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Kepler, Newton and Einstein

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Abstract

This essay describes orbits of particles passing near the sun, first according to Newton's celestial mechanics and then according to Einstein's general theory of relativity. Attempts to verify the latter by observing the deflection of light during total solar eclipses are mentioned. Confirmation also comes from mea- surements of the delay in the arrival of reflected light, as well as from the use of Very Long Baseline Interferometry (VLBI). A comparison of explanations of these phenomena reveals contradicting statements.

Keywords: Celestial mechanics · gravitation · the general theory of relativity · deflection of light rays · delay in the arrival of reflected light · Kepler · Newton · Einstein

1. Introduction

According to Newton's celestial mechanics, a particle of positive mass that passes with high speed close to the sun travels along a branch of a hyperbola. The faster it goes, the smaller is the angle between the direction of the incoming path and the direction of the outgoing path. How does this angle compare with the deflection angle by which the sun bends a light ray?

We consider three phenomena which might influence a light ray: Newtonian gravi- tation; Einsteinian curvature of spacetime; refraction in the sun's atmosphere.

The approach is both astronomical, geometric, and historical.

The most important persons in this essay are Johannes Kepler (1571–1630), Isaac Newton (1642 old style – 1727), and Albert Einstein (1879–1955).

2. Sources

Of utmost importance for the understanding of the deflection of light rays passing close to the sun are two papers published by Einstein (1911, 1916a), the latter reprinted as a book (1916b). His book (1917) was intended to be popular ("*Gemeinverständlich*"), but its translation (1920) was provided with three more technical and quite useful appendices, the third of which, entitled "The experimental confirmation of the general theory of relativity," is authored by Einstein at the request of the translator according to the Translator's Note on page ix.

A book by Julian Schwinger (1986) gives a most readable account of the devel- opment. A book chapter by Joseph Weber (1964) offers good explanations. Shlomo Sternberg's book (2012) has been important here.

This study originates in my interest in astronomy and the history of science in general. Let me mention as a background that I learnt some celestial mechanics from Gunnar Larsson-Leander (1918–2020) and Bertil Lindblad (1895–1965) at Stockholm University College and Saltsjöbaden Observatory during the academic year 1957–1958. I worked as an assistant at this observatory in the northern summer of 1957.

3. Kepler and Newton

Kepler's third law says that the periods p_1 and p_2 of two planets at mean distances r_1 and r_2 from the sun satisfy $(p_2/p_1)^2 = (r_2/r_1)^3$, assuming that the masses of the two planets are equal—approximately so if these masses are negligible in comparison with that of the sun. This implies that the speeds $v_j = 2\pi r_j/p_j$ of planets in circular orbits satisfy

$$\frac{v_2}{v_1} = \sqrt{\frac{r_1}{r_2}}. \tag{3.1}$$

The mean angular speeds are of course inversely proportional to the periods, but to determine the actual average speed in an elliptic orbit we have to use the circumference of an ellipse, which is

$$a\int_0^{2\pi}\sqrt{1-\varepsilon^2\sin^2\theta}\,\mathrm{d}\theta \leqslant 2\pi a\Bigl(1-\tfrac{1}{4}\varepsilon^2\Bigr),$$

where a is the major semi-axis and ε the excentricity. The right-hand side is a good approximation for small values of ε.

Let us now consider a particle of positive mass which passes at time $t = 0$ at a point with distance s from the sun and at that moment has a velocity orthogonal to the radius vector to the sun. At a certain speed, which we shall call $v_{\mathrm{circ}}(s)$, the orbit is a circle with the sun at its center. If the speed v satisfies $0 < v < v_{\mathrm{circ}}(s)$, the orbit is an ellipse with the sun at the focus farthest away from the particle. If the speed satisfies $v_{\mathrm{circ}}(s) < v < \sqrt{2}\,v_{\mathrm{circ}}(s)$, the orbit is an ellipse with the sun at the focus which is closest to the particle. If the speed is equal to $\sqrt{2}\,v_{\mathrm{circ}}(s)$, the orbit is a parabola, and, finally, if the speed is larger than $\sqrt{2}\,v_{\mathrm{circ}}(s)$, the orbit is a branch of a hyperbola. As v increases, the excentricity of the orbit first decreases to zero and then increases again; the exact expression being given in Theorem 12.1.

All this was known to Kepler, at least for elliptical orbits, and follows from Newton's celestial mechanics.

Comets can move along a branch of a hyperbola ... but only approximately so: Göran Henriksson (personal communication 2002 May 28) warns against us thinking of comets, since they warm up when approaching the sun and then let out gases, a process which perturbs the orbit. They are also subject to pressure from the solar wind. So we should think about a tiny particle, a little bullet—an ideal comet.

The hyperbolic comet orbits that have been observed typically have exentricities between 1.0002 and 1.005, being close to a parabolic orbit, whereas for speeds close to that of light, the excentricity is large as we shall see.

Steven S. Shapiro and Irwin L. Shapiro (2010) write:

> The first calculation of the deflection of light by mass was published by the German astronomer Johann Georg von Soldner in 1801. Soldner showed that rays from a distant

star skimming the Sun's surface would be deflected through an angle of about 0.9 seconds of arc [...] (Shapiro & Shapiro 2010)

In the Wikipedia page on von Soldner the angle is given as $0''.84$. Let us call this angle $\sigma(s, c)$ with a sigma in honor of von Soldner, where s is the distance to the sun's center (at least equal to $R_\odot$, the sun's radius), and c the speed of light. We shall redo his calculation in Sections 14 and 15. The assumption is that a photon has a positive mass—it is claimed that the photon has mass zero, but we are free to imagine differently.

Shapiro & Shapiro (2010) add that, as far as they know, "neither Soldner nor later astronomers attempted to verify this prediction." Einstein and Poor made the same calculations in 1911 and 1927, respectively.

4. Einstein in 1911

For the influence of gravitation on a light ray, Einstein (1911:908, 1973a:139) gives in an article submitted on 1911 June 21 the formula for the angle α of deflection for a light ray passing at distance Δ from the sun:

$$\alpha = \frac{2kM}{c^2\Delta},$$

wobei k die Gravitationskonstante, M die Masse des Himmelskörpers, Δ den Abstand des Lichtstrahles vom Mittelpunkt des Himmelskörpers bedeutet. *Ein an der Sonne vorbeigehender Lichtstrahl erlitte demnach eine Ablenkung vom Betrage $4 \cdot 10^{-6} = 0{,}83$ Bogensekunden.* (Einstein 1911:908)

He does not say here whether k is Newton's gravitational constant or his own gravitational constant mentioned in Subsection 5.1 below, but if we calculate

$$k = \frac{\alpha c^2 \Delta}{2M},$$

we obtain $k \approx 6.287\,\mathrm{cm}^2{\cdot}\mathrm{s}^{-2}{\cdot}\mathrm{g}^{-1}$, in good agreement with the present value of Newton's gravitational constant, now called G (see Table 1 in Section 14).

In our notation from Section 3 we have

$$\sigma(r, c) = \frac{2GM_\odot}{c^2 r}, \tag{4.1}$$

where G is Newton's gravitational constant,[1] c the speed of light in vacuum, $M_\odot$ the mass of the sun, and r the distance to the sun's center. This formula yields an angle of $4 \cdot 10^{-6}$ radians, equal to $0''.83$. This is just half of the amount in the later formula (5.2) below. He stresses that the formula is a first approximation only (1911:898, 1973a:129)—maybe in anticipation of his later article (1916a).

