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Partial Algebraic Systems of type $(t_n,(n))$

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Abstract

In this paper, we define the set $(CF_{(t_n,(n))}(X_n))^{\underline{A}^s}$ of all n-ary C-formulas on the partial algebraic system $\underline{A}^s = (A;(f_i^A)_{i\in I},r^A)$ of type $(t_n,(n))$ and define the operation $R^{n,A}$ on the set $(W_{t_n}^C(X_n))^{\underline{A}^s} \cup (CF_{(t_n,(n))}(X_n))^{\underline{A}^s}$. After this definition we have a unitary Menger algebra $((W_{t_n}^C(X_n))^{\underline{A}^s} \cup (CF_{(t_n,(n))}(X_n))^{\underline{A}^s};R^{n,A},x_1^{\underline{A}^s},\ldots,x_n^{\underline{A}^s})$ of rank n. Finally, we show that the set of all C-hypersubstitutions for an algebraic system of the type $(t_n,(n))$ with a binary operation on this set and the identity element forms a monoid.

Keywords: term, superposition, unitary Menger algebra of rank n.

AMS Subject Classification : 08A40, 08A55, 08A70.

1. Introduction

Let $P^n(A) := \{f^A : A^n \multimap A\}$ be the set of all n -ary partial operations defined on the non-empty set A and let $P(A) := \bigcup_{n=1}^{\infty} P^n(A)$ be the set of all partial operations on A. A partial algebra $\underline{A} = (A;(f_i^A)_{i \in I})$ of type $\tau = (n_i)_{i \in I}$ is a pair consisting of a set A and an indexed set $(f_i^A)_{i \in I}$ of partial operations where f_i^A is n_i -ary. Let $PAlg(\tau)$ be the class of all partial algebras of type τ.

For $n \in \mathbb{N}^+ := \mathbb{N} \setminus \{0\}$ we define the superposition operation $S^{n,A} : (P^n(A))^{n+1} \to P^n(A)$

$$S^{n,A}(f^A, g_1^A, \ldots, g_n^A)(a_1, \ldots, a_n) := f^A(g_1^A(a_1, \ldots, a_n), \ldots, g_n^A(a_1, \ldots, a_n)).$$

Here $(a_1, \ldots, a_n) \in domS^{n,A}(f^A, g_1^A, \ldots, g_n^A)$ iff $(a_1, \ldots, a_n) \in \bigcap_{j=1}^{n} domg_j^A$ and for all $j = 1, \ldots, n$, $g_j^A(a_1, \ldots, a_n) = b_j$, we have $(b_1, \ldots, b_n) \in domf^A$, i.e. $domS^{n,A}(f^A, g_1^A, \ldots, g_n^A) := \{(a_1, \ldots, a_n) \in A^n \mid (a_1, \ldots, a_n) \in \bigcap_{j=1}^{n} domg_j^A$ and for all $j = 1, \ldots, n$,

$$g_j^A(a_1, \ldots, a_n) = b_j \text{, we have } (b_1, \ldots, b_n) \in domf^A\}$$

Special n -ary(total) operations are the projections to the i -th component, where $1 \leq i \leq n$: $e_i^{n,A} : A^n \to A$ by $e_i^{n,A}(a_1, \ldots, a_n) := a_i$.

Definition 1.1 ([7]) An algebra $(M; S^n, e_1, \ldots, e_n)$ of type $\tau = (n+1, 0, \ldots, 0)$ is called a unitary Menger algebra of rank n if it satisfies the axioms (C1),(C2),(C3):

(C1) $S^n(X_0, S^n(Y_1, X_1, \ldots, X_n), \ldots, S^n(Y_n, X_1, \ldots, X_n)) \approx S^n(S^n(X_0, Y_1, \ldots, Y_n), X_1, \ldots, X_n)$, (C2) $\bar{S}^n(I_j, X_1, \dots, X_n) \approx X_j$ for $1 \le j \le n$,

(C3) $\bar{S}^n(X_j, I_1, \dots, I_n) \approx X_j$ for $1 \le j \le n$.

(Here $\bar{S}^n$ is an $(n+1)$-ary operation symbol, $I_1, \dots, I_n$ are nullary operation symbols and $X_0, X_1, \dots, X_n, Y_1, \dots, Y_n$ are variables.) An algebra $(M; S^n)$ of type $\tau = (n+1)$ which satisfies (C1) is called a Menger algebra of rank n.

Theorem 1.2 The algebra $(P^n(A); S^{n,A})$ is a Menger algebra of rank n.

We have proven already that the algebra $(P^n(A); S^{n,A})$ satisfies the axiom(C1) (see [4]).

Let $W_\tau(X_n)$ be the set of all n-ary terms of type τ and let $W_\tau(X_n)^{\underline{A}}$ be the set of all n-ary term operations induced by n-ary terms on the partial algebra $\underline{A}$. For the definition of a term operation $t^{\underline{A}}$ induced by the term t on the partial algebra $\underline{A}$ (see [11],[1]). Different from the total case, the set $W_\tau(X_n)^{\underline{A}}$ is, in general, a proper subset of $P^n(A)$ (see some examples in [11]). In 1989, W. Craig [6] introduced concept of terms for partial algebras by additional symbols e and in 1996, F. Börner [1] introduced another concept of terms for partial algebras.

Let X be an alphabet and let $\{f_i \mid i \in I\}$ be a set of operation symbols of type τ, where each f_i has arity n_i and $X \cap \{f_i \mid i \in I\} = \emptyset$. We need additional symbols $e_j^k \notin X$, for every $k \in \mathbb{N}^+ := \mathbb{N} \setminus \{0\}$ and $1 \le j \le k$. Let $X_n = \{x_1, \dots, x_n\}$ be an n-element alphabet. The set of all n-ary C-terms of type τ over X_n is defined inductively as follows (see [1]):

(i) every $x_i \in X_n$ is an n-ary C-term of type τ;

(ii) if $w_1, \dots, w_k$ are n-ary C-terms of type τ, then $e_j^k(w_1, \dots, w_k)$ is an n-ary C-term of type τ for all $1 \le j \le k$ and all $k \in \mathbb{N}^+$;

(iii) if $w_1, \ldots, w_n$ are n-ary C-terms of type τ and if f_i is an n_i-ary operation symbol, then $f_i(w_1, \ldots, w_{n_i})$ is an n-ary C-term of type τ.

Let $W_\tau^C(X_n)$ be the set of all n-ary C-terms of type τ defined in this way. Then $W_\tau^C(X) := \bigcup_{n=1}^{\infty} W_\tau^C(X_n)$ denotes the set of all C-terms of this type.

Now we consider a type $\tau_n = (n, n, \ldots, n)$ consisting of n-ary operation symbols only.

On the sets $W_{\tau_n}^C(X_n)$ we introduce the following superposition operations. Let $t, w_1, \ldots, w_n$ be n-ary C-terms. Then we define an n-ary C-term $S^n(t, w_1, \ldots, w_n)$ inductively by the following steps:

(i) For $t = x_j$, $1 \leq j \leq n$, we define $S^n(x_j, w_1, \ldots, w_n) = w_j$.

(ii) For $t = e_j^k(s_1, \ldots, s_k)$ we set $S^n(t, w_1, \ldots, w_n) = e_j^k(S^n(s_1, w_1, \ldots, w_n), \ldots, S^n(s_k, w_1, \ldots, w_n))$,

where $s_1, \ldots, s_k$ are n-ary C-terms, for all $k \in \mathbb{N}^+$ and $1 \leq j \leq k$.

(iii) For $t = f_i(s_1, \ldots, s_n)$ we set $S^n(t, w_1, \ldots, w_n) = f_i(S^n(s_1, w_1, \ldots, w_n), \ldots, S^n(s_n, w_1, \ldots, w_n))$,

where $s_1, \ldots, s_n$ are n-ary C-terms.

This defines an operation $S^n : (W_{\tau_n}^C(X_n))^{n+1} \to W_{\tau_n}^C(X_n)$, which describes the superposition of terms. Together with the nullary operations $x_1, \ldots, x_n$ one obtains an algebra

$$n\text{-}clone_{\tau_n}^C := (W_{\tau_n}^C(X_n); S^n, x_1, \ldots, x_n).$$

Theorem 1.3 The algebra $n\text{-}clone_{\tau_n}^C$ is a unitary Menger algebra of rank n.

We have proven already that the algebra $n\text{-}clone_{\tau_n}^C$ satisfies the axiom (C1), (C2), (C3) (see [5]).

