Partial Algebraic Systems of type \((t_n,(n))\)

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Abstract

In this paper, we define the set \((\mathcal{CF}_{(t_n,(n))}(X_n))^{A_s}\) of all \(n\)-ary C-formulas on the partial algebraic system \(A_s = (A; (f_i^A), f, r^A)\) of type \((t_n,(n))\) and define the operation \(R_{n,A}^{\text{rank}}\) on the set \((\mathcal{W}_{t_n}^C(X_n))^{A_s} \hat{\times} (\mathcal{CF}_{(t_n,(n))}(X_n))^{A_s}\). After this definition we have a unitary Menger algebra \((\mathcal{W}_{t_n}^C(X_n))^{A_s} \hat{\times} (\mathcal{CF}_{(t_n,(n))}(X_n))^{A_s} ; R_{n,A}, x_1^{A_s}, ..., x_n^{A_s}\) of rank \(n\). Finally, we show that the set of all C-hypersubstitutions for an algebraic system of the type \((t_n,(n))\) with a binary operation on this set and the identity element forms a monoid.

Keywords: term, superposition, unitary Menger algebra of rank \(n\).
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1. Introduction

Let \( P^n(A) := \{ f^A : A^n \to A \} \) be the set of all \( n \)-ary partial operations defined on the non-empty set \( A \) and let \( P(A) := \bigcup_{n=1}^{\infty} P^n(A) \) be the set of all partial operations on \( A \). A partial algebra \( A = (A; (f^A_i)_{i \in \mathbb{I}}) \) of type \( t = (n_i)_{i \in \mathbb{I}} \) is a pair consisting of a set \( A \) and an indexed set \((f^A_i)_{i \in \mathbb{I}}\) of partial operations where \( f^A_i \) is \( n_i \)-ary. Let \( PA_{lg}(t) \) be the class of all partial algebras of type \( t \).

For \( n \in \mathbb{I} \) and \( t = (n_i)_{i \in \mathbb{I}} \), we define the superposition operation \( S^{n,\mathbb{I}}: (P^n(A))^n \to P^n(A) \)

\[
S^{n,\mathbb{I}}(f^A, g^A_1, ..., g^A_n)(a_{i_1}, ..., a_{i_n}) := f^A(g^A_{i_1}(a_{i_1}, ..., a_{i_n}), ..., g^A_{i_n}(a_{i_1}, ..., a_{i_n}))
\]

Here \( (a_{i_1}, ..., a_{i_n}) \in \text{dom} S^{n,\mathbb{I}}(f^A, g^A_1, ..., g^A_n) \) iff \( (a_{i_1}, ..., a_{i_n}) \in \bigcap_{j=1}^{n} \text{dom} f^A_{i_j} \) and for all \( j = 1, ..., n \), \( g^A_{i_j}(a_{i_1}, ..., a_{i_n}) = b_{i_j} \), we have \( (b_{i_1}, ..., b_{i_n}) \in \text{dom} f^A_{i_1} \), i.e.

\[
\text{dom} S^{n,\mathbb{I}}(f^A, g^A_1, ..., g^A_n) := \{(a_{i_1}, ..., a_{i_n}) \in A^n | (a_{i_1}, ..., a_{i_n}) \in \bigcap_{j=1}^{n} \text{dom} f^A_{i_j} \text{ and for all } j = 1, ..., n, \quad g^A_{i_j}(a_{i_1}, ..., a_{i_n}) = b_{i_j}, \text{ we have } (b_{i_1}, ..., b_{i_n}) \in \text{dom} f^A_{i_1}\}
\]

Special \( n \)-ary (total) operations are the projections to the \( i \)-th component, where \( \mathbf{1} \in \mathbb{I} \): \( n \) \n
\[
e^{n,\mathbb{I}} : A^n \circ A \quad \text{by} \quad e^{n,\mathbb{I}}(a_{i_1}, ..., a_{i_n}) := a_{i_j}.
\]

Definition 1.1 ([7]) An algebra \((M; S^n, e_{i_1}, ..., e_{i_n})\) of type \( t = (n + 1, 0, ..., 0) \) is called a unitary Menger algebra of rank \( n \) if it satisfies the axioms (C1), (C2), (C3):

\[
(C1) \quad S^n(X_{i_1}, S^n(Y_{i_1}, X_{i_1}, ..., X_{i_n}), ..., S^n(Y_{i_n}, X_{i_1}, ..., X_{i_n})) \in S^n(S^n(X_{i_1}, Y_{i_1}, ..., Y_{i_n}), X_{i_1}, ..., X_{i_n}),
\]

63
(C2) \[ S^n(l_j, x_1, \ldots, x_n) \equiv x_j \text{ for } 1 \leq j \leq n, \]

(C3) \[ S^n(x_j, l_1, \ldots, l_n) \equiv x_j \text{ for } 1 \leq j \leq n. \]

(Here \( S^n \) is an \((n+1)\)-ary operation symbol, \( l_1, \ldots, l_n \) are nullary operation symbols and \( x_0, x_1, \ldots, x_n, y_1, \ldots, y_n \) are variables.) An algebra \((M; S^n)\) of type \( t = (n+1) \) which satisfies (C1) is called a Menger algebra of rank \( n \).