Concerning the speed of light we note a statement by Einstein from 1914:

Es erscheint mir unglaublich, daß der Ablauf irgendeines Vorganges (z. B. der Lichausbreitung im Vakuum) als unabhängig von allem übrigen Geschehen in der Welt aufgefaßt werden könne. (Einstein 1914:176)

Indeed, there is at present a discussion about possible variations in the speed of light.

[1]The International Astronomical Union (2015:2) states: "the universal constant G is currently one of the least precisely determined constants, whereas the error in the product $GM_\odot$ is five orders of magnitude smaller [...]." I shall use here the value of $GM_\odot$ given in Table 1 in Section 14.

5. Einstein in 1916

5.1. Masses cause spacetime to curve

In 1916, Albert Einstein published his general theory of relativity in *Annalen der Physik* (1916a). The manuscript was received on 1916 March 20 and the article was published on 1916 May 11; then reprinted in (1916b).

A large part of the 1916 text is devoted to Riemannian geometry, which can describe a spacetime with any kind of curvature. To get a description of the universe, it is necessary to relate the curvature to existing masses. This he does on page 818 in (1916a)—the fiftieth page in his 54-page article—by comparing his gravitational potential with that of Newton. This comparison results in formula (69) on page 818; let me quote:

> Durch Vergleich ergebt sich
>
> (69) $$\kappa = \frac{8\pi K}{c^2} = 1{,}87 \cdot 10^{-27}.$$ (Einstein 1916a:818)

The unit is $\mathrm{cm{\cdot}g^{-1}}$ here. Here κ is Einstein's gravitational constant and K Newton's gravitational constant, given the value $6.7 \cdot 10^{-8}$. With other units it can be given as

(5.1) $$\kappa = \frac{8\pi K}{c^4}.$$

5.2. Deflection of a light ray

According to Albert Einstein's general theory of relativity, the sun deflects a light ray: light follows a geodesic in spacetime, and geodesics are influenced by masses. In the words of Clifford M. Will, there are

> four main phenomena associated with the spacetime geometry that is the central concept of Einstein's theory, namely that it bends light, it warps time, it moves mass, and it makes waves. (Will 2017:19)

He determines the deflection, called "die Biegung" and denoted by B, as follows.

> Die Ausrechnung ergibt
>
> (74) $$B = \frac{2\alpha}{\Delta} = \frac{\kappa M}{4\pi\Delta}.$$
>
> Ein an der Sonne vorbeigehender Lichstrahl erfährt demnach eine Biegung von 1,7″, ein am Planeten Jupiter vorbeigehender eine solche von etwa 0,02″.
> (Einstein 1916a:822; similarly in 1916b:63)

> According to this, a ray of light going past the sun undergoes a deflection of 1·7″ and a ray going past Jupiter a deflection of about ·02″. (Einstein 1973b:171)

The angle of 1″.7 is that for the smallest possible value of Δ, i.e., $\Delta = R_\odot$, the radius of the sun, in our notation

(5.2) $$\theta(R_\odot, c) = \frac{4GM_\odot}{c^2 R_\odot},$$

where, as in (4.1), G is Newton's gravitational constant, $M_\odot$ the mass of the sun, and c the speed of light. This amount is obtained by combining formula (69) on page 818 and formula (70a) on page 819 in (Einstein 1916a); likewise by combining formula

(105a) on page 86 and formula (108) on page 89 in (Einstein 1987), and is stated also in (Weber 1964:233). The angle of $1''.7$ is twice as large as that given in (4.1) in the previous section.

Concerning the doubling, he writes:

> It may be added that, according to the theory, half of [...] this deflection is produced by the Newtonian field of attraction of the sun, and the other half by the geometrical modification ("curvature") of space caused by the sun.
>
> (Einstein 1920: Appendix III, (b), page 153)

In his book (1986), Julian Schwinger (1918–1994) gives the amount 0.875 arc seconds (1986:140) and then explains the doubling (1986:189):

> A light beam travels at speed c far from gravitating masses. As it nears the Sun, it takes longer (Einstein, 1911)[2] to go a shorter distance (Einstein, 1915).[3] The two fractional changes are equal. Light slows down by *twice* the fractional change that the consideration of time alone predicted.
> [...]
> we now find that the angular deflection of light in a grazing passage of the Sun is $4gR/c^2$, [to be corrected to $4g'R/c^2$, which in the notation used here is equal to (5.2)]; the numerical value, an angle measured in radians, is 8.48×10^{-8} [equal to $1''.75$].[4]
>
> (Schwinger 1986:209)

Joseph Weber (1919–2000) explains:

> It is easy to see why we get an extra factor of 2. The deflection is given by the contribution of two identical terms Γ^x_{oo} and Γ^y_{xx} [...]. In the classical theory, only one term is present (Γ^y_{xx}). At low velocity, $dx/ds \sim v/c$ and $\Gamma^y_{xx}v^2 \ll \Gamma^x_{oo}c^2$; at $v = c$, they contribute equally. We can say that the photon acts as if it has a gravitational mass twice its inertial mass. (Weber 1964:233)

He mentions (1964:233) that the spin of a photon does not influence the deflection.

Similarly, Callahan (2001:419) writes that the doubling is due to "a fuller account of the gravitational effect."

Charles Lane Poor (1866–1951), who was critical of relativity theory,[5] wrote:

> Einstein, himself, has given two very definite predictions as to the amount by which the light of a star should be bent, or deflected in its passage by the sun. In 1911 he fixed this amount as $0''.83$; in 1916 he doubled this and made the deflection, according to his theories, $1''.70$. But the way in which Einstein derived these two different values is not given in any general works on relativity. (Poor 1927:225)

But as already mentioned, Einstein did present this doubling in his paper (1916a).

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[2]The year refers to Einstein's thinking in 1911, manifest in his article (1911).

[3]This refers to Einstein's thinking in the year 1915, published in (1916a).

[4]Misner et al. (2002:1103) give the value $\frac{1}{2}(\gamma+1)\cdot 1''.75$, where γ is the parametrized Post-Newtonian (PPN) parameter. See also (Will 2017:20) and (Lemos et al. 2019:2). For numerical calculations, see Section 14.

[5]Milena Wazeck presents in her two books (2009, 2014) detailed accounts of the widespread resistance against relativity theory. She lists 26 persons who were "Einsteingegner" (opponents of Einstein) in her first book (2009:25). Poor is not mentioned among these. Ernst Gehrcke (1878–1960) and Arvid Reuterdahl (1876–1933) were two of the most active opponents (2009:22, 23). The network around the two, reconstructed from available correspondence, shows that they had together 34 correspondents (2009:307). See also (Goenner 1993) with his "A Hundred Authors against Einstein."

It should be noted that the angle is inversely proportional to the distance r, thus

$$\theta(r, c) = \frac{R_\odot}{r}\theta(R_\odot, c), \tag{5.3}$$

where $R_\odot$ is the radius of the sun (cf. also Soffel 1989:6).

It is also inversely proportional to the square of the speed of light—it is permissible to think of several universes with different values of c.

5.3. Numerical values in Einstein's 1916 paper

I inserted numerical values of the quantities mentioned. The quantity α in the quoted formula (74) above is given in formula (70a) on page 819 of (1916a) and likewise on page 60 in (1916b) as

$$\alpha = \frac{\kappa M}{8\pi}\,\text{cm}, \tag{70a}$$

and if we insert this value in the formula (74) quoted above, using the numerical values

$$\begin{aligned}\kappa &= 1.87 \cdot 10^{-27}\,\text{cm} \cdot \text{g}^{-1},\\ M = M_\odot &= 1.989 \cdot 10^{33}\,\text{g}, \text{ and}\\ \Delta = R_\odot^{\text{N}} &= 6.957 \cdot 10^{10}\,\text{cm},\end{aligned}$$

we obtain

$$B \approx 4,4254 \cdot 10^{-6} \text{ radians} \approx 0''.876,$$

the same as in the 1911 paper, not the double of that value.

