SCIREA Journal of Mathematics
http://www.scirea.org/journal/Mathematics
January 25, 2023
Volume 8, Issue 1, February 2023
https://doi.org/10.54647/mathematics110379

The *p*-adic valuation of $C_n(l)=\prod_{k=1}^{n}\prod_{j=0}^{l}(2k+2j-1)$

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Abstract:

The formula of the *p*-adic valuation of the sequence $C_n(l)=\prod_{k=1}^{n}\prod_{j=0}^{l}(2k+2j-1)$ is studied. It is proved that $C_n(l)$ is not a square if *l* is even and $n\geq\max\{2l,\frac{3l+19}{5}\}$. It is proved also that there are many squares in the sequences $C_n(1)$ and $C_n(3)$, while there are no squares in the sequences $C_n(5)$, $C_n(7)$ and $C_n(9)$.

Keywords: *p-adic valuation, Legendre's formula, Pell equation*

1. Introduction

For $f(x)\in Z[x]$ and an positive integer *n*, we define $C_n(f)$ by $C_n(l)=\prod_{k=1}^{n}f(k)$. The study of the sequence $C_n(f)$ that containing infinitely many squares is an important problem in number theory. So far, many important results have been achieved in this research.

Amdeberhan, Media and Moll [1] conjectured that $C_n(x^2+1)$ is not a square for any integer $n > 3$. Cilleruelo [2] confirmed the conjecture and proved that $C_n(x^2+1)$ is a square if and only if $n=3$. Hong-Liu [3] proved that there are infinitely many squares in the sequence $C_n(x^2-1)$. Fang [4] confirmed another similar conjecture posed by Amdeberhan, Media and Moll [1]. E. G\¨urel and A.U.O. Kisisel [5] proved that $C_n(x^3+1)$ is not a square for any positive n. The number of squares in the sequence $C_n(f)$ was estimated by Cilleruelo-Luca-Quiros [6] when $f(x)$ is irreducible with degree greater than 2.

Zhang-Wang [7] proved that $C_n(x^d+1)$ is not a square when $d \ge 5$ is a prime. By Yang-Zhao [8], $C_n(x^d+1)$ is not a square when d is an arbitrary odd integer. By Chen-Gong-Ren [9], $C_n(x^d+1)$ is not a powerful number when d is an odd prime power. By Chen-Gong [10], $C_n(x^d+1)$ is not a powerful number when d is odd and has at most two distinct prime factors. By E. G\"urel [11], there is an integer $N(m)$ such that $\frac{C_n(x^d+1)}{C_m(x^d+1)}$ is not a square if $n \ge N(m)$ when $d = 2$ or 3. Niu-Liu [12] showed that if $n \ge \max\{q, 1198 - q\}$, then $C_n(x^3+q^3)$ is not a powerful number for a fixed positive integer q.

For a prime p, the p-adic valuation of n is defined by $v_p(n)=e$ if $p^e \mid n$ and $p^{e+1} \nmid n$. Let $f(x)=\prod_{j=0}^{l}(2x+2j-1)$ and write $C_n(f)$ as $C_n(l)$. In this paper, an explicit formula of $v_p(C_n(l))$ is given. By the properties of some Diophantine equations, the squares in the sequences $C_n(l)$, $1 \le l \le 9$ are studied also. Let $n=\sum_{k=0}^{l} n_k p^k$ be the p-adic expansion, we define $s_p(n)=\sum_{k=0}^{l} n_k$. The main results are as follows.

Theorem 1.1. Let p be an odd prime,

$$v_p(C_n(l)) = \frac{(l+1)(n+s_p(n)-s_p(2n))}{p-1} + \sum_{j=1}^{l}(l-j+1)(v_p(2n+2j-1)-v_p(2j-1)).$$

Corollary 1.2. For an odd prime p, $v_p(C_n(l)) \sim \frac{(l+1)n}{p-1}, n \to \infty.$

Theorem 1.3. If $n \ge \max\{2l, \frac{3l+19}{5}\}$, then there exists a prime p such that $v_p(C_n(l)) = l+1$.

By Theorem 1.3, it is trivial to deduce the following two corollaries.

Corollary 1.4. When l is even, if $n \geq \max\{2l, \frac{3l+19}{5}\}$, then $C_n(l)$ is not a square.

When it turns to whether $C_n(l)$ is a square or not for $l = 1,3,5,7,9,$ we have the following results.

Theorem 1.5. (1) $C_n(1)$ is a square if and only if $n = 2m^2 + 2m$, where $m \geq 1$.

(2) $C_n(3)$ is a square if and only if $n = \frac{(\frac{1+\sqrt{5}}{2})^{2k} + (\frac{1-\sqrt{5}}{2})^{2k} - 3}{2}$, where $k \geq 2$ and $3 \vdots k$.