Theorem 1.2 The algebra \((P^n(A); S^{n,A})\) is a Menger algebra of rank \( n \).

We have proven already that the algebra \((P^n(A); S^{n,A})\) satisfies the axiom (C1) (see [4]).

Let \( W_t(x_n) \) be the set of all \( n \)-ary terms of type \( t \) and let \( \omega_t(x_n)^A \) be the set of all \( n \)-ary term operations induced by \( n \)-ary terms on the partial algebra \( A \). For the definition of a term operation \( t^A \) induced by the term \( t \) on the partial algebra \( A \) (see [11],[1]). Different from the total case, the set \( \omega_t(x_n)^A \) is, in general, a proper subset of \( P^n(A) \) (see some examples in [11]). In 1989, W. Craig [6] introduced concept of terms for partial algebras by additional symbols \( e \) and in 1996, F. Börner [1] introduced another concept of terms for partial algebras.

Let \( X \) be an alphabet and let \( \{ f_i | i \in I \} \) be a set of operation symbols of type \( t \), where each \( f_i \) has arity \( n_i \) and \( X \cup \{ f_i | i \in I \} = \mathcal{AE} \). We need additional symbols \( e^k \) for every \( k \in \mathbb{N}^* := \mathbb{N} \setminus \{0\} \) and \( 1 \leq j \leq k \). Let \( X_n = \{ x_1, \ldots, x_n \} \) be an \( n \)-element alphabet. The set of all \( n \)-ary \( C \)-terms of type \( t \) over \( X_n \) is defined inductively as follows (see [1]):

(i) every \( x_j \) is an \( n \)-ary \( C \)-term of type \( t \);

(ii) if \( w_1, \ldots, w_k \) are \( n \)-ary \( C \)-terms of type \( t \), then \( e^k(w_1, \ldots, w_k) \) is an \( n \)-ary \( C \)-term of type \( t \) for all \( 1 \leq j \leq k \) and all \( k \in \mathbb{N}^* \).
(iii) If $w_1, \ldots, w_n$ are $n$-ary $C$-terms of type $t$, and if $f_i$ is an $n_i$-ary operation symbol, then

$$f_i(w_1, \ldots, w_{n_i})$$

is an $n$-ary $C$-term of type $t$.

Let $W_t^C(X_n)$ be the set of all $n$-ary $C$-terms of type $t$ defined in this way. Then $W_t^C(X) := \bigcup_{n=1}^\infty W_t^C(X_n)$ denotes the set of all $C$-terms of this type.

Now we consider a type $t_n = (n, n, \ldots, n)$ consisting of $n$-ary operation symbols only.

On the sets $W_t^C(X_n)$ we introduce the following superposition operations. Let $t, w_1, \ldots, w_n$ be $n$-ary $C$-terms. Then we define an $n$-ary $C$-term $S^n(t, w_1, \ldots, w_n)$ inductively by the following steps:

(i) For $t = x_j, 1 \notin \mathcal{E} j \notin n$, we define $S^n(x_j, w_1, \ldots, w_n) = w_j$.

(ii) For $t = e^k_j(s_1, \ldots, s_k)$ we set $S^n(t, w_1, \ldots, w_n) = e^k_j(S^n(s_1, w_1, \ldots, w_n), \ldots, S^n(s_k, w_1, \ldots, w_n))$,

where $s_1, \ldots, s_k$ are $n$-ary $C$-terms, for all $k \notin \mathcal{Y}^* \text{ and } 1 \notin \mathcal{E} j \notin k$.

(iii) For $t = f_i(s_1, \ldots, s_n)$ we set $S^n(t, w_1, \ldots, w_n) = f_i(S^n(s_1, w_1, \ldots, w_n), \ldots, S^n(s_n, w_1, \ldots, w_n))$,

where $s_1, \ldots, s_n$ are $n$-ary $C$-terms.