In the many subsequent publications that I have seen, none mentions this, as if nobody had cared to insert numerical values in the formulas. But we can look into a later translation.

6. Einstein 1973

A part of Einstein's paper (1916a) was translated into English and published as a chapter (1973b) in the book (1973) edited by Clive William Kilmister (1924–2010). There we find on page 168

$$\alpha = \frac{\kappa M}{4\pi}, \tag{70a}$$

double the value of the original formula (70a) quoted in Subsection 5.3. Consequently

$$B = \frac{2\alpha}{\Delta} = \frac{\kappa M}{2\pi\Delta}, \tag{74}$$

twice the quantity in equation (74) quoted above in Subsection 5.2. So the deflection is twice the value from 1911, and the numerical value is just as stated by Einstein in (1916a, 1916b).

But there is no remark in this book that a change has been made in the translation.

In Volume 6 of Einstein's collected papers, edited by Kox et al. (1996), formula (74) is quoted as in Section 5.2, i.e., with the same value as in (1916a), and there is no mentioning of the doubling as given in formula (74) in the present section. Also, I have found nothing about this topic in Callahan (2001) or Pais (1982), neither in the

"Introduction to Volume 6" on pages xv–xxv in (Kox et al., Eds., 1996), nor in the notes [1]–[36] on pages 338–339 in that volume in connection with the 1916 paper.

We may conclude that those who wrote about the deflection knew about the error in the two publications in 1916—but they did not mention it, not even the editors of Einstein's collected papers.

Also Landau & Lifshitz (1975:309, formula (101.9)) give the same value for B as equation (74) quoted in the present section.

Einstein made two errors in his paper (1916c), which were both corrected in (1918: 154). The first-mentioned was explicitly corrected and said to have occurred "durch einen bedaurerlichen Rechenfehler," while the second was silently corrected; see Notes [8] and [9] in (Kox et al., Eds., 1996:357).

So the remarkable thing is not that Einstein made errors, but that the one in (1916a) went uncorrected for a long time.

7. Einstein's field equations

Einstein presented his field equations first in the academy in Berlin on 1915 November 25, stating proudly in the short article published one week later, on December 02: "Damit ist entlich die allgemeine Relativitätstheorie als logisches Gebäude abgeschlossen." (1915:847). He mentions that he has explained Mercury's perihelial movement.

The field equations are nowadays usually written

$$G_{\mu,\nu} + \Lambda g_{\mu,\nu} = KT_{\mu,\nu},$$

where $G_{\mu,\nu}$ is the Einstein tensor, $g_{\mu,\nu}$ the metric tensor, K Einstein's gravitational constant, $T_{\mu,\nu}$ the stress-energy tensor, and Λ is the cosmological constant.

7.1. Criticism—and defense

John Earman & Clark Glymour criticized Einstein's derivations for being "problematic" from "a modern point of view." (1980:52). Forty years later, Gerard Gilmore and Gudrun Tausch-Pebody (2020) returned to this criticism and pointed out various errors in Earman's and Glymour's article, defending Einstein's reasoning.

7.2. The field equations in a spherically symmetric situation

Already in the same year as Einstein's publication (1916a), Karl Schwarzschild (1873–1916) and Johannes Droste (1886–1963) presented solutions to the field equations in the case of spherical symmetry (see Schwarzschild 1916 and Droste 1916). Droste claimed in a footnote on page 200 that he saw Schwarzschild's publication only after having communicated his calculations to the academy in Amsterdam.

Both authors published the same expression for the metric in this special case: formula (14) in (Schwarzschild 1916:194) and formula (7) in (Droste 1916:200).

7.2.1. Sternberg on Schwarzschild's solution

Let us denote by u the reciprocal of the distance between the sun and a particle in orbit, and by φ the angle at the sun between the particle and an arbitrarily chosen direction. Then, according to Sternberg (2012:131, formula (4.18)), Schwarzschild's solution satisfies[6]

$$u''(\varphi) + u(\varphi) = 3GM_\odot u(\varphi)^2, \tag{7.1}$$

where the derivative is with respect to φ, $u'' = \mathrm{d}^2u/\mathrm{d}\varphi^2$, and where the constant $3GM_\odot$ in the right-hand side is directly given by the value of K in formula (69) quoted above in Subsection 5.1, or possibly formula (5.1).

Is is clear that an orbit which is a straight line satisfies $u'' + u = 0$, and when $u'' + u$ is positive, the orbit is curved towards the sun. We can choose a straight line given by u_0 so that the difference $v = u - u_0$ tends to zero when time goes to minus infinity. For the speed of interest here, v is small and easy to estimate; see Sternberg (2012:131–132).

The curve given by (7.1) is not a conic section (for which $u''+u$ would be constant): close to the sun its curvature is larger than that of a hyperbola with the same deflection, while the curve is flatter than the hyperbola far from the sun. This can be compared with the perihelial movement of Mercury: the orbit is not an ellipse but the axes move a little, $43''$ per century, implying a complete revolution of the axes in three million earth years.

Equation (7.1) is nonlinear (with a small quadratic right-hand side), so it may be of help to compare it with linear equations. This we do in the following lemma.

Lemma 7.1. *Let us consider two differentional equations on* $\mathbf{R}$*,*

$$u_j'' + u_j = f_j, \qquad j = 0, 1,$$

with initial conditions $u(0) = 1/s$*,* $u'(0) = 0$*. If* $f_0 \leqslant f_1$ *it follows that*

$$u_0 \leqslant u_1 \quad \text{and that} \quad u_0'' - f_0 \geqslant u_1'' - f_1.$$

Proof. It is well known that

$$u_j(\varphi) = \frac{1}{s} + \int_0^\varphi (f_j(\varphi) - 1/s)G(\varphi - \psi)\mathrm{d}\psi, \qquad \varphi \geqslant 0,$$

where $G(\varphi) = 0$ for $\varphi \leqslant 0$ and $G(\varphi) = \sin\varphi$ for $x > 0$, thus with $G'' + G = \delta_0$ in the sense of distribution theory. Because u is an even function, it is enough to consider $\varphi \geqslant 0$. It follows that

$$u_j'(\varphi) = (f_j(\varphi) - 1/s)G(0) + \int_0^\varphi (f_j(\psi) - 1/s)G(\varphi - \psi)\mathrm{d}\psi = \int_0^\varphi (f_j(\psi) - 1/s)G(\varphi - \psi)\mathrm{d}\psi$$

and that

$$u_j''(\varphi) - f_j = -u_j = -\frac{1}{s} - \int_0^\varphi (f_j(\psi) - 1/2)G(\varphi - \psi)\mathrm{d}\psi.$$

This expression is decreasing in f_j as long as $G(\varphi - \psi)$ is nonnegative, thus for $0 \leqslant \varphi - \psi \leqslant \pi$.

We can study approximations of the solutions to equation (7.1).

[6] Here the left-hand side has dimension *length*$^{-1}$ while $3GM_\odot$ is of dimension *length*$^3\cdot$*time*$^{-2}$ (see Table 1 in Section 14). If we multiply by u^2c^{-2} we obtain the dimension *length*$^{-1}$ also in the right-hand side.