Every n-ary C-term $w \in W^C_{\tau_n}(X_n)$ induces an n-ary C-term operation $w^{\underline{A}}$ of any partial algebra $\underline{A} = (A;(f_i^A)_{i \in I})$ of type τ_n. For $a_1, \ldots, a_n \in A$, the value $w^{\underline{A}}(a_1, \ldots, a_n)$ is defined in the following inductive way:

(i) If $w = x_i$ then $w^{\underline{A}} = x_i^{\underline{A}} = e_i^{n,A}$, where $e_i^{n,A}$ is as usual the n-ary total projection on the i-th component.

(ii) If $w = e_j^k(w_1, \ldots, w_k)$ and we assume that $w_1^{\underline{A}}, \ldots, w_k^{\underline{A}}$ are the term operations induced by the terms $w_1, \ldots, w_k$ and that the $w_i^{\underline{A}}(a_1, \ldots, a_n)$ are defined for $1 \le i \le k$, then $w^{\underline{A}}(a_1, \ldots, a_n)$ is defined and $w^{\underline{A}}(a_1, \ldots, a_n) = w_j^{\underline{A}}(a_1, \ldots, a_n)$.

(iii) Now assume that $w = f_i(w_1, \ldots, w_n)$ and that the $w_j^{\underline{A}}(a_1, \ldots, a_n)$ are defined, with values $w_j^{\underline{A}}(a_1, \ldots, a_n) = b_j$ for $1 \le j \le n$. If $f_i^A(b_1, \ldots, b_n)$ is defined, then $w^{\underline{A}}(a_1, \ldots, a_n)$ is defined and $w^{\underline{A}}(a_1, \ldots, a_n) := S^{n,A}(f_i^A, w_1^{\underline{A}}(a_1, \ldots, a_n), \ldots, w_n^{\underline{A}}(a_1, \ldots, a_n))$.

Let $W^C_{\tau_n}(X_n)^{\underline{A}}$ be the set of all n-ary C-term operations induced by the n-ary C-terms from $W^C_{\tau_n}(X_n)$ on the partial algebra $\underline{A}$ and $W^C_{\tau_n}(X)^{\underline{A}} := \bigcup_{n=1}^{\infty} W^C_{\tau_n}(X_n)^{\underline{A}}$.

Theorem 1.4 The algebra $(W^C_{\tau_n}(X_n)^{\underline{A}}; S^{n,A}, e_1^{n,A}, \ldots, e_n^{n,A})$ is a unitary Menger algebra of rank n. (i.e. $(W^C_{\tau_n}(X_n)^{\underline{A}}; S^{n,A}, e_1^{n,A}, \ldots, e_n^{n,A})$ satisfies the following identities:

$(\overline{C1})$ $S^{n,A}(s^{\underline{A}}, S^{n,A}(t_1^{\underline{A}}, s_1^{\underline{A}}, \ldots, s_n^{\underline{A}}), \ldots, S^{n,A}(t_n^{\underline{A}}, s_1^{\underline{A}}, \ldots, s_n^{\underline{A}})) = S^{n,A}(S^{n,A}(s^{\underline{A}}, t_1^{\underline{A}}, \ldots, t_n^{\underline{A}}), s_1^{\underline{A}}, \ldots, s_n^{\underline{A}})$,

$(\overline{C2})$ $S^{n,A}(e_i^{n,A}, t_1^{\underline{A}}, \ldots, t_n^{\underline{A}}) = t_i^{\underline{A}}$ for $1 \le i \le n$,

$(\overline{C3})$ $S^{n,A}(t^{\underline{A}}, e_1^{n,A}, \ldots, e_n^{n,A}) = t^{\underline{A}}$)

2. Partial Algebraic Systems

The concept of an algebraic system was first introduced by A.I. Malcev in 1973 [9]. We now recall the informal definition of algebraic systems. An algebraic system is a structure consisting of a non-empty set together with a sequence of operations and a sequence of relations on this set. In 2002, P. Burmeister [3] introduced the concept of a congruence relation on partial algebra. Using the definition of algebraic systems and of relation on a set of partial algebra, we defined the new concept of the partial algebraic system.

Definition 2.1 A partial algebraic system of type $(\tau_n, (n))$ is a triple $\underline{A}^{\sigma} = (A; (f_i^A)_{i \in I}, r^A)$ consisting of a non-empty set A, a sequence $(f_i^A)_{i \in I}$ of partial operations defined on A indexed by the index set I where f_i^A is n-ary for $i \in I$ and r^A of n-ary relation on A, which is compatible with all the partial operations $(f_i^A)_{i \in I}$, i.e.

if $(a_1^{(1)}, a_1^{(2)}, \ldots, a_1^{(n)}) \in r^A, \ldots, (a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(n)}) \in r^A$ and if all

$(a_1^{(1)}, a_2^{(1)}, \ldots, a_n^{(1)}), \ldots, (a_1^{(n)}, a_2^{(n)}, \ldots, a_n^{(n)}) \in dom\, f_i^A$

then $(f_i^A(a_1^{(1)}, a_2^{(1)}, \ldots, a_n^{(1)}), \ldots, f_i^A(a_1^{(n)}, a_2^{(n)}, \ldots, a_n^{(n)})) \in r^A$.

Written in matrix notation

$$(a_1^{(1)}, a_1^{(2)}, \ldots, a_1^{(n)}) \in r^A$$

$$\vdots$$

$$(a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(n)}) \in r^A$$

$(a_1^{(1)}, a_2^{(1)}, \ldots, a_n^{(1)}) \in dom\, f_i^A, \ldots, (a_1^{(n)}, a_2^{(n)}, \ldots, a_n^{(n)}) \in dom\, f_i^A$

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$(f_i^A(a_1^{(1)}, a_2^{(1)}, \ldots, a_n^{(1)}), \ldots, f_i^A(a_1^{(n)}, a_2^{(n)}, \ldots, a_n^{(n)})) \in r^A$.

Let $PAlgsys(\tau_n,(n))$ be the class of all partial algebraic systems of type $(\tau_n,(n))$. Every n-ary C-term $w \in W^C_{\tau_n}(X_n)$ induces an n-ary C-term operation $w^{\underline{A}^s}$ of any partial algebraic system $\underline{A}^s = (A;(f_i^A)_{i \in I}, r^A)$ of type $(\tau_n,(n))$. For $a_1,\ldots,a_n \in A$, the value $w^{\underline{A}^s}(a_1,\ldots,a_n) := w^{\underline{A}}(a_1,\ldots,a_n)$ when $\underline{A} = (A;(f_i^A)_{i \in I})$ is partial algebra of type τ_n.

Not all of the terms in the second-order language will be used to express the properties of algebraic systems. The one is called formulas, first introduced by A.I. Mal'cev in 1973. In 1986, P. Burmeister [2] introduced the concept of formulas of the language by using the logical connectives. In 2013, K. Denecke and D. Phusanga [10] introduced the concept of formulas by using terms, the logical connective $\neg$ (for negation), $\vee$ (for disjunction), and the equation symbol $\approx$. So, we introduce the concept of a C-formulas of type $(\tau_n,(n))$ by using n-ary C-terms of type τ_n.

Definition 2.2 Let $n \geq 1$. An n-ary C-formula of type $(\tau_n,(n))$ is defined in the following inductive way:

(i) If t_1, t_2 are n-ary C-terms of type τ_n, then the equation $t_1 \approx t_2$ is an n-ary C-formula of type $(\tau_n,(n))$.

(ii) If $t_1,\ldots,t_n$ are n-ary C-terms of type τ_n, then $r(t_1,\ldots,t_n)$ is an n-ary C-formula of type $(\tau_n,(n))$.

(iii) If CF is an n-ary C-formula of type $(\tau_n,(n))$, then $\neg CF$ is an n-ary C-formula of type $(\tau_n,(n))$.

(iv) If CF_1 and CF_2 are an n-ary C-formulas of type $(\tau_n,(n))$, then $CF_1 \vee CF_2$ is an n-ary C-formula of type $(\tau_n,(n))$.

Let $CF_{(t_n,(n))}(X_n)$ be the set of all n-ary C-formula of type $(t_n,(n))$.

We want to extend a superposition to n-ary C-formulas. If we substitute variables occurring in an n-ary C-formula by terms we obtain a new C-formula. We want to describe this by the following operations $R^n, n \geq 1$.