This defines an operation $S^n : (W_{t_n}^C(X_n))^{n+1} \otimes W_{t_n}^C(X_n)$, which describes the superposition of terms. Together with the nullary operations $x_1, \ldots, x_n$ one obtains an algebra

$$n - \text{clone}_{t_n}^C := (W_{t_n}^C(X_n); S^n, x_1, \ldots, x_n).$$

Theorem 1.3 The algebra $n - \text{clone}_{t_n}^C$ is a unitary Menger algebra of rank $n$.

We have proven already that the algebra $n - \text{clone}_{t_n}^C$ satisfies the axiom (C1), (C2), (C3) (see [5]).
Every \( n \)-ary \( C \)-term \( W^C_{t_n}(x_n) \) induces an \( n \)-ary \( C \)-term operation \( W^A \) of any partial algebra \( A = (A; (f^A_i)_{i \in I}) \) of type \( t_n \). For \( a_1, \ldots, a_n \), the value \( W^A(a_1, \ldots, a_n) \) is defined in the following inductive way:

(i) If \( w = x_i \) then \( W^A = x^A_i = e^n_i \), where \( e^n_i \) is as usual the \( n \)-ary total projection on the \( i \)-th component.

(ii) If \( w = e^k_j(w_1, \ldots, w_k) \) and we assume that \( W^A_1, \ldots, W^A_k \) are the term operations induced by the terms \( w_1, \ldots, w_k \) and that the \( W^A_i(a_1, \ldots, a_n) \) are defined for \( 1 \leq i \leq k \), then

\[
W^A(a_1, \ldots, a_n) = W^A_i(a_1, \ldots, a_n).
\]

(iii) Now assume that \( w = f_j(w_1, \ldots, w_n) \) and that the \( W^A_i(a_1, \ldots, a_n) \) are defined, with values

\[
w^A_j(a_1, \ldots, a_n) = b_j \quad \text{for} \quad 1 \leq j \leq n. \quad \text{If} \quad f^A_j(b_1, \ldots, b_n) \quad \text{is defined, then} \quad W^A(a_1, \ldots, a_n) \quad \text{is defined}
\]

and

\[
W^A(a_1, \ldots, a_n) := S^n(f^A_j, W^A_i(a_1, \ldots, a_n), \ldots, W^A_n(a_1, \ldots, a_n)).
\]

Let \( W^C_{t_n}(X)^A \) be the set of all \( n \)-ary \( C \)-term operations induced by the \( n \)-ary \( C \)-terms from \( W^C_{t_n}(X_n)^A \) on the partial algebra \( A \) and

\[
W^C_{t_n}(X)^A := \bigcup_{n=1} W^C_{t_n}(X_n)^A.
\]

Theorem 1.4 The algebra \( (W^C_{t_n}(X)^A; S^n, e^n_1, \ldots, e^n_n) \) is a unitary Menger algebra of rank \( n \) (i.e. \( (W^C_{t_n}(X_n)^A; S^n, e^n_1, \ldots, e^n_n) \) satisfies the following identities:

\[
(C\ 1) \quad S^n(A(s^n_1, s^n_2, \ldots, s^n_n), \ldots, S^n(A(s^n_1, s^n_2, \ldots, s^n_n)) = S^n(A(s^n_1, s^n_2, \ldots, s^n_n), s^n_1, \ldots, s^n_n),
\]

\[
(C\ 2) \quad S^n(e^n_i, t^n_1, \ldots, t^n_n) = t^n_i \quad \text{for} \quad 1 \leq i \leq n,
\]

\[
(C\ 3) \quad S^n(t^n_1, e^n_1, \ldots, e^n_n) = t^n
\]
2. Partial Algebraic Systems

The concept of an algebraic system was first introduced by A.I. Malcev in 1973 [9]. We now recall the informal definition of algebraic systems. An algebraic system is a structure consisting of a non-empty set together with a sequence of operations and a sequence of relations on this set. In 2002, P. Burmeister [3] introduced the concept of a congruence relation on partial algebra. Using the definition of algebraic systems and of relation on a set of partial algebra, we defined the new concept of the partial algebraic system.

**Definition 2.1** A partial algebraic system of type \( (t_n, (n)) \) is a triple \( A^s = (A; (f^A_i), l, r^A) \) consisting of a non-empty set \( A \), a sequence \( (f^A_i)_{i \in l} \) of partial operations defined on \( A \) indexed by the index set \( I \) where \( f^A_i \) is \( n \)-ary for \( i \in I \) and \( r^A \) of \( n \)-ary relation on \( A \), which is compatible with all the partial operations \( (f^A_i)_{i \in I} \), i.e.

\[
\text{if } \begin{pmatrix} a_1^{(1)}, a_1^{(2)}, \ldots, a_1^{(n)} \end{pmatrix} \text{ r}^A, \ldots, \begin{pmatrix} a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(n)} \end{pmatrix} \text{ r}^A \text{ and if all}
\]

\[
\begin{pmatrix} a_1^{(1)}, a_2^{(1)}, \ldots, a_n^{(1)} \end{pmatrix}, \ldots, \begin{pmatrix} a_1^{(n)}, a_2^{(n)}, \ldots, a_n^{(n)} \end{pmatrix} \in \text{dom } f^A_i
\]

then \( (f^A_i(a_1^{(1)}, a_2^{(1)}, \ldots, a_n^{(1)}), \ldots, f^A_i(a_1^{(n)}, a_2^{(n)}, \ldots, a_n^{(n)})) \text{ r}^A \).