Theorem 7.2. *We shall always use the initial conditions $u(0) = 1/s > 0$ and $u'(0) = 0$. The quantity $3GM_\odot u(\varphi)^2$ varies between 0 and its maximum $3GM_\odot u(0)^2 = \theta/s$, where for brevity we put $\theta = 3GM_\odot$. So let us compare the solution u to (7.1) with the solutions u_0 and u_1 to the equations*

$$u_0'' + u_0 = 0 \text{ and } u_1'' + u_1 = \theta/s^2.$$

We assume that θ is small, $0 < \theta < \frac{1}{2}$, and define $\varepsilon = s/\theta - 1 > 1$.

For a suitable choice of direction, the solution of the first is

$$u_0(\varphi) = u(0)\cos\varphi, \qquad -\tfrac{1}{2}\pi \leqslant \varphi \leqslant \tfrac{1}{2}\pi,$$

representing a straight line; to the second

$$u_1(\varphi) = u(0)\frac{1+\varepsilon\cos\varphi}{1+\varepsilon}, \qquad -\tfrac{1}{2}\beta\pi \leqslant \varphi \leqslant \tfrac{1}{2}\beta\pi,$$

representing a hyperbola. Here $\beta > 1$ is defined by the condition $1 + \varepsilon\cos(\frac{1}{2}\beta\pi) = 0$. Then

$$u_0(\varphi) = u(0)\cos\varphi \leqslant u(\varphi) \leqslant u_1(\varphi) = u(0)\frac{1+\varepsilon\cos\varphi}{1+\varepsilon}, \qquad -\tfrac{1}{2}\beta\pi \leqslant \varphi \leqslant \tfrac{1}{2}\beta\pi,$$

and

$$-u(0)\cos\varphi \geqslant u''(\varphi) - f \geqslant -u(0)\frac{1+\varepsilon\cos\varphi}{1+\varepsilon}, \qquad -\tfrac{1}{2}\beta\pi \leqslant \varphi \leqslant \tfrac{1}{2}\beta\pi.$$

We can then go on, comparing u with the solutions u_3 and u_4 of the equations

$$u_3'' + u_3 = \theta u_0^2 \text{ and } u_4'' + u_4 = \theta u_1^2,$$

which will give a better approximation.

Proof. We just have to apply Lemma 7.1.

7.3. Gödel's solutions to the field equations

Kurt Gödel (1906–1978) produced an original solution to Einstein's field equations. There is one publication (1949) and also unpublished notes "almost ready for print," mentioned by Rebecca Goldstein in (2005:213, footnote 3).

Time is cyclic in his solution, implying that after some time we shall come back to the present. He writes:

> It is not possible to assign a time coordinate t to each space-time point in such a way that t always increases, [. . .] .
>
> (Gödel 1949), quoted from (Feferman, Editor-in-chief, 1990:191).

The time of return depends on the average density ρ of the universe:

> For the period of rotation one obtains $2 \cdot 10^{11}$ years, if for ρ the value of 10^{-30}g/cm^3 is substituted. (Gödel 1949), quoted from (Feferman, Editor-in-chief, 1990:197).

This is more than fourteen times longer than 13.8 billion years, a figure often given as the age of the universe.

Einstein confirmed the validity of Gödel's solution. He published a prudent suggestion: "It will be interesting to weigh whether these are not to be excluded on physical grounds." (Einstein 1970:688).

In a second publication (1952) Gödel presented a solution where time is not cyclical.

Several textbooks that I have checked do not mention any of Gödel's solutions to Einstein's field equations.

8. Delay in the arrival of light

While the Newtonian and the Einsteinian effects on the deflection of light are said to add to each other, the predictions concerning retardation or acceleration go in opposite directions: By Newtonian gravitation, light moves faster when passing the sun, while relativity predicts that light will be delayed (Eisenhart 1923:516; Wright 2004). Importantly, this delay can be measured more easily than the deflection, since it can be done when there is a spacecraft or planet behind the sun instead of a star—a total solar eclipse is not needed.

According to Wright's web site, Irwin Shapiro (Shapiro et al. 1977) and Bertotti et al. (2003) have confirmed the predicted delay to a very high degree of accuracy:

> Measurements of the round-trip time of flight of radio signals transmitted from the earth to the Viking spacecraft are being analyzed to test the predictions of Einstein's theory of general relativity. According to this theory the signals will be delayed by up to $\sim 250\,\mu$s owing to the direct effect of solar gravitation on the propagation. A very preliminary qualitative analysis of the Viking data obtained near the 1976 superior conjunction of Mars indicates agreement with the predictions to within the estimated uncertainty of 0.5%. (Shapiro et al., 1977: Abstract)

Misner et al. (2002:1108) report that in radar time delay measurements performed in 1971, the value of the Parametrized Post-Newtonian (PPN) parameter γ was found to be $\gamma = 1.02 \pm 0.08$, thus in agreement with the (more precise) value given in the following quotation.

> According to general relativity, photons are deflected and delayed by the curvature of space-time produced by any mass [...] The bending and delay are proportional to $\gamma + 1$, where the parameter γ is unity in general relativity but zero in the newtonian model of gravitation. The quantity $\gamma - 1$ measures the degree to which gravitation is not a purely geometric effect and is affected by other fields; such fields may have strongly influenced the early Universe, but would have now weakened so as to produce tiny—but still detectable—effects. Several experiments have confirmed to an accuracy of $\sim 0.1\%$ the predictions for the deflection [...] and delay [...] of photons produced by the Sun. Here we report a measurement of the frequency shift of radio photons to and from the Cassini spacecraft as they passed near the Sun. Our result, $\gamma = 1 + (2.1 \pm 2.3) \times 10^{-5}$, agrees with the predictions of standard general relativity with a sensitivity that approaches the level at which, theoretically, deviations are expected in some cosmological models [...]. (Bertotto et al., 2003: Abstract)

9. Refraction

A light ray is also subject to refraction in the sun's atmosphere. Xu (2002) states that it amounts to $26''$ when the ray is grazing the solar limb, thus much more than the other deviations, so it can actually completely hide the other effects. Per Carlqvist (personal communication 2020 May 08) points out that the refraction varies both with time and position—there are rapid changes and also large inhomogenities in the sun's atmosphere.

However, the refraction tapers off very quickly when the distance to the sun increases: it is down to $0''.4$ when the distance is 0.7 % larger than the sun's radius (Xu

2002:247), i.e., when the distance is $6''.7$ from the sun's limb—at this distance, the decrease given by formula (5.2) is just down from $1''.75$ to $1''.75/1.007 \approx 1''.74$.

Dainis Dravins writes (personal communication 2020 May 22) that it may be possible to measure the refraction: during the course of a year, the sun passes over several bright stars hotter than the sun, and therefore brighter than the sun in ultraviolet light. The refraction for these stars can then be measured.

10. The solar eclipse of 1919 May 29

During the solar eclipse on 1919 May 29, astronomers measured the deflection for stars close to the sun. Arthur Stanley Eddington (1882–1944) reported in his book (1920:118) that the deflection at Príncipe[7] was $1''.61$, that at Sobral[8] was reported as $1''.98$, thus in agreement with the predicted amount of $1''.75$. The two expeditions are described by Eddington (1920), Lemos (2019) and Lemos et al. (2019); in great detail in the long paper by Lemos as sole author. See also Earman & Glymour (1980:73–76) and Gilmore & Tausch-Pebody (2020:5–8).

Soffel (1989:9) presents a table with observations from nine solar eclipses during the years 1919–1973 and reporting the deflection of up to a maximum of 150 stars. All in all, these observations are compatible with Einstein's prediction, although they are rather spread out.