(3) There are no squares in the sequences $C_n(5), C_n(7)$ and $C_n(9)$.

2. Proof of Theorem 1.1 and Theorem1.3

Lemma 2.1. (Legendre's formula [13]) Let n be a positive integer and p be a prime, then $v_p(n!) = \frac{n - s_p(n)}{p-1}$.

Lemma 2.2. ([14]) Let n be a positive integer and p be a prime, then $\frac{n}{p-1} - \frac{\log(1+n)}{\log p} \leq v_p(n!) \leq \frac{n}{p-1}$.

Lemma 2.3. Let n be a positive integer and p be a prime, then $\lim_{n\to\infty} \frac{s_p(n)}{n} = 0$.

Proof. By Lemmas 2.1 and 2.2, we have $0 \leq \frac{s_p(n)}{n} \leq (p-1)\frac{\log(n+1)}{n \log p}$, then the lemma follows.

Lemma 2.4. [15] For any positive integer n, there is always a prime $p \in [n, \frac{9(n+3)}{8}]$.

Proof of Theorem 1.1. Since

$$C_n(l) = \prod_{k=1}^{n} \prod_{j=0}^{l} (2k + 2j - 1) = \prod_{j=0}^{l} \frac{(2n+2j-1)!!}{(2j-1)!!},$$

we have

(1) $$C_n(l) = ((2n-1)!!)^{l+1} \prod_{j=1}^{l} (\frac{2n+2j-1}{2j-1})^{l-j+1}.$$

Since $(2n-1)!!=\frac{(2n)!}{n!2^n}$, by taking the p-adic valuations on both sides of (1) we have

$$v_p(C_n(l))=(l+1)(v_p((2n)!)-v_p(n!))+\sum_{j=1}^{l}(l-j+1)(v_p(2n+2j-1)-v_p(2j-1)).$$

Therefore the proof is complete by Legendre's formula in Lemma 2.1.

Proof of Corollary 1.2. By Theorem 1.1,

$$\frac{(p-1)v_p(C_n(l))}{(l+1)n}=1+\frac{1}{n}(s_p(n)-s_p(2n))+\frac{p-1}{(l+1)n}\sum_{j=1}^{l}(l-j+1)(v_p(2n+2j-1)-v_p(2j-1)). \tag{2}$$

Since for each $j = 1, . . . , l$, we have $0\le\frac{v_p(2n+2j-1)}{n}\le\frac{\log_p(2n+2j-1)}{n}$. Therefore the right hand of (2) goes to 1 as n tends to infinity by Lemma 2.3. The proof is done.

Proof of Theorem 1.3

Proof. Since $n\ge\frac{3l+19}{5}$, then $\frac{9}{8}(\frac{2n+2l+1}{3}+3)\le 2n-1$, hence there is a prime $p\in[\frac{2n+2l+1}{3}, 2n-1]$ by Lemma 2.4. Thus $\frac{p+1}{2}\le n\le\frac{3p-1}{2}-l$, and we have $p>2l$ since $n\ge 2l$. Thus the equation

$$(2x-1)(2x+1)(2x+3)\cdots(2x+2l-1)\equiv 0 \pmod p$$

has exactly $l + 1$ solutions $x_i=\frac{p-2l+2i+1}{2}$, $i=1,2,\cdots,l$, in the interval $[1, n]$, therefore

$$v_p(C_n(l))=\sum_{i=0}^{l}v_p((2x_i-1)(2x_i+1)\cdots(2x_i+2l-1)).$$

For each i, if $p\mid 2x_i+2j_1-1, p\mid 2x_i+2j_2-1$ with $0\le j_1\ne j_2\le l$, then $p\mid j_1-j_2$, which contradicts $p>2l.$ Then we have

$$v_p((2x_i-1)\cdots(2x_i+2l-1))=v_p(2x_i+2j_i-1),$$

for a fixed j_i with $0\le j_i\le l.$ Since $p^2\ge 3p\ge 2n+2l+1>2x_i+2j_i-1,$ we have

$$v_p((2x_i-1)\cdots(2x_i+2l-1))=v_p(2x_i+2j_i-1)=1.$$

Therefore, $v_p(C_n(l))=\sum_{i=0}^{l}v_p((2x_i-1)(2x_i+1)\cdots(2x_i+2l-1))=l+1.$

The proof of Theorem 1.3 is complete.

3. Proof of Theorem 1.5

In this section, we discuss the *p*-adic valuations of $C_n(l)$ when $l = 1,3,5,7,9$ and prove Theorem 1.5.

