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Linearized Stability and Hopf Bifurcations for a Nonautonomous Delayed Predator-prey System

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ABSTRACT

In past many years, biomathematics population models are constructed based on plausible explicit and implicit biological assumptions. In the case that not enough analysis is carried out for a well-motivated and plausible model, the result is no or minimum insights gained. In this study, existence of Hopf bifurcations of a nonautonomous delayed predator-prey system with stage-structure for predator is proposed. Furthermore, conditions of linearized stability and Hopf bifurcations for this system are established. Numerical simulations are presented it illustrate the feasibility of our main result.

Key words: Hopf bifurcations; stage-structure; positive periodic solation; linearized stability.

1. Introduction

The predator-prey system is a major mathematics concern in biomathematics field. Over the past year, a few studies were undertaken to address in predator-prey system [1-3, 6-8]. It is assumed in the model that each individual predator admits the same ability to attack prey.

However, they did not classified individuals of predator as belonging to either the immature or the mature and supposed that the immature population does not feed on prey. This seems reasonable for a number of mammals, where immature predators are raised by their parents, the rate they attack at prey and the reproductive rate can be ignored. This study is an extension of previous efforts, emphasizing on the development of predator-prey system. Stage-structured models have also been studied by several authors [4, 5, 9-11]. The model of [11] considered was a stage-structured model of one species' growth consisting of immature and mature individuals was analyzed. And the model of [11] considered further assumed that the time from immaturity to maturity is itself state dependent. An equilibrium analysis and eventual lower bound and eventual upper bound of positive solutions for that model were given. In [9], the authors considered the following predator prey system with stage-structure for predator:

$$\begin{cases} \dot{x}(t) = x(t)(r - ax(t-\tau_1) - by_2(t)), \\ \dot{y}_1(t) = kbx(t-\tau_2)y_2(t-\tau_2) - (D+v_1)y_1(t), \\ \dot{y}_2(t) = Dy_1(t) - v_2 y_2(t). \end{cases} \tag{1}$$

where $x(t)$ is the density of prey of prey at time t; $y_1(t)$ is the density of immature predator at time t; $y_2(t)$ is the density of mature predator at time t; r is the intrinsic growth rate of prey, v_1 is the death rate of immature predator and v_2 the death rate of mature predator, constant $k > 0$ denotes the coefficient in conversing prey into new immature predator, constant $D > 0$ denotes the rate of immature predator becoming mature predator. It is assumed that this rate is proportional to the density of immature predator. In [9], the authors studied the asymptotic behavior of system (1). When time delay due to gestation of predator and time delay from crowding effect of prey are incorporated, they establish the condition for the permanence of populations and sufficient conditions under which positive equilibrium of system (1) is globally stable.

However, the effect of prevention functions from predator itself on its growth in number and changing environment is not considered in system (1). For more complication conditions, such as nonautonomous delayed predator-prey system with stage-structure for predator are

desired:

$$\begin{cases} \dot{x}(t) = x(t)(r - ax(t-\tau_1) - by_2(t)), \\ \dot{y}_1(t) = kbx(t-\tau_2)y_2(t-\tau_2) - (D+v_1)y_1(t) - k_1 y_1^2(t), \\ \dot{y}_2(t) = Dy_1(t) - v_2 y_2(t) - k_2 y_2^2. \end{cases} \quad (2)$$

where all parameters are position constants.

In this paper, our objective is to derive the existence conditions for Hopf bifurcation when $\tau_1 = 0$. Linearized stability and Hopf bifurcations for a three-dimensional system are scarcely studied. Therefore, linearized stability and Hopf bifurcation of system (2) is based on environmental factors in theory and application.

2. The existence of Hopf bifurcation

Let (x^*, y_1^*, y_2^*) is a positive equilibrium of system (2) with $\tau_1 = 0$.Then (x^*, y_1^*, y_2^*) satisfies the following equations:

$$\begin{cases} r = ax + by_2, \\ kbxy_2 = (D+v_1)y_1 + k_1 y_1^2, \\ Dy_1 = v_2 y_2 + k_2 y_2^2. \end{cases} \quad (3)$$

When system (3) has a unique positive solution, system (2) has a unique positive equilibrium.