Definition 2.3 The operations

$$R^n : (W_{t_n}^C(X_n) \cup CF_{(t_n,(n))}(X_n)) \times W_{t_n}^C(X_n)^n \to W_{t_n}^C(X_n) \cup CF_{(t_n,(n))}(X_n)$$

where $n \in \mathbb{N}^+$, are defined by the following inductive steps: Let $s_1,\ldots,s_n \in W_{t_n}^C(X_n)$.

(i) If $t \in W_{t_n}^C(X_n)$, then we define $R^n(t,s_1,\ldots,s_n) := S^n(t,s_1,\ldots,s_n)$ (which is the superposition of terms).

(ii) If $t_1,t_2 \in W_{t_n}^C(X_n)$, then $R^n(t_1 \approx t_2,s_1,\ldots,s_n) := R^n(t_1,s_1,\ldots,s_n) \approx R^n(t_2,s_1,\ldots,s_n)$.

(iii) If $t_1,\ldots,t_n \in W_{t_n}^C(X_n)$, then

$$R^n(r(t_1,\ldots,t_n),s_1,\ldots,s_n) := r(R^n(t_1,s_1,\ldots,s_n),\ldots,R^n(t_n,s_1,\ldots,s_n)).$$

(iv) If $CF \in CF_{(t_n,(n))}(X_n)$, then $R^n(\neg \mathrm{CF},s_1,\ldots,s_n) := \neg R^n(\mathrm{CF},s_1,\ldots,s_n)$.

(v) If $CF_1,CF_2 \in CF_{(t_n,(n))}(X_n)$, then

$$R^n(\mathrm{CF}_1 \vee \mathrm{CF}_2,s_1,\ldots,s_n) := R^n(\mathrm{CF}_1,s_1,\ldots,s_n) \vee R^n(\mathrm{CF}_2,s_1,\ldots,s_n).$$

These operations define an algebra

$$C-Formclone(t_n,(n)) := (CF_{(t_n,(n))}(X_n) \cup W_{t_n}^C(X_n); R^n, x_1,\ldots x_n)$$

which is called the C-formula-term clone of type $(t_n,(n))$.

This algebra keeps the properties of the term clones.

Theorem 2.4 The algebra $C-Formclone(t_n,(n))$ is a unitary Menger algebra of rank n. (i.e. the algebra $C-Formclone(t_n,(n))$ satisfies the following identities:

(CFC1) $R^n(b,R^n(t_1,s_1,\ldots,s_n),\ldots,R^n(t_n,s_1,\ldots,s_n))$

$\approx R^n(R^n(b,t_1,\ldots,t_n),s_1,\ldots,s_n)$ whenever $s_1,\ldots,s_n,t_1,\ldots,t_n \in W^C_{t_n}(X_n)$,

(CFC2) $R^n(x_i,s_1,\ldots,s_n) \approx s_i$ whenever $s_1,\ldots,s_n \in W^C_{t_n}(X_n)$ for $1 \le i \le n$,

(CFC3) $R^n(b,x_1,\ldots,x_n) \approx b$)

Proof If $b \in W^C_{t_n}(X_n)$, then the assertion is clear by (C1). Now let $b \in CF_{(t_n,(n))}(X_n)$. We give a proof of (CFC1) by induction on the complexity of an n-ary C-formula as the following the steps:

(i) If CF have the form $s \approx t$, then

$R^n(s \approx t, R^n(t_1,s_1,\ldots,s_n),\ldots,R^n(t_n,s_1,\ldots,s_n))$

$= R^n(s,R^n(t_1,s_1,\ldots,s_n),\ldots,R^n(t_1,s_1,\ldots,s_n)) \approx R^n(t,R^n(t_1,s_1,\ldots,s_n),\ldots,R^n(t_1,s_1,\ldots,s_n))$
$= R^n(R^n(s,t_1,\ldots,t_n),s_1,\ldots,s_n)) \approx R^n(R^n(t,t_1,\ldots,t_n),s_1,\ldots,s_n))$

$= R^n(R^n(s,t_1,\ldots,t_n) \approx R^n(t,t_1,\ldots,t_n),s_1,\ldots,s_n)$

$= R^n(R^n(s \approx t,t_1,\ldots,t_n),s_1,\ldots,s_n)$.

(ii) If CF have the form $r(p_1,\ldots,p_n)$, then

$R^n(r(p_1,\ldots p_n),R^n(t_1,s_1,\ldots,s_n),\ldots,R^n(t_n,s_1,\ldots,s_n))$

$= r(R^n(p_1,R^n(t_1,s_1,\ldots,s_n),\ldots,R^n(t_n,s_1,\ldots,s_n)),\ldots,R^n(p_n,R^n(t_1,s_1,\ldots,s_n),\ldots,R^n(t_n,s_1,\ldots,s_n)))$

$= r(R^n(R^n(p_1,t_1,\ldots,t_n),s_1,\ldots,s_n),\ldots,R^n(R^n(p_n,t_1,\ldots,t_n),s_1,\ldots,s_n))$

$= R^n(r(R^n(p_1,t_1,\ldots,t_n),\ldots,R^n(p_n,t_1,\ldots,t_n)),s_1,\ldots,s_n)$

$= R^n(R^n(r(p_1, \ldots, p_n), t_1, \ldots, t_n), s_1, \ldots, s_n)$.

(iii) We have to show that if (CFC1) is satisfied for the C -formula CF , then it is also satisfied for $\neg CF$. In fact,

$R^n(\neg CF, R^n(t_1, s_1, \ldots, s_n), \ldots, R^n(t_n, s_1, \ldots, s_n))$

$= \neg(R^n(CF, R^n(t_1, s_1, \ldots, s_n), \ldots, R^n(t_n, s_1, \ldots, s_n)))$

$= \neg(R^n(R^n(CF, t_1, \ldots, t_n), s_1, \ldots, s_n))$

$= R^n(\neg(R^n(CF, t_1, \ldots, t_n)), s_1, \ldots, s_n)$

$= R^n(R^n(\neg CF, t_1, \ldots, t_n), s_1, \ldots, s_n)$.

(iv) Here we have to show that if CF_1 and CF_2 satisfy (CFC1), then also $CF_1 \vee CF_2$ satisfies (CFC1). In fact,

$R^n(CF_1 \vee CF_2, R^n(t_1, s_1, \ldots, s_n), \ldots, R^n(t_n, s_1, \ldots, s_n))$

$= R^n(CF_1, R^n(t_1, s_1, \ldots, s_n), \ldots, R^n(t_1, s_1, \ldots, s_n)) \vee R^n(CF_2, R^n(t_1, s_1, \ldots, s_n), \ldots, R^n(t_1, s_1, \ldots, s_n))$

$= R^n(R^n(CF_1, t_1, \ldots, t_n), s_1, \ldots, s_n)) \vee R^n(R^n(CF_2, t_1, \ldots, t_n), s_1, \ldots, s_n))$

$= R^n(R^n(CF_1, t_1, \ldots, t_n) \vee R^n(CF_2, t_1, \ldots, t_n), s_1, \ldots, s_n)$

$= R^n(R^n(CF_1 \vee CF_2, t_1, \ldots, t_n), s_1, \ldots, s_n)$.

This finishes the proof of (CFC1).

(CFC2) is clearly by (C2) and the fact that $R^n(x_i, s_1, \ldots, s_n) = S^n(x_i, s_1, \ldots, s_n)$.

(CFC3) If $b \in W_{t_n}^C(X_n)$, then the assertion is clear by (C3). It is left to consider the case that b is an n -ary C -formula CF . We will proceed in a similar way considering the completely of an n -ary C -formula CF .

(i) If CF is an equation $s \approx t$, then

$$R^n(s \approx t, x_1, \ldots, x_n) = R^n(s, x_1, \ldots, x_n) \approx R^n(t, x_1, \ldots, x_n)$$

$$= S^n(s, x_1, \ldots, x_n) \approx S^n(t, x_1, \ldots, x_n)$$

$$= s \approx t \text{ by (C3).}$$

(ii) If CF have the form $r(t_1, \ldots, t_n)$, then

$$R^n(r(t_1, \ldots, t_n), x_1, \ldots, x_n) = r(R^n(t_1, x_1, \ldots, x_n), \ldots, R^n(t_n, x_1, \ldots, x_n)) = r(t_1, \ldots, t_n).$$

(iii) Assume that (CFC3) is satisfied for CF. We show that it is also satisfied for $\neg CF$.

By definition $R^n(\neg CF, x_1, \ldots, x_n)$ is the formula $\neg R^n(CF, x_1, \ldots, x_n)$ i.e. it is equal to $\neg CF$.