Lemma 3.1. [16] The positive integer solutions of the Pell equation

$$x^2 - 5y^2 = 4$$

are (x_m, y_m), where $\frac{x_m + \sqrt{5}y_m}{2} = (\frac{1+\sqrt{5}}{2})^{2m}, m = 1, 2, \ldots.$

Lemma 3.2. [17] The positive integer solutions of Pell equation

$$x^2 - 5y^2 = -4$$

are (x_m, y_m), where $\frac{x_m + \sqrt{5}y_m}{2} = (\frac{1+\sqrt{5}}{2})^{2m-1}, m = 1, 2, \ldots.$

Lemma 3.3. Let $x_1 = \frac{1+\sqrt{5}}{2}$ and $x_2 = \frac{1-\sqrt{5}}{2}$.

(1) Write $S_k = x_1^{2k} + x_2^{2k}$, $k = 1, 2, \ldots,$ then $S_k \equiv 1$ or $7 \pmod{10}$ if and only if $3 \vdots k$.

(2) Write $T_k = x_1^{2k+1} + x_2^{2k+1}$, $k = 1, 2, \ldots,$ then $T_k \equiv 1$ or $9 \pmod{10}$ if and only if $3 \vdots (k-1)$.

Proof. By direct computations, we have $S_1 = 3$ and $S_2 = 7$. When $k > 2$, by the recursion formula

$$S_k = S_{k-1}S_1 - S_{k-2} = 3S_{k-1} - S_{k-2},$$

we have $S_3 \equiv 8,\ S_4 \equiv 7,\ S_5 \equiv 3,\ S_6 \equiv 2,\ S_7 \equiv 3,\ S_8 \equiv 7 \pmod{10}$ and hence the first statement follows.

For the second statement , we have $T_1 = 4, T_2 = 11$. When $k > 2$, by the recursion formula

$$T_k = T_{k-1}S_1 - T_{k-2} = 3T_{k-1} - T_{k-2},$$

we have $T_3 \equiv 9, T_4 \equiv 6, T_5 \equiv 9, T_6 \equiv 1, T_7 \equiv 4, T_8 \equiv 1 \pmod{10}$, and hence the second statement follows.

Lemma 3.4. (1) When $n \geq 1$, there is at most one square of the numbers $2n+1, 2n+5, 2n+9, 2n+13$.

(2) When $n \geq 5$, there is at most one square of the numbers $2n+1, 2n+5, 2n+9, 2n+13, 2n+17$.

(3) When $k \geq 1$, there is at most one square of the numbers $2k+1,\ 6k+7,\ 6k+11,\ 2k+5,\ 6k+19$.

(4) When $k \geq 1$, there is at most one square of the numbers $6k+5,\ 2k+3,\ 6k+13,\ 6k+17,\ 2k+7$.

Proof. Since the Diophantine equations $x^2 - y^2 = k_0$, have no solutions with $x > 5$ when $k_0 = 4,8,12,16,$ then statements 3.4 (1) and (2) are proved. Since the Diophantine equations $x^2 - 3y^2 = \pm 4,\ x^2 - 3y^2 = \pm 8$ and $x^2 - 3y^2 = \pm 16$ have no odd solutions, then statements 3.4 (3) and (4) are proved.

Lemma 3.5. (1) $C_n(1)$ is a square if and only if $z_n(1) = 2n+1$ is a square.

(2) $C_n(3)$ is a square if and only if $z_n(3) = \dfrac{(2n+1)(2n+5)}{5}$ is a square.

(3) $C_n(5)$ is a square if and only if $z_n(5) = \dfrac{(2n+1)(2n+5)(2n+9)}{5}$ is a square.

(4) $C_n(7)$ is a square if and only if $z_n(7) = \dfrac{(2n+1)(2n+5)(2n+9)(2n+13)}{5 \cdot 13}$ is a square.

(5) $C_n(9)$ is a square if and only if $z_n(9) = \dfrac{(2n+1)(2n+5)(2n+9)(2n+13)(2n+17)}{5 \cdot 13 \cdot 17}$ is a square.

Proof. As the proofs are similar, we show statement (2) only. By (1),
$$C_n(3) = z_n(3)\left(\frac{((2n-1)!!)^2(2n+1)(2n+3)}{3}\right)^2,$$

hence the if part is trivial. For the only if part, assume that $C_n(3)$ is a square. If $z_n(3) \notin Z$, then $v_5(C_n(3))$ is odd. which is impossible, hence $z_n(3) \in Z$, which implies that $z_n(3)$ is a square also. The lemma is proved.

Proof of Theorem 1.5.

(1) $l = 1$.

It is trivial to see that $2n+1$ is a square if and only if $n = 2m^2 + 2m$, $m = 1,2, \ldots,$ then statement (1) follows.

(2) $l = 3$.