Lemma 2.1. Assume that

$$(H1): \ \frac{r}{a} > \frac{(D+v_1)v_2}{kbD} \quad (4)$$

Then system (2) has a unique positive equilibrium.

Proof. Form (3), we have

$$[aD(D+v_1)k_2 + D^2kb^2 + ak_1v_2^2]y_2 + ak_1y_2^3 + 2ak_1v_2k_2y_2^2 + aD(D+v_1)v_2 - rD^2kb = 0$$

Let

$$F(y) = [aD(D+v_1)k_2 + D^2kb^2 + ak_1v_2^2]y_2 + ak_1y_2^3 + 2ak_1v_2k_2y_2^2 + aD(D+v_1)v_2 - rD^2kb,$$

for $y > 0$.

Since

$$F(0) = aD(D+v_1)v_2 - rD^2kb < 0$$

and

$$b^3F(\frac{r}{b}) = rb^2[aD(D+v_1)k_2 + D^2kb^2 + ak_1v_2^2] + ak_1r^3 + 2ak_1k_2r^2b + aD(D+v_1)b^3v_2 - rD^2kb^4$$ then

$$= rb^2[aD(D+v_1)k_2 + ak_1v_2^2] + ak_1r^3 + 2ak_1k_2v_2r^2b + aDb^3v_2(D+v_1) > 0,$$

there exists a point $\xi \in (0, \frac{r}{b})$ such that $F(\xi) = 0$. By derivatives, it shows that $F'(y) > 0$ for $y > 0$. Hence $F(y)$ has a unique positive zero point, that is (3) has a unique positive solution y_2^*. Thus from $Dy_1 = v_2y_2 + k_2y_2^2$, we obtain $y_1^* > 0$. From $r = ax + by_2$, we have $x = \frac{r - by_2^*}{a} > 0$. Therefore, system (2) has a unique positive equilibrium.

Then consider the existence conditions of Hopf bifurcation when $\tau_1 = 0$.

Taking $\bar{x} = x - x^*$, $\bar{y}_1 = y_1 - y_1^*$, $\bar{y}_2 = y_2 - y_2^*$ and replacing $\bar{x}, \bar{y}_1, \bar{y}_2$ by x, y_1,y_2, respectively, and taking $\tau_2 = \tau$, then system (2) with $\tau_1 = 0$, it becomes:

$$\begin{cases} \dot{x}(t) = -ax^*(t) - bx^*y_2(t), \\ \dot{y}_1(t) = kby_2^*x(t-\tau) - (D + v_1 + 2k_1y_1^*)y_1(t) + kbx^*y_2(t-\tau), \\ \dot{y}_2(t) = Dy_1 - (v_2 + 2k_2y_2^*)y_2. \end{cases} \quad (5)$$

Let

$$B_1 = D + v_1 + 2k_1y_1^* + v_2 + 2k_2y_2^* + ax^*,$$

$$B_2 = (D + v_1 + 2k_1y_1^*)(v_2 + k_2y_2^*) + ax^*(D + v_1 + 2k_1y_1^* + v_2 + k_2y_2^*),$$

$$B_3 = -kbDx^*,$$

$$B_4 = kb^2Dx^*y^* - akbd(x^*)^2,$$

$$B_5 = ax^*(D + v_1 + 2k_1y_1^*)(v_2 + 2k_2y_2^*).$$

We assume that

($H2$): $B_1^2 > 2B_2$,

($H3$): $B_4 > B_5, B4 > B5$;

$(H4)$: $(B_1^2 - 2B_2)^2 < 3(B_2^2 - B_1B_5 - B_3^2)$.