10.1. Criticism of the results—and defense

Eddington's results have been criticized, notably by Earman and Glymour (1980), who write that Eddington's book (1921) was "not entirely rigorous" (1980:56). They go on:

> Now the eclipse expeditions confirmed the theory only if part of the observations were thrown out and the discrepancies in the remainder ignored; Dyson[9] and Eddington, who presented the results to the scientific world, threw out a good part of the data and ignored the discrepancies. (Earman & Glymour 1980:95)

In a recent paper, Gilmore and Tausch-Pebody (2020) present a detailed analysis of the various interpretations made of the measurements from 1919, and of the criticism that has appeared since then, in particular that by Earman & Glymour. They write:

> [...] while the original analysis was statistically and methodologically robust, the 1980 re-analysis was not. (Gilmore & Tausch-Pebody 2020:4)

> Our research shows that these strong claims are based entirely on methological error. Earman and Glymour failed to understand the difference between the dispersion of a set of measurements and an uncertainty, random plus systematic, on the value of the parameter being measured. They speculated but did not calculate, and their conclusions are not supported by evidence. (Gilmore & Tausch-Pebody 2020:13)

[7] An island in the Gulf of Guinea at 1°37′N, 7°24′E. At the time this island was Portuguese territory; now it belongs to the Democratic Republic of São Tomé and Príncipe.

[8] In Ceará, Brazil, at 3°40′S, 40°14′W.

[9] Sir Frank Watson Dyson, Astronomer Royal, (1868–1939).

10.2. Very Long Baseline Interferometry

Using Very Long Baseline Interferometry (VLBI), it has been possible to measure the deflection with high accuracy; see the quotation from Bertotti et al. in Section 8. Among the many publications on VLBI, see also those by Robertson et al. (1991), Shapiro et al. (2004), and Titov et al. (2018). They all confirm Einstein's prediction, the results showing that the parameter γ mentioned in footnote 4 is very close to 1.

Formula (5.2) for the deflection yields a value of $1''.75$ (see Section 14), thus without taking into account the refraction in the sun's atmosphere. Maybe this refraction is so small that it is unimportant—see Sections 8 and 9 above.

11. Curvature of conic sections

A circle has the same curvature at every point; a parabola has a single point (the vertex) where the curvature is maximal, and all other conic sections possess two points where the curvature is maximal.

Consider an ellipse with the usual equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1. \tag{11.1}$$

Its semi-axes are $a, b > 0$, and its excentricity is

$$\varepsilon = \sqrt{1 - \frac{(a \wedge b)^2}{(a \vee b)^2}},$$

which is equal to $\sqrt{1 - b^2/a^2}$ if we assume that $b \leqslant a$, which we shall do from now on. We have $0 \leqslant \varepsilon < 1$.

Example 11.1. For the earth's orbit with excentricity equal to 0.0167, the quotient between the semi-axes is $b/a = 0.99986$; for Mercury's orbit with excentricity 0.2056, $b/a = 0.97864$.

The foci are located at $(\pm\varepsilon a, 0)$. The smallest radius of curvature is obtained at the vertices $(\pm a, 0)$ and is $\rho = b^2/a = a(1 - \varepsilon^2)$. The center of curvature to the right is at $\gamma = (a - b^2/a, 0) = (\varepsilon^2 a, 0)$. We thus have $0 \leqslant \varepsilon^2 a \leqslant \varepsilon a < a$, defining three different points on the x-axis if the excentricity is positive.

A hyperbola with the usual equation

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1 \tag{11.2}$$

has semi-axes $a, b > 0$, and its excentricity is $\varepsilon = \sqrt{1 + b^2/a^2} > 1$. Here we can have $b \geqslant a$ as well as $b \leqslant a$. It intersects the x-axis (the transverse axis) in $(a, 0)$ and $(-a, 0)$ but not the y-axis (the conjugate axis). The foci are given by the same formula as for the ellipse; they are $(\pm\varepsilon a, 0)$, and also here the smallest radius of curvature is obtained at the vertices $(\pm a, 0)$ and is $\rho = b^2/a = a(\varepsilon^2 - 1)$. The center of curvature of the left branch is at $\gamma = (-a - b^2/a, 0) = (-\varepsilon^2 a, 0)$. So we have three different points with abscissae $-\varepsilon^2 a < -\varepsilon a < -a < 0$.

Between the two curves mentioned we have the parabola, of equation $x = s - \mu y^2$ for a positive parameter μ, whose focus is at $(s - (4\mu)^{-1}, 0)$, with radius of curvature $\rho = (2\mu)^{-1}$ and center of curvature at $\gamma = (s - (2\mu)^{-1}, 0)$.

Lemma 11.2. *To determine the radius of curvature ρ at a point (p,q) on a curve given by $x = f(y)$, $y \in \mathbf{R}$, it is convenient to search for the best-fitting parabola of equation $x = p + A(y-q) + \frac{1}{2}B(y-q)^2$, where $A \in \mathbf{R}$ and $B < 0$ are constants. It follows that $f(p) = p$, $f'(p) = A$, and that $f''(p) = B$. We find that the radius of curvature ρ and the curvature $\kappa = 1/\rho$ are, respectively*

$$\rho = -B^{-1}(1+A^2)^{3/2} \text{ and } \kappa = -B(1+A^2)^{-3/2},$$

and that the center of curvature is

$$\gamma = (\gamma_1, \gamma_2) = (p - \rho(1+A^2)^{-1/2}, q + \rho A(1+A^2)^{-1/2}).$$

Proof. The osculating circle at (p,q) has the equation

$$(x-\gamma_1)^2 + (y-\gamma_2)^2 = \rho^2,$$

from which we deduce that, near (p,q) where x is a function of y: $x = g(y)$, we have

$$2(g(y)-\gamma_1)g'(y) + 2(y-\gamma_2) = 0 \text{ and } 2g'(y)^2 + 2(g(y)-\gamma_1)g''(y) + 2 = 0.$$

We impose the conditions $g(q) = f(q) = p$, $g'(q) = f'(q) = A$ and $g''(q) = f''(q) = B$ and can then determine γ and ρ.

In the following lemma we choose coordinates so that the sun is always at the origin and the ideal comet at the point $(s,0)$ at time $t = 0$. The original question concerns actually only case 5, but it is not so much more work to include all conic sections.

Proposition 11.3. *The semi-axes a, b, the radius of curvature ρ and the center of curvature γ are the following for the conic sections considered.*

1. *For the ellipse of equation*

   (11.3) $$\left(\frac{x+a-s}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1,$$

   we have

   $$a = \frac{s}{1+\varepsilon}, \quad b = s\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}, \quad \rho = s(1-\varepsilon), \quad \gamma = (s\varepsilon, 0).$$

   The origin is at the focus to the left.

2. *For the circle of equation*

   (11.4) $$x^2 + y^2 = s^2,$$

   we have

   $$\varepsilon = 0, \quad a = b = \rho = s, \quad \gamma = (0,0).$$

3. *For the ellipse of equation*

   (11.5) $$\left(\frac{x+a-s}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1,$$

   we have

   $$a = \frac{s}{1-\varepsilon}, \quad b = s\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}, \quad \rho = s(1+\varepsilon), \quad \gamma = (-s\varepsilon, 0).$$

   The origin is now at the focus to the right.