(iv) Assume that (CFC3) is satisfied for CF_1 and CF_2. Then $R^n(CF_1 \vee CF_2, x_1, \ldots, x_n)$ is the formula $R^n(CF_1, x_1, \ldots, x_n) \vee R^n(CF_2, x_1, \ldots, x_n)$ and this is to equal $CF_1 \vee CF_2$.

Now we will define the realization of an n-ary C-formula of type $(\tau_n(n))$ on the partial algebraic system $\underline{A}^{\sigma}$ of the same type.

Definition 2.5 Let $\underline{A} = (A; (f_i^A)_{i \in I})$ be a partial algebra of type τ_n and CF be an n-ary C-formula of type $(\tau_n(n))$. Then the realization of CF on the partial algebraic system $\underline{A}^{\sigma} = (A; (f_i^A)_{i \in I}, r^A)$ denoted by $CF^{\underline{A}^{\sigma}}$ is defined as follows:

(i) If CF has the form $t_1 \approx t_2$, then

$$(t_1 \approx t_2)^{\underline{A}^{\sigma}} := \{(a_1, \ldots, a_n) \in A^n \mid t_1^{\underline{A}}(a_1, \ldots, a_n) \text{ and } t_2^{\underline{A}}(a_1, \ldots, a_n) \text{ both exist and}$$

$$t_1^{\underline{A}}(a_1, \ldots, a_n) = t_2^{\underline{A}}(a_1, \ldots, a_n)\}.$$

(ii) If CF has the form $r(t_1, \ldots, t_n)$, then

$(r(t_1,\ldots,t_n))^{\underline{A}^s} := \{(a_1,\ldots,a_n) \in A^n \mid (a_1,\ldots,a_n) \in dom\, t_i^{\underline{A}}$ for $i = 1,\ldots,n$ and

$$(t_1^{\underline{A}}(a_1,\ldots,a_n),\ldots,t_n^{\underline{A}}(a_1,\ldots,a_n)) \in r^A\}.$$

(iii) If $CF \in CF_{(t_n,(n))}(X_n)$ and suppose that $CF^{\underline{A}^s}$ is already defined, then

$$(\neg CF)^{\underline{A}^s} := \{(a_1,\ldots,a_n) \in A^n \mid (a_1,\ldots,a_n) \notin CF^{\underline{A}^s}\}.$$

(iv) If $CF_1, CF_2 \in CF_{(t_n,(n))}(X_n)$ and suppose that $CF_1^{\underline{A}^s}$ and $CF_2^{\underline{A}^s}$ are already defined, then

$$(CF_1 \vee CF_2)^{\underline{A}^s} := \{(a_1,\ldots,a_n) \in A^n \mid (a_1,\ldots,a_n) \in CF_1^{\underline{A}^s} \text{ or } (a_1,\ldots,a_n) \in CF_2^{\underline{A}^s}\}.$$

Let $(CF_{(t_n,(n))}(X_n))^{\underline{A}^s} := \{CF^{\underline{A}^s} \mid CF \in CF_{(t_n,(n))}(X_n)\}$ be the set of all "realizations" of n-ary C-formulas on the partial algebraic system $\underline{A}^s = (A,(f_i^A)_{i\in I}, r^A)$ and let $(CF_{(t_n,(n))}(X))^{\underline{A}^s} := \bigcup_{n\geq 1}(CF_{(t_n,(n))}(X_n))^{\underline{A}^s}$ be the set of all "realizations" of C-formulas on the partial algebraic system $\underline{A}^s = (A,(f_i^A)_{i\in I}, r^A)$.

Extending this idea to algebraic systems we can define an operation

$$R^{n,A} : ((W_{t_n}^C(X_n))^{\underline{A}^s} \cup (CF_{(t_n,(n))}(X_n))^{\underline{A}^s}) \times ((W_{t_n}^C(X_n))^{\underline{A}^s})^n \to (W_{t_n}^C(X_n))^{\underline{A}^s} \cup (CF_{(t_n,(n))}(X_n))^{\underline{A}^s}$$

as follows:

Definition 2.6 For any $b^{\underline{A}^s} \in (W_{t_n}^C(X_n))^{\underline{A}^s} \cup (CF_{(t_n,(n))}(X_n))^{\underline{A}^s}$ and n-tuple $(t_1^{\underline{A}^s},\ldots,t_n^{\underline{A}^s})$ of n-ary C-term operations, we define

$R^{n,A}(b^{\underline{A}^s}, t_1^{\underline{A}^s},\ldots,t_n^{\underline{A}^s}) := S^{n,A}(b^{\underline{A}}, t_1^{\underline{A}},\ldots,t_n^{\underline{A}})$ if $b^{A^s} \in (W_{t_n}^C(X_n))^{A^s}$

and if $b^{\underline{A}^S} \in (CF_{(t_n,(n))}(X_n))^{\underline{A}^S}$, we define

$$R^{n,A}(b^{\underline{A}^S}, t_1^{\underline{A}^S}, \ldots, t_n^{\underline{A}^S}) = R^{n,A}(b^{\underline{A}^S}, t_1^{\underline{A}}, \ldots, t_n^{\underline{A}}) := \{(a_1, \ldots, a_n) \in A^n \mid (a_1, \ldots, a_n) \in \bigcap_{i=1}^{n} dom\, t_i^{\underline{A}} \text{ and } (t_1^{\underline{A}}(a_1, \ldots, a_n), \ldots, t_n^{\underline{A}}(a_1, \ldots, a_n)) \in b^{\underline{A}^S}\}.$$

Lemma 2.7 For $b \in W_{t_n}^C(X_n) \cup CF_{(t_n,(n))}(X_n)$ and $s_1, \ldots, s_n \in W_{t_n}^C(X_n)$ we have

$$R^{n,A}(b^{\underline{A}^S}, s_1^{\underline{A}^S}, \ldots, s_n^{\underline{A}^S}) = (R^n(b, s_1, \ldots, s))^{\underline{A}^S}.$$

Proof In the first case $b = t \in W_{t_n}^C(X_n)$. We can be proved that

$R^{n,A}(b^{\underline{A}^S}, s_1^{\underline{A}^S}, \ldots, s_n^{\underline{A}^S}) = S^{n,A}(t^{\underline{A}}, s_1^{\underline{A}}, \ldots, s_n^{\underline{A}}) = (S^n(t, s_1, \ldots, s_n))^{\underline{A}} = (R^n(b, s_1, \ldots, s_n))^{\underline{A}^S}$ by induction on the complexity of the n-ary C-term t.

In the second case $b = CF \in CF_{(t_n(n))}(X_n)$. We show it by induction on the complexity of n-ary C-formula CF. Let $s_1, \ldots, s_n \in W_{t_n}^C(X_n)$.

(i) IF CF has the form $t_1 \approx t_2$, then $(a_1, \ldots, a_n) \in R^{n,A}((t_1 \approx t_n)^{\underline{A}^S}, s_1^{\underline{A}^S}, \ldots, s_n^{\underline{A}^S})$

$\Leftrightarrow$ $(a_1, \ldots, a_n) \in \bigcap_{i=1}^{n} dom\, s_i^{\underline{A}}$ and $(s_1^{\underline{A}}(a_1, \ldots, a_n), \ldots, s_n^{\underline{A}}(a_1, \ldots, a_n)) \in (t_1 \approx t_2)^{\underline{A}^S}$

$\Leftrightarrow$ $t_i^{\underline{A}}(s_1^{\underline{A}}(a_1, \ldots, a_n), \ldots, s_n^{\underline{A}}(a_1, \ldots, a_n))$ exist for $i = 1, 2$ and

$t_1^{\underline{A}}(s_1^{\underline{A}}(a_1, \ldots, a_n), \ldots, s_n^{\underline{A}}(a_1, \ldots, a_n)) = t_2^{\underline{A}}(s_1^{\underline{A}}(a_1, \ldots, a_n), \ldots, s_n^{\underline{A}}(a_1, \ldots, a_n))$ (by Definition 2.5 (i))

$\Leftrightarrow$ $S^{n,A}(t_i^{\underline{A}}, s_1^{\underline{A}}, \ldots, s_n^{\underline{A}})(a_1, \ldots, a_n)$ exist for $i = 1, 2$ and

$S^{n,A}(t_1^{\underline{A}}, s_1^{\underline{A}}, \ldots, s_n^{\underline{A}})(a_1, \ldots, a_n) = S^{n,A}(t_2^{\underline{A}}, s_1^{\underline{A}}, \ldots, s_n^{\underline{A}})(a_1, \ldots, a_n)$

