkl}, X_0) < \frac{\varepsilon}{2}\right\}$ and $B = \left\{(n,k,l) : \bar{d}(X_{nkl}, \bar{0}) < \frac{\varepsilon}{2}\right\}$.

Then clearly

$\delta_3(A) = \delta_3(B) = 1$. Let $E = A \cap B$, then $\delta_3(E) = 1$

$$\therefore \delta_3\left(\{(n,k,l) \in N \times N \times N : d(X_{nkl}, X_0) < \varepsilon\}\right) \ge \delta_3(E) = 1.$$

$$\Rightarrow \delta_3\left(\{(n,k,l) \in N \times N \times N : d(X_{nkl}, X_0) \ge \varepsilon\}\right) = 0.$$

Hence $stat_3 - \lim_{n,k,l\to\infty} X_{nkl} = X_0$. ■

Theorem 3.2. *A triple sequence* $\langle X_{nlk} \rangle$ *of fuzzy real numbers is statistically convergent, if and only if* $\langle X_{nlk} \rangle$ *is a statistical Cauchy sequence*

Proof**. Let** $stat_3 - \lim_{n,k,l\to\infty} X_{nkl} = X_0.$

Then for each $\varepsilon \geq 0, \quad \delta_3(\{(n,k,l) \in N \times N \times N : d(X_{nkl}, X_0) \geq \varepsilon\}) = 0.$

Choosing numbers p, q and r such that $d(X_{pqr}, X_0) \geq \varepsilon.$

Now let

$A = \{(n,k,l) \in N \times N \times N, n \leq u, k \leq v, l \leq w : d(X_{nkl}, X_{pqr}) \geq \varepsilon\};$

$B = \{(n,k,l) \in N \times N \times N, n \leq u, k \leq v, l \leq w : d(X_{nkl}, X_0) \geq \varepsilon\};$

$C = \{(p,q,r) \in N \times N \times N : d(X_{pqr}, X_0) \geq \varepsilon\},$

then $A \subseteq B \cup C$ and therefore $\rho(A) \leq \rho(B) + \rho(C) = 0.$ Hence $\langle X_{nkl} \rangle$ is statistically Cauchy.

Conversely, let $\langle X_{nkl} \rangle$ be statistically Cauchy. Then for a given $\varepsilon > 0,$ there exists $n_0(\varepsilon), k_0(\varepsilon)$ and $l_0(\varepsilon) \in N$ such that $\delta_3(\{(n,k,l) \in N \times N \times N : d(X_{nkl}, X_{pqr}) \geq \varepsilon\}) = 0.$

If possible let $\langle X_{nkl} \rangle$ be not statistically convergent. Then there exist $p, q, r \in N$ such that $\delta_3(A) = 0,$ i.e. A has triple natural density zero, where

$A = \{(n,k,l) \in N \times N \times N, n \leq u, k \leq v, l \leq w : d(X_{nkl}, X_{pqr}) \geq \varepsilon\}.$

Hence $\delta_3(E) = 1,$ where $E = \{(n,k,l) \in N \times N \times N, n \leq u, k \leq v, l \leq w : d(X_{nkl}, X_{pqr}) < \varepsilon\}.$

In particular, we can write

$$d(X_{nkl}, X_{pqr}) \leq 2d(X_{nkl}, X_0) < \varepsilon \qquad ....(1),$$

if $d(X_{nkl}, X_0) < \frac{\varepsilon}{2}.$

Since $\langle X_{nkl} \rangle$ is not statistically convergent, $\delta_3(B) = 1,$ where

$B = \{(n,k,l) \in N \times N \times N, n \leq u, k \leq v, l \leq w : d(X_{nkl}, X_0) \geq \varepsilon\};$

i.e. $\delta_3(\{(n,k,l) \in N \times N \times N, n \leq u, k \leq v, l \leq w : d(X_{nkl}, X_0) < \varepsilon\}) = 0.$