Written in matrix notation

\[
\begin{pmatrix} a_1^{(1)}, a_1^{(2)}, \ldots, a_1^{(n)} \end{pmatrix} \text{ r}^A
\]

\[
\wedge
\]

\[
\begin{pmatrix} a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(n)} \end{pmatrix} \text{ r}^A
\]

\[
\begin{pmatrix} a_1^{(1)}, a_2^{(1)}, \ldots, a_n^{(1)} \end{pmatrix} \in \text{dom } f^A_i, \ldots, \begin{pmatrix} a_1^{(n)}, a_2^{(n)}, \ldots, a_n^{(n)} \end{pmatrix} \in \text{dom } f^A_i
\]

\[
(f^A_i(a_1^{(1)}, a_2^{(1)}, \ldots, a_n^{(1)}), \ldots, f^A_i(a_1^{(n)}, a_2^{(n)}, \ldots, a_n^{(n)})) \text{ r}^A.
\]
Let $\mathcal{PA_{lgsys}}(t_1'(n))$ be the class of all partial algebraic systems of type $(t_1'(n))$. Every $n$-ary $A$-term $w^\mathcal{A}_n(\bar{x}_n)$ induces an $n$-ary $A$-term operation $w^\mathcal{A}_s$ of any partial algebraic system $A = (A; (f_i^A)_{i \in I}, r^A)$ of type $(t_1'(n))$. For $a_1, \ldots, a_n \in A$, the value $w^\mathcal{A}_s(a_1, \ldots, a_n) = w^A(a_1, \ldots, a_n)$ when $A = (A; (f_i^A)_{i \in I})$ is partial algebra of type $t_n$.

Not all of the terms in the second-order language will be used to express the properties of algebraic systems. The one is called formulas, first introduced by A.I. Mal’cev in 1973. In 1986, P. Burmeister [2] introduced the concept of formulas of the language by using the logical connectives. In 2013, K. Denecke and D. Phusanga [10] introduced the concept of formulas by using terms, the logical connective $\varnothing$ (for negation), $\mathcal{U}$ (for disjunction), and the equation symbol $\Rightarrow$. So, we introduce the concept of a $C$-formulas of type $(t_1'(n))$ by using $n$-ary $C$-terms of type $t_n$.

**Definition 2.2** Let $n^A \geq 1$. An $n$-ary $C$-formula of type $(t_1'(n))$ is defined in the following inductive way:

(i) If $t_1, t_2$ are $n$-ary $C$-terms of type $t_n$, then the equation $t_1 \Rightarrow t_2$ is an $n$-ary $C$-formula of type $(t_1'(n))$.

(ii) If $t_1, \ldots, t_n$ are $n$-ary $C$-terms of type $t_n$, then $r(t_1, \ldots, t_n)$ is an $n$-ary $C$-formula of type $(t_1'(n))$.

(iii) If $CF$ is an $n$-ary $C$-formula of type $(t_1'(n))$, then $\varnothing CF$ is an $n$-ary $C$-formula of type $(t_1'(n))$.

(iv) If $CF_1$ and $CF_2$ are an $n$-ary $C$-formulas of type $(t_1'(n))$, then $CF_1 \mathcal{U} CF_2$ is an $n$-ary $C$-formula of type $(t_1'(n))$. 

68
Let $CF_{(t_n, (n))}(X_n)$ be the set of all $n$-ary $C$-formula of type $(t_n, (n))$.

We want to extend a superposition to $n$-ary $C$-formulas. If we substitute variables occurring in an $n$-ary $C$-formula by terms we obtain a new $C$-formula. We want to describe this by the following operations $R^n, n \geq 1$.

**Definition 2.3** The operations

$$R^n : (W^C_{t_n}(X_n)) \ast CF_{(t_n, (n))}(X_n)) \ast W^C_{t_n}(X_n)^n \ast W^C_{t_n}(X_n) \ast CF_{(t_n, (n))}(X_n)$$

where $n \in \mathbb{N}^+$, are defined by the following inductive steps: Let $s_1, \ldots, s_n \in W^C_{t_n}(X_n)$.