Then the characteristic equation of (5) is

$$\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3e^{-\lambda\tau}\lambda + B_4e^{-\lambda\tau} + B_5 = 0 \qquad (6)$$

Letting , $\lambda = i\omega$ $(\omega > 0)$,we obtain

$$\begin{cases} -B_1\omega^2 + B_3\omega\sin\omega\tau + B_4\omega\cos\omega\tau + B_5 = 0, \\ -\omega^3 + B_2\omega + B_3\omega\cos\omega\tau - B_4\omega\sin\omega\tau = 0. \end{cases} \qquad (7)$$

From (7), we obtain

$$\omega^6 + (B_1^2 - 2B_2)\omega^4 + (B_2^2 - 2B_1B_5 - B_3^2)\omega^2 = B_4^2 - B_5^2. \qquad (8)$$

Let $\omega^2 = z$, then

$$z^3 + (B_1^2 - 2B_2)z^2 + (B_2^2 - 2B_1B_5 - B_3^2)z = B_4^2 - B_5^2 \qquad (9)$$

Set $F(z) = z^3 + (B_1^2 - 2B_2)z^2 + (B_2^2 - 2B_1B_5 - B_3^2)z - B_4^2 + B_5^2$, for $z > 0$.

Since $F(0) = B_5^2 - B_4^2 < 0$, $F(+\infty) > 0$, then there exists a point $\eta \in (0, +\infty)$ such that $F(\eta) = 0$.

Assume that $(z - \eta)(z^2 + A_1z + A_2) = F(z)$, where A_1, A_2 are two constants determined later.

Then from (H3) and (H4), we have $A_1 = \eta + B_1^2 - 2B_2 > 0$ and $A_2 = \dfrac{B_4^2 - B_5^2}{\eta} > 0$. Therefore

$F(z)$ has a unique positive zero point $z = \eta$, that is equation (9) has a unique positive root

$z = \eta$. From (7), we have $\sin(\omega\tau + \theta) = \dfrac{B_1\omega^2 - B_5}{\sqrt{B_3^2\omega^2 + B_4^2}} \doteq H$, where

$\tan\theta = \dfrac{B_4}{B_3\omega}$. Thus $\omega_n = \dfrac{-\arcsin H + \theta + 2n\pi}{\tau_n}$ or $\tau_n = \dfrac{-\arcsin H + \theta + 2n\pi}{\eta}$, $n = 1, 2, \cdots$.

Letting, $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be the roof of *Eq.* (6), we have the following result.

Lemma 2.2 Assume the (H2), (H3) and (H4) hold. Then the following transversally condition hold:

$$\left.\frac{dRe\lambda(\tau)}{d\tau}\right/_{\tau = \tau_n} > 0.$$

Proof. By (6), differentiating with respect to τ , derive that

$$\frac{d\lambda}{d\tau}=\frac{B_4\lambda e^{-\lambda\tau}+B_3\lambda^2 e^{-\lambda\tau}}{3\lambda^2+2B_1\lambda+B_2-B_4\tau e^{-\lambda\tau}+B_3 e^{-\lambda\tau}-B_3\lambda\tau e^{-\lambda\tau}}.$$

It is obvious that the value of $\frac{d\,\mathrm{Re}\,\lambda}{d\tau}$ at $\lambda=i\omega_n$ is

$$\frac{d\,\mathrm{Re}\,\lambda}{d\tau}\Big/_{\lambda=iw_n}=\frac{AC_1+BD_1}{A^2+B^2},$$

where

$$A=B_2-3\omega_n^2-\omega_n\tau\cos\omega_n\tau+B_3\cos\omega_n\tau-B_3\omega_n\tau\sin\omega_n\tau$$

$$=B_2-3\omega_n^2+\tau(B_5-B_1\omega_n^2)+B_3\cos\omega_n\tau,$$

$$B=2B_1\omega_n+B_4\tau\sin\omega_n\tau-B_3\sin\omega_n\tau-B_5\omega_n\tau\cos\omega_n\tau$$

$$=2B_1\omega_n^2-B_3\sin\omega_n\tau+\omega_n\tau(B_2-\omega_n^2),$$

$$C_1=B_4\omega_n\tau\sin\omega_n\tau-B_3\omega_n^2\cos\omega_n\tau=\omega_n^2(B_2-\omega_n^2),$$

$$D_1=B_4\omega_n\tau\cos\omega_n\tau+B_3\omega_n^2\sin\omega_n\tau=\omega_n(B_1\omega_n^2-B_5).$$