Therefore by (1),

$\delta_3(\{(n,k,l) \in N \times N \times N, n \le u, k \le v, l \le w : d(X_{nkl}, X_{pqr}) < \varepsilon\}) = 0.$

i.e. $\delta_3(A) = 1,$ a contradiction. Hence $\langle X_{nkl} \rangle$ is statistically convergent. ■

Theorem 3.3. (i) Let $p \in (0,\infty).$ If a triple sequence $\langle X_{nkl} \rangle$ of fuzzy real number is *strongly p-Cesáro summable to a fuzzy number* $X_0,$ *then it is also statistically convergent to* $X_0.$

(*ii*) *Let* $p \in (0,\infty).$ *If a bounded triple sequence* $\langle X_{nkl} \rangle$ *of fuzzy real number is statistically convergent to* $X_0,$ *then it is strongly p-Cesáro summable to* $X_0.$

Proof. (*i*) Let $\langle X_{nkl} \rangle \in (w^F)_3^P$ be such that $\langle X_{nkl} \rangle$ is strongly *p*-Cesáro summable to a fuzzy number $X_0.$ Then $\lim_{u,v,w\to\infty} d\left(\frac{1}{uvw}(\sum_{n=1}^{u}\sum_{l=1}^{v}\sum_{k=1}^{w} X_{nkl}), X_0 \right) = 0.$

Now for any $\varepsilon > 0,$

$$\frac{1}{uvw}\sum_{n=1}^{u}\sum_{l=1}^{v}\sum_{k=1}^{w}(d(X_{nkl}, X_0)^p) \ge \frac{1}{uvw}\left|\{(n,k,l) \in N \times N \times N : n \le u; l \le v, k \le w : d(X_{nkl}, X_0)^p) \ge \varepsilon\}\right|.$$

$$\therefore \lim_{u,v,w\to\infty} \frac{1}{uvw}\left|\{(n,k,l) \in N \times N \times N : n \le u; k \le v, l \le w; d(X_{nkl}, X_0)^p) \ge \varepsilon\}\right| = 0.$$

i.e. $\delta_3(\{(n,k,l) \in N \times N \times N : d(X_{nkl}, X_0)^p \ge \varepsilon\}) = 0.$

$\therefore stat_3 - \lim_{n,k,l\to\infty} X_{nkl} = X_0.$

Again for a bounded triple sequence $\langle X_{nkl} \rangle,$ let $stat_3 - \lim_{n,k,l\to\infty} X_{nkl} = X_0.$

Let $K = d(X_{nkl}, \bar{0}) + d(X_0, \bar{0}).$

Let $\varepsilon > 0$ be given, then there exist u_0, v_0, w_0 such that

$$\frac{1}{uvw}\left|\left\{(n,k,l) \in N \times N \times N : n \le u; k \le v, l \le w; d(X_{nkl}, X_0) \ge \left(\frac{\varepsilon}{2}\right)^{\frac{1}{p}}\right\}\right| < \frac{\varepsilon}{2K^p},$$

for all $u > u_0, v > v_0$ and $w > w_0.$

Let $L_{uvw} = \left\{(n,k,l) \in N \times N \times N : n \le u; k \le v; l \le w; d(X_{nkl}, X_0)^p \ge \frac{\varepsilon}{2}\right\}$

Now for all $u > u_0, v > v_0$ and $w > w_0$,

$$\frac{1}{uvw}\sum_{n=1}^{u}\sum_{l=1}^{v}\sum_{k=1}^{w}(d(X_{nkl}, X_0)^p)$$

$$= \frac{1}{uvw}\left\{\sum\sum\sum_{(n,k,l)\in L_{uvw}}(d(X_{nkl}, X_0)^p)\right\} + \frac{1}{uvw}\left\{\sum\sum\sum_{(n,k,l)\notin L_{uvw}}(d(X_{nkl}, X_0)^p)\right\}$$

$$< \frac{1}{uvw}\left\{uvw\frac{\varepsilon}{2K^p}.K^p + uvw\frac{\varepsilon}{2}\right\} < \varepsilon.$$

$$\therefore \lim_{u,v,w\to\infty}\frac{1}{uvw}\sum_{n=1}^{u}\sum_{k=1}^{v}\sum_{l=1}^{w}\left[d(X_{nkl}, X_0)\right]^p = 0.$$

Hence $\langle X_{nkl}\rangle$ is strongly p-Cesáro summable to X_0. ■

In view of the above theorem and using standard technique, the following results can be obtained.