(i) If $t \in W^C_{t_n}(X_n)$, then we define $R^n(t, s_1, \ldots, s_n) := S^n(t, s_1, \ldots, s_n)$ (which is the superposition of terms).

(ii) If $t_1, t_2 \in W^C_{t_n}(X_n)$, then $R^n(t_1 \cdot t_2, s_1, \ldots, s_n) := R^n(t_1, s_1, \ldots, s_n) \ast R^n(t_2, s_1, \ldots, s_n)$.

(iii) If $t_1, \ldots, t_n \in W^C_{t_n}(X_n)$, then

$$R^n(r(t_1, \ldots, t_n), s_1, \ldots, s_n) := r(R^n(t_1, s_1, \ldots, s_n), \ldots, R^n(t_n, s_1, \ldots, s_n)).$$

(iv) If $CF \in CF_{(t_n, (n))}(X_n)$, then $R^n(\emptyset CF, s_1, \ldots, s_n) := \emptyset R^n(CF, s_1, \ldots, s_n)$.

(v) If $CF_1, CF_2 \in CF_{(t_n, (n))}(X_n)$, then

$$R^n(CF_1 \cup CF_2, s_1, \ldots, s_n) := R^n(CF_1, s_1, \ldots, s_n) \cup R^n(CF_2, s_1, \ldots, s_n).$$

These operations define an algebra

$$\mathcal{C} \cdot Formclone(t_n, (n)) := (CF_{(t_n, (n))}(X_n) \ast W^C_{t_n}(X_n)) \ast R^n, x_1 \ldots x_n$$

which is called the $\mathcal{C}$-formula-term clone of type $(t_n, (n))$.

This algebra keeps the properties of the term clones.
Theorem 2.4 The algebra $C - \text{Formclone}(t_n, (n))$ is a unitary Menger algebra of rank $n$. (i.e. the algebra $C - \text{Formclone}(t_n, (n))$ satisfies the following identities:

(CFC1) $R^n(b, R^n(t_1, s_1, ..., s_n), ..., R^n(t_n, s_1, ..., s_n))$

$$R^n(R^n(b, t_1, ..., t_n), s_1, ..., s_n) \text{ whenever } s_1, ..., s_n, t_1, ..., t_n \hat{\in} W^c_{t_n}(X_n),$$

(CFC2) $R^n(x_1, s_1, ..., s_n) \to s_i \text{ whenever } s_1, ..., s_n \hat{\in} W^c_{t_n}(X_n)$ for $1 \leq i \leq n$.

(CFC3) $R^n(b, x_1, ..., x_n) \to b$

Proof: If $b \hat{\in} W^c_{t_n}(X_n)$, the assertion is clear by (C1). Now let $b \hat{\in} CF_{(t_n, (n))}(X_n)$. We give a proof of (CFC1) by induction on the complexity of an $n$-ary $C$-formula as the following the steps:

(i) If $CF$ have the form $s \to t$, then

$$R^n(s \to t, R^n(t_1, s_1, ..., s_n), ..., R^n(t_n, s_1, ..., s_n))$$

$$= R^n(s, R^n(t_1, s_1, ..., s_n), ..., R^n(t_n, s_1, ..., s_n)) \to R^n(t, R^n(t_1, s_1, ..., s_n), ..., R^n(t_n, s_1, ..., s_n))$$

$$= R^n(R^n(s, t_1, ..., t_n), s_1, ..., s_n) \to R^n(R^n(t, t_1, ..., t_n), s_1, ..., s_n)$$

$$= R^n(R^n(s \to t_1, ..., t_n), s_1, ..., s_n)$$

$$= R^n(R^n(s \to t_1, ..., t_n), s_1, ..., s_n).$$

(ii) If $CF$ have the form $r(p_1, ..., p_n)$, then

$$R^n(r(p_1, ..., p_n), R^n(t_1, s_1, ..., s_n), ..., R^n(t_n, s_1, ..., s_n))$$

$$= r(R^n(p_1, R^n(t_1, s_1, ..., s_n), ..., R^n(t_n, s_1, ..., s_n)), ..., R^n(p_n, R^n(t_1, s_1, ..., s_n), ..., R^n(t_n, s_1, ..., s_n)))$$

$$= r(R^n(p_1, t_1, ..., t_n), s_1, ..., s_n), ..., R^n(p_n, t_1, ..., t_n), s_1, ..., s_n)$$

$$= R^n(r(R^n(p_1, t_1, ..., t_n), ..., R^n(p_n, t_1, ..., t_n)), s_1, ..., s_n).$$
= R^n(R^n(r(p_1, ..., p_n), t_1, ..., t_n), s_1, ..., s_n).