First, we show the necessity. Let $C_n(3)$ be a square. Then by Lemma 3.5 (2), $z_n(3)$ is a square also. Since $(2n+1,2n+5)=1$, then either $\frac{2n+1}{5}$ and $2n+5$, or $2n+1$ and $\frac{2n+5}{5}$ are squares.

Suppose $2n+5$ and $\frac{2n+1}{5}$ are squares. Let $(a,b)=\left(\sqrt{2n+5}, \sqrt{\frac{2n+1}{5}}\right)$, then (a,b) is a positive integer solution of $x^2 - 5y^2 = 4$. Hence by Lemma 3.1, $a=(\frac{1+\sqrt{5}}{2})^{2k}+(\frac{1-\sqrt{5}}{2})^{2k}$ for some positive integer k. Therefore

(3) $$n=\frac{a^2-5}{2}=\frac{(\frac{1+\sqrt{5}}{2})^{4k}+(\frac{1-\sqrt{5}}{2})^{4k}-3}{2}$$

for some positive integer k. Since $5\mid 2n+1$, we have $a^2 \equiv 4$ (mod 5). Thus $a\equiv 3$ or $7 \pmod{10}$ since a is odd. Therefore , $3 ⋮ k$ by Lemma 3.3 and then $3 ⋮ 2k$.

Suppose $2n+1$ and $\frac{2n+5}{5}$ are squares. Let $(a,b)=(\sqrt{2n+1}, \sqrt{\frac{2n+5}{5}})$, then (a,b) is a positive integer solution of $x^2 - 5y^2 = -4$. Hence by Lemma 3.2, $a=(\frac{1+\sqrt{5}}{2})^{2k+1}+(\frac{1-\sqrt{5}}{2})^{2k+1}$ for some positive integer k. Therefore

(4) $$n=\frac{a^2-1}{2}=\frac{(\frac{1+\sqrt{5}}{2})^{4k+2}+(\frac{1-\sqrt{5}}{2})^{4k+2}-3}{2}$$

for some positive integer k. Since $5\mid(2n+5)$,we have $a^2 \equiv 1 \pmod 5$. Thus $a\equiv 1$ or $9 \pmod{10}$ since a is odd. Therefore , $3 ⋮ (k-1)$ by Lemma 3.3, and then $3 ⋮ (2k+1)$.

By combing equations (3) and (4), we have $n=\frac{(\frac{1+\sqrt{5}}{2})^{2k}+(\frac{1-\sqrt{5}}{2})^{2k}-3}{2}$ for some positive integer $k\ge 2$ with $3 ⋮ k$. If $k=1$, then $n=0$, thus $k\neq 1$. The necessity follows.

Conversely, if $n=\frac{(\frac{1+\sqrt{5}}{2})^{2k}+(\frac{1-\sqrt{5}}{2})^{2k}-3}{2}$ for some positive integer $k\ge 2$ with $3 ⋮ k$, then

$$z_n(3)=\frac{1}{5}(2n+1)(2n+5)=\left(\frac{1}{\sqrt{5}}\left((\frac{1+\sqrt{5}}{2})^{2k}-(\frac{1-\sqrt{5}}{2})^{2k}\right)\right)^2,$$

which is a square, therefore the sufficiency follows. The proof is done.

(3) $l = 5,7,9.$

The proofs for $l = 5,7,9$ are similar. In order to make our proof clean and efficient, we give the proof of case $l = 5$ in detail, and for other cases, only those points that are different are mentioned.

By Lemma 3.4, there is at most one square of the numbers $2n+1, 2n+5, 2n+9$. Since any two integers of $2n+1, 2n+5, 2n+9$ are co-prime, then $z_n(5) = a^2 \frac{D}{5}$, where D is a product of at least two different primes. Hence $z_n(5)$ is not a square. By Lemma 3.5, $C_n(5)$ is not a square.

For $l = 7$, the difference is that when $(2n+1, 2n+13) = 3$, we need to rewrite $z_n(7)$ as

$$z_n(7) = \frac{(6k+3)(6k+7)(6k+11)(6k+15)}{5\cdot 13} = \frac{3^2(2k+1)(6k+7)(6k+11)(2k+5)}{5\cdot 13}.$$

For $l = 9$, when $n = 1,2,3,4,$ we need to compute $z_n(9)$ directly. When $n \geq 5,$ if $(2n+1, 2n+13) = 3,$ then

$$z_n(9) = \frac{3^2(2k+1)(6k+7)(6k+11)(2k+5)(6k+19)}{5\cdot 13\cdot 17},$$

and if $(2n+5, 2n+17) = 3,$ then

$$z_n(9) = \frac{3^2(6k+5)(2k+3)(6k+13)(6k+17)(2k+7)}{5\cdot 13\cdot 17}.$$

The proof of Theorem 1.5 is complete.

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