Since

$$\begin{aligned}AC_1+BD_1&=[B_2-3\omega_n^2+\tau(B_5-B_1\omega_n^2)+B_5\cos\omega_n\tau][\omega_n^2(B_2-\omega_n^2)]\\&\quad+[2B_1\omega_n^2-B_5\sin\omega_n\tau+\omega_n\tau(B_2-\omega_n^2)][\omega_n(B_1\omega_n^2-B_5)]\\&=\omega_n^2(B_2-\omega_n^2)(B_2-3\omega_n^2)+B_3\omega_n^2(B_2-\omega_n^2)\cos\omega\\&\quad-2B_1\omega_n^2(B_5-B_1\omega_n^2)+B_3\omega_n^2(B_5-B_1\omega_n^2)\sin\omega\\&=\omega_n^2[(B_2-\omega_n^2)(B_2-3\omega_n^2)+B_2B_3(-B_1\omega_n^2)+\\&=\omega_n^2[3\omega_n^4+2(B_1^2-B_2)\omega_n^2+B_2^2-B_1B_5-\end{aligned}$$

Therefore

$$\frac{d\,\mathrm{Re}\,\lambda}{d\tau}\Big/_{\lambda=i\omega_n}=\omega_n^2[3\omega_n^4+2(B_1^2-2B_2)\omega_n^2+B_2^2-2B_1B_5-B_3^2].$$

Since

$$\Delta=4(B_1^2-2B_2)^2\omega_n^2)-12(B_2^2-2B_1B_5-B_3^2].$$

Then

$$\frac{d\,\mathrm{Re}\,\lambda}{\tau}/_{\tau=\tau_n} > 0.$$

This completes the proof of Lemma 2.2.

By above discussion,the Lemma below is derived.

Lemma 2.3 If (H1)-(H4) hold, then there exists real sequence $\{\tau_n\}$, n = 0, 1, 2, …, satisfying,

$\tau_{n+1} > \tau_n$,and $\tau_n = \dfrac{2n\pi + \theta - \arcsin H}{\eta}, n = 0,1,2,...,$

such that

(i) all root of *Eq* (6) have strictly negative real parts for $\tau \in (0, \tau_0)$;

(ii) when $\tau = \tau_0$, *Eq* (6) has a pair of pure imaginary root $\pm i\omega_0$ which are simply, and all other roots have negative parts;

(iii) when $\tau > \tau_0$, *Eq* (6) has at least one root with strictly positive position real part.

Applying Lemma 2.3 above and Theorem 1.1 in [5], we have

Theorem 2.1 If (H1)-(H4) hold, then system (2) when $\tau_1 = 0$ has a Hopf bifucation at

$\tau = \tau_n$; n = 0,1, 2,….

Theorem 2.2 In system (2) with $\tau_1 = 0$, let (H1)-(H4) hold.

(1) If $0 < \tau < \tau_0$, then (x^*, y_1^*, y_2^*) is asymptotically stable.

(2) If $\tau > \tau_0$, then (x^*, y_1^*, y_2^*) is unstable.

Moreover, the characteristic equation (6) always has a root $\lambda = \alpha + i\omega$ such $\alpha > 0$.

3. Conclusions

In this study, predator-prey system with stage-structure for predator has been developed. It has been combined with the effect of prevention functions from predator itself on its growth in number and changing environment on the nonautonomous delayed predator-prey system. Then existence of Hopf bifurcations of the system (2) is proposed. Furthermore, conditions of linearized stability and Hopf bifurcations for this system are established.

We would like to mention here that an interesting but maybe challenging problem associated

with the study of system (2) should be the uniqueness and global stability of positive solution. We leave this to future work.

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