(iii) We have to show that if (CFC1) is satisfied for the C-formula CF, then it is also satisfied for \( \emptyset CF \). In fact,

\[
R^n(\emptyset CF, R^n(t_1, s_1, ..., s_n), ..., R^n(t_n, s_1, ..., s_n))
\]

= \( \emptyset(R^n(CF, R^n(t_1, s_1, ..., s_n), ..., R^n(t_n, s_1, ..., s_n))) \)

= \( \emptyset(R^n(CF, t_1, ..., t_n), s_1, ..., s_n) \)

= \( R^n(\emptyset(CF, t_1, ..., t_n), s_1, ..., s_n) \)

= \( R^n(\emptyset CF, t_1, ..., t_n), s_1, ..., s_n) \).

(iv) Here we have to show that if \( CF_1 \) and \( CF_2 \) satisfy (CFC1), then also \( CF_1 \cup CF_2 \) satisfies (CFC1). In fact,

\[
R^n(CF_1 \cup CF_2, R^n(t_1, s_1, ..., s_n), ..., R^n(t_n, s_1, ..., s_n))
\]

= \( R^n(CF_1, R^n(t_1, s_1, ..., s_n), ..., R^n(t_n, s_1, ..., s_n)) \cup R^n(CF_2, R^n(t_1, s_1, ..., s_n), ..., R^n(t_n, s_1, ..., s_n)) \)

= \( R^n(R^n(CF_1, t_1, ..., t_n), s_1, ..., s_n)) \cup R^n(R^n(CF_2, t_1, ..., t_n), s_1, ..., s_n)) \)

= \( R^n(R^n(CF_1, t_1, ..., t_n), s_1, ..., s_n)) \cup R^n(R^n(CF_2, t_1, ..., t_n), s_1, ..., s_n)) \)

= \( R^n(R^n(CF_1 \cup CF_2, t_1, ..., t_n), s_1, ..., s_n) \).

This finishes the proof of (CFC1).

(CFC2) is clearly by (C2) and the fact that \( R^n(x_1, s_1, ..., s_n) = S^n(x_1, s_1, ..., s_n) \).

(CFC3) If \( b \xi W^C_{n,t}(X^n) \), then the assertion is clear by (C3). It is left to consider the case that \( b \) is an \( n \)-ary C-formula CF. We will proceed in a similar way considering the completely of an \( n \)-ary C-formula CF.

(i) If \( CF \) is an equation \( s \equiv t \), then
\[ R^n(s \cdot t, x_1, ..., x_n) = R^n(s, x_1, ..., x_n) \cdot R^n(t, x_1, ..., x_n) \]

\[ = S^n(s, x_1, ..., x_n) \cdot S^n(t, x_1, ..., x_n) \]

\[ = s \cdot t \text{ by } (C3). \]

(ii) If \( CF \) have the form \( r(t_1, ..., t_n) \), then

\[ R^n(r(t_1, ..., t_n), x_1, ..., x_n) = r(R^n(t_1, x_1, ..., x_n), ..., R^n(t_n, x_1, ..., x_n)) = r(t_1, ..., t_n). \]

(iii) Assume that \( (CFC3) \) is satisfied for \( CF \). We show that it is also satisfied for \( \emptyset CF \).

By definition \( R^n(\emptyset CF, x_1, ..., x_n) \) is the formula \( \emptyset R^n(CF, x_1, ..., x_n) \) i.e. it is equal to \( \emptyset CF \).

(iv) Assume that \( (CFC3) \) is satisfied for \( CF_1 \) and \( CF_2 \). Then \( R^n(CF_1 \cup CF_2, x_1, ..., x_n) \) is the formula \( R^n(CF_1, x_1, ..., x_n) \cup R^n(CF_2, x_1, ..., x_n) \) and this is to equal \( CF_1 \cup CF_2 \).

Now we will define the realization of an \( n \)-ary \( C \)-formula of type \( (t_n(n)) \) on the partial algebraic system \( A^s \) of the same type.