4. *For the parabola of equation*

$$x = s - \frac{1}{4s}y^2, \tag{11.6}$$

the excentricity is $\varepsilon = 1$*, the focus is at the origin, and* $\rho = 2s$*,* $\gamma = (-s, 0)$*.*

5. *Finally, for the hyperbola of equation*

$$\left(\frac{x-a-s}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1, \tag{11.7}$$

we have

$$a = \frac{s}{\varepsilon - 1}, \quad b = s\sqrt{\frac{\varepsilon+1}{\varepsilon-1}}, \quad \rho = s(1+\varepsilon), \quad \gamma = (-s\varepsilon, 0),$$

thus with ρ *and* γ *as in case 3. The origin is at the focus to the left.*

We note that

$$\frac{b}{a} = \sqrt{\varepsilon^2 - 1} = \frac{\rho}{s}\sqrt{1 - 2\frac{s}{\rho}}.$$

Proof. We try $x = s + \frac{1}{2}By^2$ in the first case, and find the equation

$$\frac{B^2y^4}{4a^2} + \frac{By^2}{a} + \frac{y^2}{b^2} = 0.$$

Here the term of order four in y is small when y is small, which, if we neglect it, yields the result that $B = -a/b^2$. This determines the best-fitting parabola, which, in view of Proposition 11.2 is as indicated and yielding $\rho = -1/B = b^2/a$.

Similarly in all the other cases.

Lemma 11.4. *The best-fitting parabola at a point* (p, q) *on the left branch of the hyperbola of equation* (11.7) *has the equation* $x = p + A(y-q) + \frac{1}{2}B(y-q)^2$*, where*

$$A = \frac{a^2}{b^2}\frac{q}{p-a-s} = -\frac{aq}{b^2}(1+q^2b^{-2})^{-1/2} \text{ and}$$
$$B = \frac{a^4}{b^2}\frac{1}{(p-a-s)^3} = -\frac{a}{b^2}(1+q^2b^{-2})^{-3/2}.$$

It follows that the curvature is

$$\kappa = -B(1+A^2)^{-3/2} = ab^{-2}(1+q^2b^{-2})^{-3/2} + a^3b^{-6}q^2(1+q^2b^{-2})^{-5/2}.$$

Proof. We apply Lemma 11.2 to the equation (11.7).

As $q \to +\infty$ on the left branch of the hyperbola (and consequently $p \to -\infty$), we see that $A \to -a/b$, $q^3B \to -ab$, and that the curvature κ behaves like a constant times q^{-3}, more precisely like $abq^{-3}(1+a^2b^{-2})^{-3/2} = abq^{-3}(1-\varepsilon^{-2})^{3/2}$.

12. Orbits according to Newton

We now compare the results of Lemma 11.3 with the orbit an ideal comet takes according to Newtonian gravitation under various initial conditions.

A particle of positive mass which passes at time $t = 0$ the point $(s, 0)$ and with velocity vector $(0, v)$ follows an orbit which is one of the conic sections. The parabola of equation $x = s - \mu y^2$ which best fits the orbit is that with

$$\mu = \frac{1}{2s}\left(\frac{v_{\text{circ}}(s)}{v}\right)^2.$$

For a circular orbit, thus when $v = v_{\text{circ}}(s)$, we get $\mu = 1/(2s)$.

Theorem 12.1. *For orbits along a conic section, the excentricity is given by the formula*

$$\varepsilon = \left|1 - \frac{v^2}{v_{\text{circ}}(s)^2}\right|.$$

Conversely, the relative speed $v/v_{\text{circ}}(s) > 0$ is determined from the excentricity if $1 \leqslant \varepsilon$ by the formula

$$\frac{v}{v_{\text{circ}}(s)} = \sqrt{1+\varepsilon},$$

and with two possible values $\sqrt{1-\varepsilon}$ and $\sqrt{1+\varepsilon}$ if $0 < \varepsilon < 1$.

Let the sun be placed at $(0, 0)$ and let our ideal comet pass at time $t = 0$ the point $(s, 0)$ with velocity vector $(0, v)$.

1. *If $0 < v < v_{\text{circ}}(s)$, then the orbit is the ellipse* (11.3).
2. *If $v = v_{\text{circ}}(s)$, then the orbit is a circle of equation* (11.4).
3. *If $v_{\text{circ}}(s) < v < \sqrt{2}\, v_{\text{circ}}(s)$, the orbit is an ellipse of equation* (11.5)*, with the sun in the focus closest to the particle.*
4. *If $v = \sqrt{2}\, v_{\text{circ}}(s)$, then the orbit is a parabola of equation* (11.6).
5. *If $v > \sqrt{2}\, v_{\text{circ}}(s)$, finally, then the orbit is the leftmost branch of the hyperbola of equation* (11.7).

   The asymptotic lines have the equations $y = \pm b a^{-1}(x - a - s)$, and the angle between the incoming asymptotic line in the lower half plane and the outgoing one in the upper half plane is
   (12.1)
   $$\sigma(r, v) = 2\arctan(a/b) = 2\arctan\left(\frac{1}{\sqrt{\varepsilon^2 - 1}}\right) = 2\arctan\left(\frac{s/\rho}{\sqrt{1 - 2s/\rho}}\right).$$

   When a/b is small, a good approximation is

   $$\sigma(r, v) \approx \frac{2}{\varepsilon} \approx 2\left(\frac{v_{\text{circ}}(s)}{v}\right)^2 \approx \frac{2s}{\rho}. \qquad (12.2)$$

Example 12.2. For the earth, with orbit excentricity 0.0167, we have at aphelium in early July case 1 with $v/v_{\text{circ}}(R_\oplus) = \sqrt{1-\varepsilon} \approx 0.9916$. At perihelium in early January, we have case 3 with $v/v_{\text{circ}}(R_\oplus) = \sqrt{1+\varepsilon} \approx 1.0083$.

For Mercury, with orbit excentricty equal to 0.2056, the two relative speeds are $\sqrt{1-\varepsilon} = 0.8913$ and $\sqrt{1+\varepsilon} = 1.0980$.

13. Orbits in polar coordinates

It is sometimes convenient to describe orbits using polar coordinates with the sun at the origin. This is closely related to the description of conic sections using a directrix.

Proposition 13.1. *The orbits studied are the following in polar coordinates. We denote by $u(\varphi)$ one over the distance from the sun to the particle, where φ is the angle at the sun between the radius vector and a conveniently chosen direction.*

1. *For speeds v satisfying $0 < v \leqslant v_{\mathrm{circ}}(s)$, thus with excentricities*

$$\varepsilon = 1 - \left(\frac{v}{v_{\mathrm{circ}}(s)}\right)^2, \qquad 1 > \varepsilon \geqslant 0,$$

we have

$$u(\varphi) = \frac{1 - \varepsilon \cos\varphi}{(1-\varepsilon)s} \text{ and } u'' + u = \frac{1}{(1-\varepsilon)s}.$$

Here φ is the angle between the radius vector and the major axis—any direction if $v = v_{\mathrm{circ}}(s)$.

2. *For speeds v satisfying $v_{\mathrm{circ}}(s) \leqslant v < +\infty$, thus with excentricities*

$$\varepsilon = \left(\frac{v}{v_{\mathrm{circ}}(s)}\right)^2 - 1, \qquad 0 \leqslant \varepsilon < +\infty,$$

we have

$$u(\varphi) = \frac{1 + \varepsilon \cos\varphi}{(\varepsilon+1)s} \text{ and } u'' + u = \frac{1}{(\varepsilon+1)s}.$$

Here φ is the angle between the radius vector and the major axis if $0 < \varepsilon < 1$; the axis of the parabola if $\varepsilon = 1$; the transverse axis of the hyperbola if $\varepsilon > 1$.

3. *For a straight line with $s > 0$ we have*

$$u(\varphi) = \frac{\cos\varphi}{s} \text{ and } u'' + u = 0,$$

corresponding to an infinite speed and an infinite excentricity. Here φ is the angle between the radius vector and the normal to the straight line.

Thus in all cases $u'' + u$ is a constant depending on the excentricity ε and the closest distance s to the sun.