$\leftrightarrow \quad R^{n,A}(t_i^{\underline{A}}, s_1^{\underline{A}}, \ldots, s_n^{\underline{A}})(a_1, \ldots, a_n)$ exist for $i = 1, 2$ and

$R^{n,A}(t_1^{\underline{A}}, s_1^{\underline{A}}, \ldots, s_n^{\underline{A}})(a_1, \ldots, a_n) = R^{n,A}(t_2^{\underline{A}}, s_1^{\underline{A}}, \ldots, s_n^{\underline{A}})(a_1, \ldots, a_n)$

$\leftrightarrow \quad (R^n(t_i, s_1, \ldots, s_n))^{\underline{A}}(a_1, \ldots, a_n)$ exist for $i = 1, 2$ and

$(R^n(t_1, s_1, \ldots, s_n))^{\underline{A}}(a_1, \ldots, a_n) = (R^n(t_2, s_1, \ldots, s_n))^{\underline{A}}(a_1, \ldots, a_n)$

$\leftrightarrow \quad (a_1, \ldots, a_n) \in (R^n(t_1, s_1, \ldots, s_n) \approx R^n(t_2, s_1, \ldots, s_n))^{\underline{A}^s}$

$\leftrightarrow \quad (a_1, \ldots, a_n) \in (R^n(t_1 \approx t_2, s_1, \ldots, s_n))^{\underline{A}^s}$ by Definition 2.3(ii).

This shows $R^{n,A}((t_1 \approx t_2)^{\underline{A}^s}, s_1^{\underline{A}^s}, \ldots, s_n^{\underline{A}^s}) = (R^n(t_1 \approx t_2, s_1, \ldots, s_n))^{\underline{A}^s}$.

(ii) IF CF has the form $r(t_1, \ldots, t_n)$, then $(a_1, \ldots, a_n) \in R^{n,A}((r(t_1, \ldots, t_n))^{\underline{A}^s}, s_1^{\underline{A}^s}, \ldots, s_n^{\underline{A}^s})$

$\leftrightarrow \quad (a_1, \ldots, a_n) \in \bigcap_{i=1}^{n} dom\, s_i^{\underline{A}}$ and $(s_1^{\underline{A}}(a_1, \ldots, a_n), \ldots, s_n^{\underline{A}}(a_1, \ldots, a_n)) \in (r(t_1, \ldots, t_n))^{\underline{A}^s}$

$\leftrightarrow \quad t_i^{\underline{A}}(s_1^{\underline{A}}(a_1, \ldots, a_n), \ldots, s_n^{\underline{A}}(a_1, \ldots, a_n))$ exist for $i = 1, \ldots, n$ and

$(t_1^{\underline{A}}(s_1^{\underline{A}}(a_1, \ldots, a_n), \ldots, s_n^{\underline{A}}(a_1, \ldots, a_n)), \ldots, t_n^{\underline{A}}(s_1^{\underline{A}}(a_1, \ldots, a_n), \ldots, s_n^{\underline{A}}(a_1, \ldots, a_n))) \in r^A$

(by Definition 2.5(ii))

$\leftrightarrow \quad S^{n,A}(t_i^{\underline{A}}, s_1^{\underline{A}}, \ldots, s_n^{\underline{A}})(a_1, \ldots, a_n)$ exist for $i = 1, \ldots, n$ and

$(S^{n,A}(t_1^{\underline{A}}, s_1^{\underline{A}}, \ldots, s_n^{\underline{A}})(a_1, \ldots, a_n), \ldots, S^{n,A}(t_n^{\underline{A}}, s_1^{\underline{A}}, \ldots, s_n^{\underline{A}})(a_1, \ldots, a_n)) \in r^A$

$\leftrightarrow \quad R^{n,A}(t_i^{\underline{A}}, s_1^{\underline{A}}, \ldots, s_n^{\underline{A}})(a_1, \ldots, a_n)$ exist for $i = 1, \ldots, n$ and

$(R^{n,A}(t_1^{\underline{A}}, s_1^{\underline{A}}, \ldots, s_n^{\underline{A}})(a_1, \ldots, a_n), \ldots, R^{n,A}(t_n^{\underline{A}}, s_1^{\underline{A}}, \ldots, s_n^{\underline{A}})(a_1, \ldots, a_n)) \in r^A$

$\leftrightarrow \quad (R^n(t_i, s_1, \ldots, s_n))^{\underline{A}}(a_1, \ldots, a_n)$ exist for $i = 1, \ldots, n$ and

$$((R^n(t_1, s_1, \dots, s_n))^{\underline{A}}(a_1, \dots, a_n), \dots, (R^n(t_n, s_1, \dots, s_n))^{\underline{A}}(a_1, \dots, a_n)) \in r^A$$

$$\Leftrightarrow \quad (a_1, \dots, a_n) \in (r(R^n(t_1, s_1, \dots, s_n), \dots, R^n(t_n, s_1, \dots, s_n)))^{\underline{A}^s}$$

$$\Leftrightarrow \quad (a_1, \dots, a_n) \in (R^n(r(t_1, \dots, t_n), s_1, \dots, s_n))^{\underline{A}^s}$$ by Definition 2.3(iii).

Hence $R^{n,A}((r(t_1, \dots, t_n))^{\underline{A}^s}, s_1^{\underline{A}^s}, \dots, s_n^{\underline{A}^s}) = (R^n(r(t_1, \dots, t_n), s_1, \dots, s_n))^{\underline{A}^s}$.

(iii) Let $CF \in CF_{(t_n,(n))}(X_n)$ and assume that $R^{n,A}(CF^{\underline{A}^s}, s_1^{\underline{A}^s}, \dots, s_n^{\underline{A}^s}) = (R^n(CF, s_1, \dots, s_n))^{\underline{A}^s}$.

Then we have $(a_1, \dots, a_n) \in R^{n,A}((\neg CF)^{\underline{A}^s}, s_1^{\underline{A}^s}, \dots, s_n^{\underline{A}^s})$

$$\Leftrightarrow \quad (a_1, \dots, a_n) \in \bigcap_{i=1}^{n} dom\, s_i^{\underline{A}} \text{ and } (s_1^{\underline{A}}(a_1, \dots, a_n), \dots, s_n^{\underline{A}}(a_1, \dots, a_n)) \in (\neg CF)^{\underline{A}^s}$$

$$\Leftrightarrow \quad (a_1, \dots, a_n) \in \bigcap_{i=1}^{n} dom\, s_i^{\underline{A}} \text{ and } (s_1^{\underline{A}}(a_1, \dots, a_n), \dots, s_n^{\underline{A}}(a_1, \dots, a_n)) \notin CF^{\underline{A}^s}$$

$$\Leftrightarrow \quad (a_1, \dots, a_n) \notin R^{n,A}(CF^{\underline{A}^s}, s_1^{\underline{A}^s}, \dots, s_n^{\underline{A}^s})$$

$$\Leftrightarrow \quad (a_1, \dots, a_n) \notin (R^n(CF, s_1, \dots, s_n))^{\underline{A}^s}$$

$$\Leftrightarrow \quad (a_1, \dots, a_n) \in (\neg R^n(CF, s_1, \dots, s_n)))^{\underline{A}^s}$$

$$\Leftrightarrow \quad (a_1, \dots, a_n) \in (R^n(\neg CF, s_1, \dots, s_n))^{\underline{A}^s}$$ by Definition 2.3(iv).