**Definition 2.5** Let \( A = (A; (t^A_i)_i) \) be a partial algebra of type \( t_n \) and \( CF \) be an \( n \)-ary \( C \)-formula of type \( (t_n(n)) \). Then the realization of \( CF \) on the partial algebraic system \( A^s = (A; (t^A_i)_i, (r^A)) \) denoted by \( CF \upharpoonright A^s \) is defined as follows:

(i) If \( CF \) has the form \( t_1 \gg t_2 \), then

\[ (t_1 \gg t_2)^s \upharpoonright A^s = \{(a_1, ..., a_n) \mid t^A_1(a_1, ..., a_n) \text{ and } t^A_2(a_1, ..., a_n) \text{ both exist and} \]

\[ t^A_1(a_1, ..., a_n) = t^A_2(a_1, ..., a_n) \} \]

(ii) If \( CF \) has the form \( r(t_1, ..., t_n) \), then
\[(r(t_1, \ldots, t_n))^{A^g} := \{(a_{i_1}, a^g_{i_1}) \mid (a_{i_1}, a^g_{i_1}) \in \text{dom} t^A_i \text{ for } i = 1, \ldots, n \text{ and }\]
\[(t^A_i(a_{i_1}, \ldots, a^g_{i_1}), \ldots, t^A_n(a_{i_1}, \ldots, a^g_{i_1})) \in r^A\}.
\]

(iii) If \(CF \hat{=} CF_{(t_n(n))}(X_n)\) and suppose that \(CF^{A^g}\) is already defined, then
\[(\emptyset CF)^{A^g} := \{(a_{i_1}, a^g_{i_1}) \mid (a_{i_1}, a^g_{i_1}) \in \text{dom} \ CF^{A^g}\}.
\]

(iv) If \(CF_1 \hat{=} CF_{(t_n(n))}(X_n)\) and suppose that \(CF_1^{A^g}\) and \(CF_2^{A^g}\) are already defined, then
\[(CF_1 \cup CF_2)^{A^g} := \{(a_{i_1}, a^g_{i_1}) \mid (a_{i_1}, a^g_{i_1}) \in \text{dom} \ CF_1^{A^g} \text{ or } (a_{i_1}, a^g_{i_1}) \in \text{dom} \ CF_2^{A^g}\}.
\]

Let \((CF_{(t_n(n))}(X_n))^{A^g} := \{CF \hat{=} CF_{(t_n(n))}(X_n)\}\) be the set of all "realizations" of \(n\)-ary \(C\)-formulas on the partial algebraic system \(A^g = (A, (f^A_i)_{i \in I}, r^A)\) and let
\[(CF_{(t_n(n))}(X_n))^{A^g} := \bigcup_{n^+} (CF_{(t_n(n))}(X_n))^{A^g}\]
be the set of all "realizations" of \(C\)-formulas on the partial algebraic system \(A^g = (A, (f^A_i)_{i \in I}, r^A)\).

Extending this idea to algebraic systems we can define an operation
\[R^{n,A} : (W^C_{t_n}(X_n))^{A^g} \rightarrow (W^C_{t_n}(X_n))^{A^g} \hat{=} (CF_{(t_n(n))}(X_n))^{A^g} \hat{=} (CF_{(t_n(n))}(X_n))^{A^g}\]
as follows:

**Definition 2.6** For any \(b^{A^g} \hat{=} (W^C_{t_n}(X_n))^{A^g}\) and \(n\)-tuple \((t^A_i, \ldots, t^A_n)\) of \(n\)-ary \(C\)-term operations, we define
\[R^{n,A}(b^{A^g}, t^A_1, \ldots, t^A_n) := S^{n,A}(b^{A^g}, t^A_1, \ldots, t^A_n) \hat{=} (W^C_{t_n}(X_n))^{A^g}\]
and if \( b^A \vdash (CF(t_n,(n))(X_n))^A \), we define

\[ R^{n,A}(b^A,t^A_1,...,t^A_n) = R^{n,A}(b^A,t^A_1,...,t^A_n) = (\{a_1,...,a_n\} \vdash A_n \setminus \{a_1,...,a_n\} \vdash \prod_{i=1}^n \text{dom } t^A_i \) and

\[ \{t^A_1(a_1,...,a_n),...,t^A_n(a_1,...,a_n)\} \vdash b^A. \]

**Lemma 2.7** For \( b \vdash W^C_{t_n}(X_n) \in CF(t_n,(n))(X_n) \) and \( s_1,...,s_n \vdash W^C_{t_n}(X_n) \), we have

\[ R^{n,A}(b^A,s^A_1,...,s^A_n) = (R^n(b,s,...,s))^A. \]

**Proof** In the first case \( b = t \vdash W^C_{t_n}(X_n) \). We can be proved that

\[ R^{n,A}(b^A,s^A_1,...,s^A_n) = S^{n,A}(t^A,s^A_1,...,s^A_n) = (S^n(t,s,...,s))^A = (R^n(b,s,...,s))^A \]

by induction on the complexity of the \( n \)-ary \( C \)-term \( t \).

In the second case \( b = CF \vdash CF(t_n,(n))(X_n) \). We show it by induction on the complexity of \( n \)-ary \( C \)-formula \( CF \). Let \( s_1,...,s_n \vdash W^C_{t_n}(X_n) \).