Proof. For $v = v_{\mathrm{circ}}(s)$ we have a circle and the result is obvious. For all other cases, the conic section is given by a directrix: u is an affine function of $\cos\varphi$, which implies the result.

14. Astronomical data for the sun and the earth

The International Astronomical Union (2015) recommends so called nominal values of certain solar and planetary conversion constants. Of these the following are of interest here; the superscript $^{\mathrm{N}}$ stands for *nominal.*

Table 1. Nominal values of some conversion constants.

Quantity	*Symbol*	*Nominal value*
The nominal solar radius	$R_\odot^{\mathrm{N}}$	$6.957 \cdot 10^8\,\mathrm{m}$
The universal gravitational constant times the mass of the sun	$(GM)_\odot^{\mathrm{N}}$	$1.3271244 \cdot 10^{20}\,\mathrm{m^3 s^{-2}}$
The equatorial radius of the earth	$R_{e\mathrm{E}}^{\mathrm{N}}$	$6.3781 \cdot 10^6\,\mathrm{m}$
The polar radius of the earth	$R_{p\mathrm{E}}^{\mathrm{N}}$	$6.3568 \cdot 10^6\,\mathrm{m}$

Let us now insert numerical values in the formulas for the deflection.

In the formula (5.2) for the deflection $\theta(r,c)$, the value of the universal gravitational constant is equal to[10]

$$G = 6.67430\,(\pm 0.00015) \cdot 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}.$$

The mass of the sun is equal to

$$M_\odot = 1.989 \cdot 10^{30}\,\mathrm{kg}.$$

For the product of G and $M_\odot$, see Table 1.

The shortest distance s from the sun to a comet or a light ray is at least equal to the sun's radius $R_\odot^{\mathrm{N}}$ given in Table 1.

Finally c is the speed of light in vacuum, which—by the definition of the meter in terms of the second introduced in 1983—is equal to

$$c = v_{\mathrm{light}} = 2.9979245 \cdot 10^8\,\mathrm{m\,s^{-1}}.$$

When the ray passes as closely as possible to the sun, thus with $r = R_\odot$, the deflection given by (5.2) is

$$\theta(R_\odot, c) = \frac{4 \cdot 6.67430 \cdot 10^{-11} \cdot 1.989 \cdot 10^{30}}{6.957 \cdot 10^8 \cdot (2.9979245 \cdot 10^8)^2} \approx 8.4925 \cdot 10^{-6}\,\text{radians} \approx 1''.75.$$

A sideral year is

$$p(R_\oplus) = 365.256363004 \cdot 8.64 \cdot 10^4\,\mathrm{s} \approx 3.1558 \cdot 10^7\,\mathrm{s}.$$

The astronomical unit, 1 au, originally defined to be the mean distance between the sun and the earth, is now a conventional unit of length defined by the International Astronomical Union (2012) to be exactly equal to

$$1\,\mathrm{au} = R_\oplus = 1.495978707 \cdot 10^{11}\,\mathrm{m}.$$

The speed of the earth around the sun, assuming a circular orbit, is

$$v_{\mathrm{circ}}(R_\oplus) = \frac{2\pi R_\oplus}{p(R_\oplus)} = \frac{2\pi \cdot 1.495978707 \cdot 10^{11}\,\mathrm{m}}{3.155814976 \cdot 10^7\,\mathrm{s}} \approx 2.9784 \cdot 10^4\,\mathrm{m\,s^{-1}}.$$

[10] This is the 2018 CODATA value, which is larger than the earlier 2014 CODATA value $G = 6.67408\,(\pm 0.00031) \cdot 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$.

We conclude that the factor $c/v_{\mathrm{circ}}(R_\oplus)$ is about $1.00653 \cdot 10^4$; its square—which is very close to the excentricity of the hyperbola—is about $1.01310 \cdot 10^8$.

The deflection of von Soldner's angle is inversely proportional to the distance r:

$$\sigma(r,c) = \frac{R_\odot}{r}\sigma(R_\odot, c), \tag{14.1}$$

just like for $\theta(r,c)$ in (5.3).

The nominal radius of the sun is given in Table 1. It follows that an astronomical unit is a factor $q = 215.03$ larger than the former, implying a factor of

$$q^{3/2} = \frac{p(R_\oplus)}{p(R_\odot)} \approx 3153.226103$$

in the periods of a particle at one astronomical unit from the sun compared with a particle which passes at the sun's surface, thus when $r = R_\odot$. The quotient for the speeds is $\sqrt{q} \approx 14.6640$, thus $v_{\mathrm{circ}}(R_\odot)/v_{\mathrm{circ}}(R_\oplus) \approx 14.6640$.

The speed v which gives a circular orbit for a particle passing at the distance $R_\odot$ is thus equal to

$$v_{\mathrm{circ}}(R_\odot) = \sqrt{q} \approx 14.6640 \cdot v_{\mathrm{circ}}(R_\oplus),$$

where $v_{\mathrm{circ}}(R_\oplus)$ is the speed of the earth in its orbit around the sun.

The period at this radius $R_\odot$ is equal to

$$p(R_\odot) = \frac{3.155814976 \cdot 10^7\,\mathrm{s}}{q^{3/2}} \approx \frac{3.15581497 \cdot 10^7\,\mathrm{s}}{3153.226103} \approx 1.00082 \cdot 10^4\,\mathrm{s},$$

about 2 hours and 47 minutes.

In view of (3.1),

$$\frac{v_{\mathrm{circ}}(R_\odot)}{v_{\mathrm{circ}}(R_\oplus)} = \sqrt{q} \approx 14.66340 \text{ and } \frac{c}{v_{\mathrm{circ}}(R_\oplus)} \approx 1.00653 \cdot 10^4,$$

resulting in the quotient $v_{\mathrm{circ}}(R_\odot)/c$ being

$$\frac{v_{\mathrm{circ}}(R_\odot)}{c} = \frac{v_{\mathrm{circ}}(R_\odot)}{v_{\mathrm{circ}}(R_\oplus)} \cdot \frac{v_{\mathrm{circ}}(R_\oplus)}{c} \approx 1.45688 \cdot 10^{-3},$$

and

$$\sigma(R_\odot, c) \approx 2\frac{v(R_\odot)^2}{c^2} \approx 4.24502 \cdot 10^{-6}\,\text{radians}, \tag{14.2}$$

which is

$$\frac{4.24502 \cdot 10^{-6} \cdot 180 \cdot 60 \cdot 60}{\pi} \approx 0.876 \text{ arc seconds}$$

The value $\sigma(R_\odot, c) = 0''.876$ is in good agreement with the angle Shapiro & Shapiro reported on von Soldner's result (see Section 3).

With the values given here, the right-hand side in (7.1) is equal to

$$3(GM_\odot)^{\mathrm{N}} u^2 \approx 3.9814\, u^2 \cdot 10^{20}\,\mathrm{m\,s^{-2}}.$$

The resulting deflection is $1''.75$; see (Sternberg 2012:132).

15. Deflection due to Newtonian gravitation

The force of gravitation between two bodies of masses M and m and the distance s from each other is $F = GMm/r^2$. The Euclidean norm α of the acceleration vector caused by this force is

$$\alpha = \frac{F}{m} = \frac{GMm}{mr^2}.$$

In the last expression, the m in the numerator is the gravitational mass, the m in the denominator is the inertial mass. It is agreed that these are equal,[11] which means that the acceleration is equal to GM/r^2 and independent of m as long as it is positive. But if $m = 0$, there is no force and no acceleration.

This means that if we assign a positive mass, however small, to the photon, say 10^{-59} kg or 10^{-200} kg or whatever, it will be subject to gravitational forces.[12]

The parabola which fits best to the orbit at $(s, 0)$ is $x = s - \frac{1}{2}\alpha y^2$.