(iv) Let $CF_1, CF_2 \in CF_{(t_n,(n))}(X_n)$ and assume that

$R^{n,A}(CF_i^{\underline{A}^s}, s_1^{\underline{A}^s}, \dots, s_n^{\underline{A}^s}) = (R^n(CF_i, s_1, \dots, s_n))^{\underline{A}^s}$; $i \in \{1, 2\}$. Then we have

$$(a_1, \dots, a_n) \in R^{n,A}((CF_1 \vee CF_2)^{\underline{A}^s}, s_1^{\underline{A}^s}, \dots, s_n^{\underline{A}^s})$$

$$\Leftrightarrow \quad (a_1, \dots, a_n) \in \bigcap_{i=1}^{n} dom\, s_i^{\underline{A}} \text{ and } (s_1^{\underline{A}}(a_1, \dots, a_n), \dots, s_n^{\underline{A}}(a_1, \dots, a_n)) \in (CF_1 \vee CF_2)^{\underline{A}^s}$$

$\leftrightarrow \quad (a_1, \ldots, a_n) \in \bigcap_{i=1}^{n} dom\, s_i^{\underline{A}}$ and $[(s_1^{\underline{A}}(a_1, \ldots, a_n), \ldots, s_n^{\underline{A}}(a_1, \ldots, a_n)) \in CF_1^{\underline{A}^s}$ or

$(s_1^{\underline{A}}(a_1, \ldots, a_n), \ldots, s_n^{\underline{A}}(a_1, \ldots, a_n)) \in CF_2^{\underline{A}^s}]$ (by Definition 2.5(iv))

$\leftrightarrow \quad (a_1, \ldots, a_n) \in \bigcap_{i=1}^{n} dom\, s_i^{\underline{A}}$ and $(s_1^{\underline{A}}(a_1, \ldots, a_n), \ldots, s_n^{\underline{A}}(a_1, \ldots, a_n)) \in CF_1^{\underline{A}^s}$

or $(a_1, \ldots, a_n) \in \bigcap_{i=1}^{n} dom\, s_i^{\underline{A}}$ and $(s_1^{\underline{A}}(a_1, \ldots, a_n), \ldots, s_n^{\underline{A}}(a_1, \ldots, a_n)) \in CF_2^{\underline{A}^s}$

$\leftrightarrow \quad (a_1, \ldots, a_n) \in R^{n,A}(CF_1^{\underline{A}^s}, s_1^{\underline{A}^s}, \ldots, s_n^{\underline{A}^s})$ or $(a_1, \ldots, a_n) \in R^{n,A}(CF_2^{\underline{A}^s}, s_1^{\underline{A}^s}, \ldots, s_n^{\underline{A}^s})$

$\leftrightarrow \quad (a_1, \ldots, a_n) \in (R^n(CF_1, s_1, \ldots, s_n))^{\underline{A}^s}$ or $(a_1, \ldots, a_n) \in (R^n(CF_2, s_1, \ldots, s_n))^{\underline{A}^s}$

$\leftrightarrow \quad (a_1, \ldots, a_n) \in (R^n(CF_1, s_1, \ldots, s_n) \vee R^n(CF_2, s_1, \ldots, s_n))^{\underline{A}^s}$

$\leftrightarrow \quad (a_1, \ldots, a_n) \in (R^n(CF_1 \vee CF_2, s_1, \ldots, s_n))^{\underline{A}^s}$ by Definition 2.3(v).

The operation $R^{n,A}$ satisfies the following equation (see also $(\overline{C1})$,$(\overline{C2})$,$(\overline{C3})$):

Theorem 2.8 The algebra $((W_{t_n}^{C}(X_n))^{\underline{A}^s} \cup (CF_{(t_n,(n))}(X_n))^{\underline{A}^s}; R^{n,A}, x_1^{A^s}, \ldots, x_n^{A^s})$ is a unitary Menger algebra of rank n. (i.e. $((W_{t_n}^{C}(X_n))^{\underline{A}^s} \cup (CF_{(t_n,(n))}(X_n))^{\underline{A}^s}; R^{n,A}, x_1^{A^s}, \ldots, x_n^{A^s})$ satisfies the following identities:

$(\overline{CFC1})$ $R^{n,A}(b^{A^s}, R^{n,A}(t_1^{A^s}, s_1^{A^s}, \ldots, s_n^{A^s}), \ldots, R^{n,A}(t_n^{A^s}, s_1^{A^s}, \ldots, s_n^{A^s}))$

$= R^{n,A}(R^{n,A}(b^{A^s}, t_1^{A^s}, \ldots, t_n^{A^s}), s_1^{A^s}, \ldots, s_n^{A^s})$,

$(\overline{CFC2})$ $R^{n,A}(x_i^{A^s}, s_1^{A^s}, \ldots, s_n^{A^s}) = t_i^{A^s}$ for $1 \leq i \leq n$,

$(\overline{CFC3})$ $R^{n,A}(b^{A^s}, x_1^{A^s}, \ldots, x_n^{A^s}) = b^{A^s}$)

Proof Using Lemma 2.7, we can prove following way:

for $(\overline{CFC1})$, if $b \in W^{C}_{t_n}(X_n)$ is clear by $\overline{C1}$. Let now $b \in CF_{(t_n,(n))}(X_n)$.

We will give a proof by induction on the complexity of an n-ary C-formula CF.

(i) If CF has the form $s \approx t$, then

$$R^{n,A}(R^{n,A}((s \approx t)^{\underline{A}^s}, t_1^{\underline{A}^s}, \dots, t_n^{\underline{A}^s}), s_1^{\underline{A}^s}, \dots, s_n^{\underline{A}^s})$$

$$= R^{n,A}(R^{n}((s \approx t), t_1, \dots, t_n)^{\underline{A}^s}, s_1^{\underline{A}^s}, \dots, s_n^{\underline{A}^s})$$

$$= (R^{n}(R^{n}((s \approx t), t_1, \dots, t_n), s_1, \dots, s_n))^{\underline{A}^s}$$

the operation R^n satisfies $(CFC1)$, hence the previous line yields

$$= (R^{n}((s \approx t), R^{n}(t_1, s_1, \dots, s_n), \dots, R^{n}(t_n, s_1, \dots, s_n)))^{\underline{A}^s}$$

$$= R^{n,A}((s \approx t)^{\underline{A}^s}, R^{n,A}(t_1^{\underline{A}^s}, s_1^{\underline{A}^s}, \dots, s_n^{\underline{A}^s}), \dots, R^{n,A}(t_n^{\underline{A}^s}, s_1^{\underline{A}^s}, \dots, s_n^{\underline{A}^s})).$$

(ii) If CF has the form $r(l_1, \dots, l_n)$, then

$$R^{n,A}(R^{n,A}((r(l_1, \dots, l_n))^{\underline{A}^s}, t_1^{\underline{A}^s}, \dots, t_n^{\underline{A}^s}), s_1^{\underline{A}^s}, \dots, s_n^{\underline{A}^s})$$

$$= R^{n,A}((R^{n}(r(l_1, \dots, l_n), t_1, \dots, t_n))^{\underline{A}^s}, s_1^{\underline{A}^s}, \dots, s_n^{\underline{A}^s})$$

$$= (R^{n}(R^{n}(r(l_1, \dots, l_n), t_1, \dots, t_n), s_1, \dots, s_n))^{\underline{A}^s}$$

$$= (R^{n}(r(l_1, \dots, l_n), R^{n}(t_1, s_1, \dots, s_n), \dots, R^{n}(t_n, s_1, \dots, s_n)))^{\underline{A}^s}$$

$$= R^{n,A}(r(l_1, \dots, l_n)^{\underline{A}^s}, R^{n,A}(t_1^{\underline{A}^s}, s_1^{\underline{A}^s}, \dots, s_n^{\underline{A}^s}), \dots, R^{n,A}(t_n^{\underline{A}^s}, s_1^{\underline{A}^s}, \dots, s_n^{\underline{A}^s})).$$

(iii) We have to show that if $(\overline{CFC1})$ is satisfied for an n-ary C-formula CF, then it is also satisfied for $\neg CF$. In fact,

$$R^{n,A}(R^{n,A}((\neg CF)^{\underline{A}^s}, t_1^{\underline{A}^s}, \dots, t_n^{\underline{A}^s}), s_1^{\underline{A}^s}, \dots, s_n^{\underline{A}^s})$$

$$= R^{n,A}((R^n(\neg CF, t_1, \ldots, t_n))^{\underline{A}^s}, s_1^{\underline{A}^s}, \ldots, s_n^{\underline{A}^s})$$

$$= (R^n(R^n(\neg CF, t_1, \ldots, t_n), s_1, \ldots, s_n))^{\underline{A}^s}$$

$$= (R^n(\neg CF, R^n(t_1, s_1, \ldots, s_n), \ldots, R^n(t_n, s_1, \ldots, s_n)))^{\underline{A}^s}$$

$$= R^{n,A}((\neg CF)^{\underline{A}^s}, R^{n,A}(t_1^{\underline{A}^s}, s_1^{\underline{A}^s}, \ldots, s_n^{\underline{A}^s}), \ldots, R^{n,A}(t_n^{\underline{A}^s}, s_1^{\underline{A}^s}, \ldots, s_n^{\underline{A}^s})).$$