Corollary 1. *Let* $0 < p < q < \infty$. *Then* $(w^F)_3^q \subset (w^F)_3^P$ *and* $(w^F)_3^P \cap (\ell^F)_3^\infty = (w^F)_3^q \cap (\ell^F)_3^\infty$.

Corollary 2. *If a bounded triple sequence* $\langle X_{nkl}\rangle$ *of fuzzy real number is statistically convergent to* X_0, *then it is Cesáro summable to* X_0.

Remark 3.1. *The converse of the Corollary 2 is not necessarily true.*

This follows from the following example.

Example 3.1. Let $X = \langle X_{nkl}\rangle$ be defined as $X_{nkl} = (-1)^n$, for all k , l. Then

$$\lim_{u,v,w\to\infty}\frac{1}{uvw}\sum_{n=1}^{u}\sum_{k=1}^{v}\sum_{l=1}^{w}X_{nkl} = 0.$$

i.e. X is Cesáro summable but obviously X is not statistically convergent. ■

Remark 3.2. *If* $X = \langle X_{nkl}\rangle$ *is unbounded triple sequence of fuzzy real number, then X is statistically convergent but X may not be strongly p-Cesáro summable.*

This remark follows from the following example.

Example 3.2. Let $X = \langle X_{nkl}\rangle \in R(L)$ be defined as

For $n = i^2, i \in N$ and for all $k, l \in N$,

$$X_{nkl}(t)=\begin{cases}1+\dfrac{t}{\sqrt{n}}, & for\ \ -\sqrt{n}\le t\le 0\\ 1-\dfrac{t}{\sqrt{n}}, & for\ \ 0\le t\le \sqrt{n}\\ 0, & otherwise.\end{cases}$$

For $n\neq i^2, i\in N$ and for all $k,l\in N,\quad X_{nkl}(t)=\bar{0}.$

The sequence X is unbounded and statistically convergent to $\bar{0}$.

But for p = 1, $\lim_{u,v,w\to\infty}\dfrac{1}{uvw}\sum_{n=1}^{u}\sum_{k=1}^{v}\sum_{l=1}^{w}(d(X_{nkl},\bar{0})^p)=1$

Hence X is not strongly Cesáro summable to $\bar{0}$. ■

Theorem 3.4. *The triple sequence spaces* $(\bar{c}_0^F)_3^P,(\bar{c}_0^F)_3^R,(\bar{c}_0^F)_3^{RB}$ *are complete under the metric* ρ *defined by* $\rho(X,Y)=\sup_{n,k,l}d(X_{nkl},Y_{nkl})$.

Proof. Let $\langle X_{nkl}\rangle$ be a Cauchy sequence in $(\bar{c}_0^F)_3^P$. Let $\varepsilon>0$ be given.

Then for each $X^{(i)}=\left\langle X_{nkl}{}^{(i)}\right\rangle\in(\bar{c}_0^F)_3^R$,

$$\rho\left(X^{(i)},X^{(j)}\right)<\varepsilon,\ \text{for each } i,j\ge n_0 \qquad (1)$$

$\Rightarrow \sup_{n,k,l}d(X_{nkl}^{(i)},X_{nkl}^{(j)})<\dfrac{\varepsilon}{3}$, for each $i,j\ge n_0$.

$\Rightarrow d(X_{nkl}^{(i)},X_{nkl}^{(j)})<\dfrac{\varepsilon}{3}$, for each $i,j\ge n_0$ and $n,k,l\in N$.