This implies that

$$v(s)^2 = \rho\alpha = \frac{\rho (GM)^{\mathrm{N}}_{\odot}}{s^2},$$

so for the earth we have $\rho = R_{\oplus}$ and

$$v_{\mathrm{circ}}(R_{\oplus})^2 = \frac{(GM)^{\mathrm{N}}_{\odot}}{R_{\oplus}}.$$

For an ideal comet with hyperbolic orbit we get

$$v_{\mathrm{circ}}(R_{\odot})^2 = \frac{(GM)^{\mathrm{N}}_{\odot}}{R_{\odot}} \approx 1.90761 \cdot 10^{11}\,\mathrm{m}^2\,\mathrm{s}^{-2},$$

in particular with $v = c$,

$$\sigma(R_{\odot}, c) \approx 2\frac{v(R_{\odot})^2}{c^2} \approx 4.2450 \cdot 10^{-6}\,\text{radians},$$

which in arc seconds is

$$\sigma(R_{\odot}, c) \approx \frac{4.2450 \cdot 10^{-6} \cdot 180 \cdot 60 \cdot 60}{\pi} \approx 0''.876.$$

This is also the result Einstein found in (1911:908, 1973a:139). A key to the understanding of the fact that he lets a photon be attracted is the following statement.

> Der Zuwachs an *schwerer* Masse ist also gelich E/c^2, also gleich dem aus der Relativitätstheorie sich ergebenden Zuwachs and *träger* Masse. (Einstein 1911:903)
>
> The increase in gravitational mass is thus equal to E/c^2, and therefore equal to the increase in inertial mass as given by the theory of relativity. (Einstein 1973a:134)

Jan Boman (personal communication of 2020 June 23) found a very rapid way to deduce formula (4.1) on page 3 by calculating the acceleration of a particle as done above.

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[11] Einstein said in a lecture in London in 1921: "The general theory of relativity owes its existence ... to the empirical equality of the inertial and gravitational masses of bodies ..." (quoted from Schwinger 1986:238). See, however, the quotation from (Weber 1964) on page 5.

[12] The question of the mass of the photon seems to be similar to the question of a possible nonzero mass of the graviton—Göran Henriksson (2017, 2020 MS) finds this mass to be $1.306\,(\pm 0.009) \cdot 10^{-56}$ g.

16. Summing up

16.1. Three phenomena

Starting with considerations in the year 1801, we have studied three possible ways a light ray might be influenced by the sun: Newtonian gravitation, assuming that a photon has a positive mass; curvature of spacetime in accordance with Einstein's general theory of relativity; refraction in the sun's atmosphere.

16.2. Newtonian gravitational attraction

Gravitation according to Newton's celestial mechanics can act only if the photon is assigned a positive mass—von Soldner and Einstein did so.

Gravitation would accelerate light, whereas this has not been confirmed by mea- surements. On the contrary, observations have confirmed with high precision that there is a delay, in complete agreement with the general theory of relativity.

So we have two arguments against gravitation, the first being questioned (since the photon is claimed to have mass zero), and the second, based on observations, being strong.

16.3. Variation with the distance from the sun

We have seen that deflection varies with the angular distance to the sun in the same way for Einstein's formula (5.3) for the deflection due to the sun's mass and von Soldner's formula (4.1). Both yield deflections which are inversely proportional to the distance to the sun.

16.4. Variation with speed

Also both formulas (4.1) and (5.2) give a deflection which is inversely proportional to the square of the speed—in the first case proportional to c^{-2} (allowing for imagined—or perhaps real—different speeds of light); in von Soldner's case proportional to $v(s)^{-2}$, where $v(s)$ is the speed of the ideal comet when it is closest to the sun; this distance being s.

16.5. Numerical results for the deflection due to gravitation

The Newtonian gravitation results in an angle of 0″.9 or 0″.84 according to von Soldner (1801); of 0″.83 according to Einstein (1911:908); of 0″.876 as calculated using two different methods in Sections 14 and 15.

The agreement is good, which is not surprising, since the calculations are based on well-established Newtonian celestial mechanics and, in the case of Section 14, on the geometry of hyperbolas. It also agrees well with Poor's calculation (1927:231) using wave fronts, yielding 0″.83.

16.6. Refraction in the sun's atmoshere

As to refraction, Xu's results gives the value of 26″, which does not agreee with reported observations. However, for $r/R_{\odot}^{\mathrm{N}}$ just a little larger than 1 it is insignificant. So it may well be that no observation has been made sufficiently close to the limb as to show such a large refraction.

16.7. A photon at rest and moving with the speed of light

Johann Georg von Soldner and Albert Einstein assert that Newtonian gravitation causes a deflection of light rays passing close to the sun, presupposing a positive mass of the photon.

A solution might be that the photon has mass zero at rest—it is never at rest—but that it can have a nonzero mass when moving with the speed of light—which it always does.

17. Questions

During the writing I have come across a couple of questions. Some of them I could answer in one way or another. But some of them remain. I do hope you can help me.

17.1. An error in Einstein's most well-known paper

The error I found in Einstein's papers (1916a) and (1916b) was silently corrected in the translation that appeared in 1973.

Question 17.1. *Is there any publication with the correction earlier than 1973?*

In Volume 6 of Einstein's collected papers (1996:334), the deflection is given as in the original paper from 1916, viz. formula (70a) in Subsection 5.3, thus without the correction published in 1973.

17.2. Newtonian attraction vs. the general theory of relativity

There are two phenomena of a rather different nature that affect light rays passing close to the sun: the deflection of the orbit from a straight line and the delay in arrival of reflected light.

There are two theories that apply to these situations: Newton's theory of gravitation and Einstein's general theory of relativity. The first gives good results in the absence of heavy celestial bodies and for particles with small speeds, but for particles with a high speed or passing close to the sun, it is not accurate.

These two theories are totally different, one could say even disjoint, and should not be mixed. The deflection of a light ray is caused by the curvature of spacetime—or so I imagined.

For light rays passing close to the sun, Einstein predicted in 1916 a deflection based, as he wrote, on both Newtonian gravitation and the geometry of spacetime, each contributing half of the deflection. The amount has been confirmed by measurements during several solar eclipses, and with high precision for radio sources using Very Long Baseline Interferometry (VLBI).

Why did he mention Newtonian gravitation here? He states the same in his popular book (1917:87), reprinted in (Kox et al., Eds., 1996:511). Is it just to try to be under- standable in this popular presentation? If so, it creates problems in the understanding of the curvature of spacetime. On the other hand, observations confirm a delay in the arrival of light passing close to the sun, in accordance with the general theory of relativity, and contradicting the hypothesis of Newtonian attraction, which instead would speed up particles passing the sun.

Question 17.2. *Newtonian attraction is said to provide an indispensable contribution to deflection—for example by Albert Einstein himself* (1917:87) *and* (1920, Appendix III, (b), page 153) *and by Julian Schwinger* (1986:209) *quoted above—but cannot be allowed in the calculation of the delay in arrival.*

Is there an explanation?

18. Conclusion

Einstein's general theory of relativity has been confirmed by observations during total solar eclipses of the deflection of light rays passing close to the sun, as well as—with high precision—by measurements of the delay in the arrival of reflected light and by using Very Long Baseline Interferometry.

An intriguing contradiction has been discovered: For predictions of the deflection of light it is stated that Newtonian gravitation is an indispensable component to be combined with the general theory of relativity, but that, on the contrary, consideration of Newtonian attraction is inadmissible when it comes to predicting the delay in the arrival of reflected light.

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