(iv) Here we have to show that if CF_1, CF_2 satisfy ($\overline{CFC\,1}$), then it is also satisfied for $CF_1 \vee CF_2$. In fact

$$R^{n,A}(R^{n,A}((CF_1 \vee CF_2)^{\underline{A}^s}, t_1^{\underline{A}^s}, \ldots, t_n^{\underline{A}^s}), s_1^{\underline{A}^s}, \ldots, s_n^{\underline{A}^s})$$

$$= R^{n,A}((R^n(CF_1 \vee CF_2, t_1, \ldots, t_n))^{\underline{A}^s}, s_1^{\underline{A}^s}, \ldots, s_n^{\underline{A}^s})$$

$$= (R^n(R^n(CF_1 \vee CF_2, t_1, \ldots, t_n), s_1, \ldots, s_n))^{\underline{A}^s}$$

$$= (R^n(CF_1 \vee CF_2, R^n(t_1, s_1, \ldots, s_n), \ldots, R^n(t_n, s_1, \ldots, s_n)))^{\underline{A}^s}$$

$$= R^{n,A}((CF_1 \vee CF_2)^{\underline{A}^s}, R^{n,A}(t_1^{\underline{A}^s}, s_1^{\underline{A}^s}, \ldots, s_n^{\underline{A}^s}), \ldots, R^{n,A}(t_n^{\underline{A}^s}, s_1^{\underline{A}^s}, \ldots, s_n^{\underline{A}^s})).$$

The proof of ($\overline{CFC\,2}$) is clear by ($CFC\,2$) and the fact that

$$R^{n,A}(x_i^{\underline{A}^s}, t_1^{\underline{A}^s}, \ldots, t_n^{\underline{A}^s}) = (R^n(x_i, t_1, \ldots, t_n))^{\underline{A}^s}.$$

For ($\overline{CFC\,3}$), we can calculate formula as follows. If $CF \in CF_{(t_n,(n))}(X_n)$, then

$$R^{n,A}(CF^{A^s}, x_1^{A^s}, \ldots, x_n^{A^s})$$

$$= \{(a_1, \ldots, a_n) \in A^n \mid (a_1, \ldots, a_n) \in \bigcap_{i=1}^{n} dom\, x_i^{\underline{A}} \text{ and } (x_1^{\underline{A}}(a_1, \ldots, a_n), \ldots, x_n^{\underline{A}}(a_1, \ldots, a_n)) \in CF^{\underline{A}^s}\}$$

$$= \{(a_1, \ldots, a_n) \in A^n \mid (a_1, \ldots, a_n) \in CF^{\underline{A}^s}\}$$

$= CF^{A^{\sigma}}$.

3. Monoid of C -Hypersubstitutions for Algebraic Systems of Type $(\tau_n,(n))$

In 1991, K. Denecke, D. Lau, R. Pöschel and D. Schweigert [8] introduced the concept of a hypersubstitution algebras. A hypersubstitution for algebraic systems was first introduced by K. Denecke and D. Phusanga [10]. It is a mapping that maps operation symbols to terms and relation symbols to formulas preserving arities. They defined a binary operation on the set of all hypersubstitutions for algebraic systems and then proved that this set the binary operation and an identity element forms a monoid. So, in this section, we would like to form the new structure of the so-called "Monoid of C -Hypersubstitution for Algebraic Systems of Type $(\tau_n(n))$". The way to approach this, we first define the based set.

Definition 3.1 Let $n \in \mathbb{N}^+$. A C -hypersubstitution for algebraic systems of type $(\tau_n,(n))$ is a mapping $\sigma : \{f_i \mid i \in I\} \cup \{r\} \to W^C_{\tau_n}(X_n) \cup CF_{(\tau_n,(n))}(X_n)$ which maps each n -ary operation symbols of type τ_n to n -ary C -term of type τ_n and maps an n -ary relation symbol r to an n -ary C -formula of type $(\tau_n,(n))$.

We denote the set of all C -hypersubstitutions for algebraic systems of type $(\tau_n,(n))$ by $Hyp^C(\tau_n,(n))$.

For every C -hypersubstitution for algebraic systems of type $(\tau_n,(n))$, we can define an extension which maps an n -ary C -term to an n -ary C -term and maps an n -ary C -formula to an n -ary C -formula as follows:

Definition 3.2 Let $n \in \mathbb{N}^+$ and $\sigma \in Hyp^C(\tau_n(n))$. Then we define a mapping

$\hat{\sigma} : W^C_{\tau_n}(X_n) \cup CF_{(\tau_n,(n))}(X_n) \to W^C_{\tau_n}(X_n) \cup CF_{(\tau_n,(n))}(X_n)$ inductively as follows:

(i) $\hat{s}[x_i] := x_i$ for every $x_i \in X_n$,

(ii) $\hat{s}[e_j^k(s_1, \dots, s_k)] := e_j^k(\hat{s}[s_1], \dots, \hat{s}[s_k])$, where $s_1, \dots, s_k \in W_{t_n}^C(X_n)$,

(iii) $\hat{s}[f_i(t_1, \dots, t_n)] := R^n(s(f_i), \hat{s}[t_1], \dots, \hat{s}[t_n])$, where $t_1, \dots, t_n \in W_{t_n}^C(X_n)$,

(iv) $\hat{s}[s \approx t] := \hat{s}[s] \approx \hat{s}[t]$ for every $s, t \in W_{t_n}^C(X_n)$,

(v) $\hat{s}[r(s_1, \dots, s_n)] := R^n(s(r), \hat{s}[s_1], \dots, \hat{s}[s_n])$, where $s_1, \dots, s_n \in W_{t_n}^C(X_n)$,

(vi) $\hat{s}[\neg CF] := \neg(\hat{s}[CF])$ for $CF \in CF_{(t_n,(n))}(X_n)$,

(vii) $\hat{s}[CF_1 \vee CF_2] := \hat{s}[CF_1] \vee \hat{s}[CF_2]$ for $CF_1, CF_2 \in CF_{(t_n,(n))}(X_n)$.

Then $\hat{s}$ is called the extension of s.

Now, we define a binary operation $\circ_r$ on $Hyp^C(t_n,(n))$ as follows:

Definition 3.3 Let $s_1, s_2 \in Hyp^C(t_n,(n))$ and $\circ$ be the usual composition of mapping. Then we define a binary operation $\circ_r$ on $Hyp^C(t_n,(n))$ by $s_1 \circ_r s_2 := \hat{s}_1 \circ s_2$.

Next, we prove that the binary operation as we already defined in Definition 3.3 satisfies associative law. To get our result, we need some preparations as follows:

Lemma 3.4 For $s \in Hyp^C(t_n,(n))$ and $b \in W_{t_n}^C(X_n) \cup CF_{(t_n,(n))}(X_n)$ we have

$\hat{s}[R^n(b, s_1, \dots, s_n)] = R^n(\hat{s}[b], \hat{s}[s_1], \dots, \hat{s}[s_n])$.

Proof The proof is straightforward and hence omitted.

As a result of Lemma 3.4, we have the following lemma.

Lemma 3.5 Let $s_1, s_2 \in Hyp^C(t_n,(n))$. Then we have $(s_1 \circ_r s_2)^\wedge = \hat{s}_1 \circ \hat{s}_2$.

Proof Let $n \in \mathbb{N}^+$ and $t \in W^{C}_{t_n}(X_n)$, we give a proof by induction on the complexity of an n-ary C-term t.

If $t = x_i$ with $i \in \{1, \ldots, n\}$, then $(s_1 \circ_r s_2)^\wedge[x_i] = x_i = \hat{s}_1[x_i] = \hat{s}_1[\hat{s}_2[x_i]] = (\hat{s}_1 \circ \hat{s}_2)[x_i]$.