$\Rightarrow \left\langle X_{nkl}^{(i)}\right\rangle_{i=1}^{\infty}$ is a Cauchy sequence in $R^n(L)$. Since $R^n(L)$ is complete, so $\left\langle X_{nkl}^{(i)}\right\rangle_{i=1}^{\infty}$ is convergent for each $n,k,l\in N$.

Let $\lim_{i\to\infty}X_{nkl}^{(i)}=X_{nkl}$, for each $n,k,l\in N$

There exists a set $A_i=\left\{(n,k,l)\in N\times N\times N: d(X_{nkl}^{(i)},X_{nkl})<\dfrac{\varepsilon}{2}\right\}\subset N\times N\times N$ such that $\delta_3(A_i)=1$, as $i\to\infty$.

Let i_0 be such $\delta_3(A_i)=1-\gamma$, for $i>i_0$,

where γ is a small positive real number. We have

$d(X_{nkl}, \bar{0}) < d(X_{nkl}^{(i)}, X_{nkl}) + d(X_{nkl}^{(i)}, \bar{0}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

Then there exists a set $B_i = \left\{(n,k,l) \in N \times N \times N : d(X_{nkl}^{(i)}, \bar{0}) < \frac{\varepsilon}{2}\right\}.$

For $i > i_0$, let $C_i = A_i \cap B_i.$

Then $\delta_3(C_i) = \delta_3(A_i) \cap \delta_3(B_i) = 1 - \gamma.$

$\therefore \lim_{i \to \infty} \delta_3(C_i) = 1.$

This implies $\langle X_{nkl} \rangle \in (\bar{c}_0^F)_3^P$. Hence $\langle X_{nkl} \rangle \in (\bar{c}_0^F)_3^P$ is complete.

Similarly the other cases can be established. ■

Theorem 3.5. *The triple sequence spaces* $(\bar{\ell}_\infty^F)_3, (\bar{c}_0^F)_3, (\bar{c}_0^F)_3^P, (\bar{c}_0^F)_3^R, (\bar{c}_0^F)_3^{RB}$ *are solid and monotone.*

Proof. Let $\langle X_{nkl} \rangle$ and $\langle Y_{nkl} \rangle$ be two fuzzy real valued triple sequences such that $d(Y_{nkl}, \bar{0}) \le d(X_{nkl}, \bar{0}),$ for all $n, k, l \in N.$

Let $\varepsilon > 0$ be given and let $\langle Y_{nkl} \rangle \in Z,$ where $Z = (\bar{c}_0^F)_3, (\bar{c}_0^F)_3^P, (\bar{c}_0^F)_3^R, (\bar{c}_0^F)_3^{RB}.$

Then the solidness of the spaces Z follows from the following inclusion relation:

$\{(n,k,l) \in N \times N \times N : d(Y_{nkl}, \bar{0}) \ge \varepsilon\} \subseteq \{(n,k,l) \in N \times N \times N : d(X_{nkl}, \bar{0}) \ge \varepsilon\},$

Also by **Remark 2.1,** it follows that the spaces Z are monotone.

Similarly, the solid and monotone property of the space $(\bar{\ell}_\infty^F)_3$ can be proved. ■

Proposition 3.6. *The triple sequence spaces* $(\bar{c}^F)_3, (\bar{c}^F)_3^P, (\bar{c}^F)_3^R, (\bar{c}^F)_3^{RB}$ *are not solid and not monotone.*

Proof. The spaces $(\bar{c}^F)_3, (\bar{c}^F)_3^P, (\bar{c}^F)_3^R, (\bar{c}^F)_3^{RB}$ are not solid. This follows from the following example.