If $t = e^k_j(w_1, \ldots, w_k)$ and assume that $(s_1 \circ_r s_2)^\wedge[w_i] = (\hat{s}_1 \circ \hat{s}_2)[w_i]$ for every $i \in \{1, \ldots, k\}$, then

$$
\begin{aligned}
(s_1 \circ_r s_2)^\wedge[e^k_j(w_1, \ldots, w_k)] &= e^k_j((s_1 \circ_r s_2)^\wedge[w_1], \ldots, (s_1 \circ_r s_2)^\wedge[w_k]) \\
&= e^k_j((\hat{s}_1 \circ \hat{s}_2)[w_1], \ldots, (\hat{s}_1 \circ \hat{s}_2)[w_k]) \\
&= e^k_j(\hat{s}_1[\hat{s}_2[w_1]], \ldots, \hat{s}_1[\hat{s}_2[w_k]]) \\
&= \hat{s}_1[e^k_j(\hat{s}_2[w_1], \ldots, \hat{s}_2[w_k])] \\
&= \hat{s}_1[\hat{s}_2[e^k_j(w_1, \ldots, w_k)]] \\
&= (\hat{s}_1 \circ \hat{s}_2)[e^k_j(w_1, \ldots, w_k)].
\end{aligned}
$$

If $t = f_i(t_1, \ldots, t_n)$ for any $i \in I$ and assume that $(s_1 \circ_r s_2)^\wedge[t_k] = (\hat{s}_1 \circ \hat{s}_2)[t_k]$ for every $k \in \{1, \ldots, n\}$, then

$$
\begin{aligned}
(s_1 \circ_r s_2)^\wedge[f_i(t_1, \ldots, t_n)] &= R^n((s_1 \circ_r s_2)(f_i), (s_1 \circ_r s_2)^\wedge[t_1], \ldots, (s_1 \circ_r s_2)^\wedge[t_n]) \\
&= R^n((\hat{s}_1 \circ s_2)(f_i), (\hat{s}_1 \circ \hat{s}_2)[t_1], \ldots, (\hat{s}_1 \circ \hat{s}_2)[t_n]) \\
&= R^n(\hat{s}_1[s_2(f_i)], \hat{s}_1[\hat{s}_2[t_1]], \ldots, \hat{s}_1[\hat{s}_2[t_n]]) \\
&= \hat{s}_1[R^n(s_2(f_i), \hat{s}_2[t_1], \ldots, \hat{s}_2[t_n])] \quad \text{(by Lemma 3.4)} \\
&= \hat{s}_1[\hat{s}_2[f_i(t_1, \ldots, t_n)]] \\
&= (\hat{s}_1 \circ \hat{s}_2)[f_i(t_1, \ldots, t_n)].
\end{aligned}
$$

For $\beta \in CF_{(t_n,(n))}(X_n)$. We will give a proof by induction on the complexity of an n-ary C-formula CF.

(i) If β has the form $s \approx t$, then

$$\widehat{(s_1 \circ_r s_2)}[s \approx t] = \widehat{(s_1 \circ_r s_2)}[s] \approx \widehat{(s_1 \circ_r s_2)}[t]$$

$$= (\hat{s}_1 \circ \hat{s}_2)[s] \approx (\hat{s}_1 \circ \hat{s}_2)[t]$$

$$= \hat{s}_1[\hat{s}_2[s]] \approx \hat{s}_1[\hat{s}_2[t]]$$

$$= \hat{s}_1[\hat{s}_2[s \approx t]]$$

$$= (\hat{s}_1 \circ \hat{s}_2)[s \approx t].$$

(ii) If β has the form $r(t_1, \ldots, t_n)$, then

$$\widehat{(s_1 \circ_r s_2)}[r(t_1, \ldots, t_n)] = R^n((s_1 \circ_r s_2)(r), \widehat{(s_1 \circ_r s_2)}[t_1], \ldots, \widehat{(s_1 \circ_r s_2)}[t_n])$$

$$= R^n((\hat{s}_1 \circ s_2)(r), (\hat{s}_1 \circ \hat{s}_2)[t_1], \ldots, (\hat{s}_1 \circ \hat{s}_2)[t_n])$$

$$= R^n(\hat{s}_1[s_2(r)], \hat{s}_1[\hat{s}_2[t_1]], \ldots, \hat{s}_1[\hat{s}_2[t_n]])$$

$$= \hat{s}_1[R^n(s_2(r), \hat{s}_2[t_1], \ldots, \hat{s}_2[t_n])] \text{ (by Lemma 3.4)}$$

$$= \hat{s}_1[\hat{s}_2[r(t_1, \ldots, t_n)]]$$

$$= (\hat{s}_1 \circ \hat{s}_2)[r(t_1, \ldots, t_n)].$$

(iii) If β has the form $\neg CF$ and if we assume that $\widehat{(s_1 \circ_r s_2)}[CF] = (\hat{s}_1 \circ \hat{s}_2)[CF]$, then

$$\widehat{(s_1 \circ_r s_2)}[\neg CF] = \neg(\widehat{(s_1 \circ_r s_2)}[CF])$$

$$= \neg((\hat{s}_1 \circ \hat{s}_2)[CF])$$

$$= \neg(\hat{s}_1[\hat{s}_2[CF]])$$

$$= \hat{s}_1[\neg(\hat{s}_2[CF])]$$

$$= \hat{s}_1[\hat{s}_2[\neg CF]]$$

$$= (\hat{s}_1 \circ \hat{s}_2)[\neg CF].$$

(iv) If b has the form $CF_1 \vee CF_2$ and if we assume that $(s_1 \circ_r s_2)^\wedge[CF_i] = (\hat{s}_1 \circ \hat{s}_2)[CF_i]$ for all $i = 1, 2$, then

$$(s_1 \circ_r s_2)^\wedge[CF_1 \vee CF_2] = (s_1 \circ_r s_2)^\wedge[CF_1] \vee (s_1 \circ_r s_2)^\wedge[CF_2]$$

$$= (\hat{s}_1 \circ \hat{s}_2)[CF_1] \vee (\hat{s}_1 \circ \hat{s}_2)[CF_2]$$

$$= \hat{s}_1[\hat{s}_2[CF_1]] \vee \hat{s}_1[\hat{s}_2[CF_2]]$$

$$= \hat{s}_1[\hat{s}_2[CF_1 \vee CF_2]]$$

$$= (\hat{s}_1 \circ \hat{s}_2)[CF_1 \vee CF_2].$$

It follows from Lemma 3.5 that the binary operation $\circ_r$ satisfies the associative law. We prove this fact in the next lemma.

Lemma 3.6 Let $s_1, s_2, s_3 \in Hyp^C(t_n,(n))$. Then we have $(s_1 \circ_r s_2) \circ_r s_3 = s_1 \circ_r (s_2 \circ_r s_3)$.

Proof By using Lemma 3.5 and the fact that $\circ$ satisfies associative law, it can be shown that $\circ_r$ satisfies associative law. In fact, we have

$$(s_1 \circ_r s_2) \circ_r s_3 = (s_1 \circ_r s_2)^\wedge \circ s_3$$

$$= (\hat{s}_1 \circ \hat{s}_2) \circ s_3$$

$$= \hat{s}_1 \circ (\hat{s}_2 \circ s_3)$$

$$= \hat{s}_1 \circ (s_2 \circ_r s_3)$$

$= s_1 \circ_r (s_2 \circ_r s_3)$.

Let s_{id} be a C -hypersubstitution for algebraic systems of type $(t_n,(n))$ which maps the operation symbol f_i to the n -ary C -term $f_i(x_1,\dots,x_n)$ for all $i \in I$, and maps the relation symbol r to the n -ary C -formulas $r(x_1,\dots,x_n)$, i.e. $s_{id}(f_i) = f_i(x_1,\dots,x_n)$ for all $i \in I$ and $s_{id}(r) = r(x_1,\dots,x_n)$.

Lemma 3.7 For any $b \in W^C_{t_n}(X_n) \cup CF_{(t_n,(n))}(X_n)$, we have $\hat{s}_{id}[b] = b$.

Proof The proof is straightforward and hence omitted.

A C -hypersubstitution s_{id} is claimed to be an identity, which we will prove this fact in the next lemma.

Lemma 3.8 Let $s_{id} \in Hyp^C(t_n,(n))$. Then s_{id} is an identity element with respect to $\circ_r$.

Proof First, we prove that s_{id} is a left identity element by using Lemma 3.7. Let $s \in Hyp^C(t_n,(n))$ and $b \in \{f_i \mid i \in I\} \cup \{r\}$. Then we have $(s_{id} \circ_r s)(b)$ $= (\hat{s}_{id} \circ s)(b) = \hat{s}_{id}[s(b)] = s(b)$. Now, we show that s_{id} is a right identity element. Let $s \in Hyp^C(t_n,(n))$. By Theorem 2.4 (CFC3), if $i \in I$, then $(s \circ_r s_{id})(f_i) = (\hat{s} \circ s_{id})(f_i) = \hat{s}[s_{id}(f_i)] = \hat{s}[f_i(x_1,\dots,x_n)] = R^n(s(f_i),x_1,\dots,x_n) = s(f_i)$ and

$$(s \circ_r s_{id})(r) = (\hat{s} \circ s_{id})(r) = \hat{s}[s_{id}(r)] = \hat{s}[r(x_1,\dots,x_n)] = R^n(s(r),x_1,\dots,x_n) = s(r).$$

Therefore, $s \circ_r s_{id} = s = s_{id} \circ_r s$.

Theorem 3.9 $yp^C(t_n,(n)) := (Hyp^C(t_n,(n)),\circ_r, s_{id})$ is a monoid.

Proof From Lemma 3.6 and 3.8, the conclusion holds.

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