Example 3.3. Consider the sequence $\langle X_{nkl} \rangle$ defined as follows:

For $n = i^2, i \in N$ and for all $k, l \in N,$

$$X_{nkl}(t)=\begin{cases}\left(1+\dfrac{t}{n+3}\right); & for \quad -(n+3)\le t\le 0,\\ \left(1-\dfrac{t}{n+3}\right); & for \quad 0\le t\le (n+3),\\ 0; & otherwise.\end{cases}$$

$X_{nkl}=\bar{1},$ *otherwise.*

Then $\langle X_{nkl}\rangle \in Z,$ where $Z=(\bar{c}^F)_3,(\bar{c}^F)_3^P,(\bar{c}^F)_3^R,(\bar{c}^F)_3^{RB}.$

Now consider the sequence $\langle Y_{nkl}\rangle \in R(L)$ defined as follows:

For $n=i^2, i\in N$ and for all $k,l\in N,$

$$Y_{nkl}(t)=\begin{cases}\left(1+\dfrac{t}{n}\right); & for \quad -n\le t\le 0,\\ \left(1-\dfrac{t}{n}\right); & for \quad 0\le t\le n,\\ 0; & otherwise.\end{cases}$$

For $n\ne i^2, i\in N$ and for all $k,l\in N,$

$$Y_{nkl}=\begin{cases}\bar{1}; & for \quad (n+k+l) \quad odd,\\ \overline{3^{-1}}; & for \quad (n+k+l) \quad even.\end{cases}$$

Then $d(Y_{nkl},\bar{0})\le d(X_{nkl},\bar{0}),$ for all $n,k,l\in N.$

But $\langle Y_{nkl}\rangle \notin Z,$ where $Z=(\bar{c}^F)_3,(\bar{c}^F)_3^P,(\bar{c}^F)_3^R,(\bar{c}^F)_3^{RB}.$

Hence the spaces are not solid. Therefore the spaces are not monotone. ■

Proposition 3.7. *The spaces* $(\bar{c}_0^F)_3,(\bar{c}_0^F)_3^P,(\bar{c}_0^F)_3^R,(\bar{c}_0^F)_3^{RB},(\bar{c}^F)_3,(\bar{c}^F)_3^P,(\bar{c}^F)_3^R,(\bar{c}^F)_3^{RB}$ *are not convergence free.*

Proof. The proof of the result follows from the following example.

Example 3.4. Consider the $\langle X_{nkl}\rangle$ in $R(L)$ defined as follows:

For $n\ne i^2, i\in N$ and for all $k,l\in N,$

$$X_{nkl}(t)=\begin{cases}1+(n+k+l)t; & for \quad -\dfrac{1}{n+k+l}\le t\le 0,\\ 1+(n+k+l)t; & for \quad 0\le t\le \dfrac{1}{n+k+l},\\ 0; & otherwise.\end{cases}$$

$X_{nkl}=\bar{0}$, *otherwise.*

Then $\langle X_{nkl}\rangle \in Z$, where $Z=(\bar{c}_0^F)_3,(\bar{c}_0^F)_3^P,(\bar{c}_0^F)_3^R,(\bar{c}_0^F)_3^{RB},(\bar{c}^F)_3,(\bar{c}^F)_3^P,(\bar{c}^F)_3^R,(\bar{c}^F)_3^{RB}$.

Now consider the sequence $\langle Y_{nkl}\rangle \in R(L)$ defined as follows:

For $n\neq i^2, i\in N$ and for all $k,l\in N$,

$$Y_{nkl}(t)=\begin{cases}1+\dfrac{t}{n+k+l}; & for \quad -(n+k+l)\le t\le 0,\\ 1+\dfrac{t}{n+k+l}; & for \quad 0\le t\le (n+k+l),\\ 0; & otherwise.\end{cases}$$

$Y_{nkl}=\bar{0}$, *otherwise.*

Then $\langle Y_{nkl}\rangle \notin Z$, where $Z=(\bar{c}_0^F)_3,(\bar{c}_0^F)_3^P,(\bar{c}_0^F)_3^R,(\bar{c}_0^F)_3^{RB},(\bar{c}^F)_3,(\bar{c}^F)_3^P,(\bar{c}^F)_3^R,(\bar{c}^F)_3^{RB}$.

Hence the spaces are not convergence free. ■

Proposition 3.8. *The spaces* $(\bar{c}_0^F)_3,(\bar{c}_0^F)_3^P,(\bar{c}_0^F)_3^R,(\bar{c}_0^F)_3^{RB},(\bar{c}^F)_3,(\bar{c}^F)_3^P,(\bar{c}^F)_3^R,(\bar{c}^F)_3^{RB}$ *are not symmetric.*

Proof. The proof of the result follows from the following example.

Example 3.5. Consider the $\langle X_{nkl}\rangle$ in $R(L)$ defined as follows:

For $n\neq i^3, i\in N$ and for all $k,l\in N$,

$$X_{nkl}(t)=\begin{cases}1+\dfrac{t}{\sqrt[3]{n}}; & for \quad -\sqrt[3]{n}\le t\le 0,\\ 1-\dfrac{t}{\sqrt[3]{n}}; & for \quad 0\le t\le \sqrt[3]{n},\\ 0; & otherwise.\end{cases}$$

$X_{nkl}=\bar{0}$, *otherwise.*

Then $\langle X_{nkl}\rangle \in Z,$ where $Z = (\bar{c}_0^F)_3, (\bar{c}_0^F)_3^P, (\bar{c}_0^F)_3^R, (\bar{c}_0^F)_3^{RB}, (\bar{c}^F)_3, (\bar{c}^F)_3^P, (\bar{c}^F)_3^R, (\bar{c}^F)_3^{RB}.$

Now let us consider the sequence $\langle Y_{nkl}\rangle$ the rearrangement of $\langle X_{nkl}\rangle$ in $R(L)$ as follows:

For *m* and *k* odd and for all $n \in N,$

$$Y_{nkl}(t) = \begin{cases} 1 + \dfrac{t}{n}; & for \quad -n \le t \le 0, \\ 1 - \dfrac{t}{n}; & for \quad 0 \le t \le n, \\ 0; & otherwise. \end{cases}$$

$Y_{nkl} = \bar{0},$ *otherwise.*

Then $\langle Y_{nkl}\rangle \notin Z,$ where $Z = (\bar{c}_0^F)_3, (\bar{c}_0^F)_3^P, (\bar{c}_0^F)_3^R, (\bar{c}_0^F)_3^{RB}, (\bar{c}^F)_3, (\bar{c}^F)_3^P, (\bar{c}^F)_3^R, (\bar{c}^F)_3^{RB}.$

Hence the spaces are not symmetric. ■

Theorem 3.9. *The spaces* $(\bar{c}_0^F)_3, (\bar{c}_0^F)_3^P, (\bar{c}_0^F)_3^R, (\bar{c}_0^F)_3^{RB}, (\bar{c}^F)_3, (\bar{c}^F)_3^P, (\bar{c}^F)_3^R, (\bar{c}^F)_3^{RB}$ *are sequence algebras.*

Proof. Consider the space $(\bar{c}_0^F)_3.$

Let $0 < \varepsilon < 1$ and $\langle X_{nkl}\rangle, \langle Y_{nkl}\rangle \in (\bar{c}_0^F)_3.$

Then the result follows from the following inclusion relation.

$$\{(n,k,l) \in N \times N \times N : d(X_{nkl} \otimes Y_{nkl}, \bar{0}) < \varepsilon\} \subseteq$$

$$\{(n,k,l) \in N \times N \times N : d(X_{nkl}, \bar{0}) < \varepsilon\} \cup \{(n,k,l) \in N \times N \times N : d(Y_{nkl}, \bar{0}) < \varepsilon\}.$$

Similarly, the result for the other sequence spaces can be established. ■

Conclusion:

We have introduced and studied the notion of different types of statistically convergent and statistically null fuzzy real-valued triple sequences. Some algebraic and topological properties of the introduced spaces are studied. Also the fuzzy real-valued Cesáro summable triple sequence space is introduced. A relation between strongly *p*-Cesáro summability and bounded statistically convergent triple sequences is derived.

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