(i) IF \( CF \) has the form \( t_1 \triangleright t_2 \), then \( (a_1,...,a_n) \vdash R^{n,A}((t_1 \triangleright t_2)^A,s^A_1,...,s^A_n) \)

\[ (a_1,...,a_n) \vdash \prod_{i=1}^n \text{dom } s^A_i \]

and

\[ (s^A_1(a_1,...,a_n),...,s^A_n(a_1,...,a_n)) \vdash (t_1 \triangleright t_2)^A. \]

\[ t^A_1(s^A_1(a_1,...,a_n),...,s^A_n(a_1,...,a_n)) \] exist for \( i = 1,2 \) and

\[ t^A_1(s^A_1(a_1,...,a_n),...,s^A_n(a_1,...,a_n)) = t^A_2(s^A_1(a_1,...,a_n),...,s^A_n(a_1,...,a_n)) \] (by Definition 2.5 (i))

\[ S^{n,A}(t^A_1,s^A_1,...,s^A_n)(a_1,...,a_n) \] exist for \( i = 1,2 \) and

\[ S^{n,A}(t^A_1,s^A_1,...,s^A_n)(a_1,...,a_n) = S^{n,A}(t^A_2,s^A_1,...,s^A_n)(a_1,...,a_n) \]
\[ R^{n,A}(t^A_i, s^A_1, ..., s^A_n)(a_1, ..., a_n) \] exist for \( i = 1, 2 \) and

\[ R^{n,A}(t^A_1, s^A_1, ..., s^A_n)(a_1, ..., a_n) = R^{n,A}(t^A_2, s^A_1, ..., s^A_n)(a_1, ..., a_n) \]

\[ (R^n(t, s_1, ..., s_n))^A(a_1, ..., a_n) \] exist for \( i = 1, 2 \) and

\[ (R^n(t_1, s_1, ..., s_n))^A(a_1, ..., a_n) = (R^n(t_2, s_1, ..., s_n))^A(a_1, ..., a_n) \]

\[ (a_1, ..., a_n) \overset{1}{\rightarrow} (R^n(t_1, s_1, ..., s_n)) \overset{A}{\rightarrow} R^n(t_2, s_1, ..., s_n)) \]

\[ (a_1, ..., a_n) \overset{1}{\rightarrow} (R^n(t_1) \rightarrow t_2, s_1, ..., s_n)) \overset{A}{\rightarrow} \]

by Definition 2.3(ii).

This shows \( R^{n,A}((t_1) \rightarrow t_2, s_1, ..., s_n) = (R^n(t_1) \rightarrow t_2, s_1, ..., s_n)) \)

(ii) IF CFI has the form \( r(t_1, ..., t_n) \), then \( (a_1, ..., a_n) \overset{1}{\rightarrow} R^{n,A}((r(t_1, ..., t_n)) \overset{A}{\rightarrow} s_1, ..., s_n) \)

\[ (a_1, ..., a_n) \overset{1}{\rightarrow} \bigcup_{i=1}^n \text{dom} s^A_i \]

and \( (s^A_i(a_1, ..., a_n), ..., s^A_n(a_1, ..., a_n)) \overset{1}{\rightarrow} (r(t_1, ..., t_n)) \)

\[ t^A_i(s^A_i(a_1, ..., a_n), ..., s^A_n(a_1, ..., a_n)) \] exist for \( i = 1, ..., n \) and

\[ (t^A_1(s^A_1(a_1, ..., a_n), ..., s^A_n(a_1, ..., a_n)), ..., t^A_n(s^A_1(a_1, ..., a_n), ..., s^A_n(a_1, ..., a_n))) \overset{1}{\rightarrow} r^A \]

(by Definition 2.5(ii))

\[ S^{n,A}(t^A_1, s^A_1, ..., s^A_n)(a_1, ..., a_n) \] exist for \( i = 1, ..., n \) and

\( (S^{n,A}(t^A_1, s^A_1, ..., s^A_n)(a_1, ..., a_n), ..., S^{n,A}(t^A_1, s^A_1, ..., s^A_n)(a_1, ..., a_n)) \overset{1}{\rightarrow} r^A \)

\[ R^{n,A}(t^A_1, s^A_1, ..., s^A_n)(a_1, ..., a_n) \] exist for \( i = 1, ..., n \) and

\( (R^{n,A}(t^A_1, s^A_1, ..., s^A_n)(a_1, ..., a_n), ..., R^{n,A}(t^A_1, s^A_1, ..., s^A_n)(a_1, ..., a_n)) \overset{1}{\rightarrow} r^A \)

\[ (R^n(t, s_1, ..., s_n))^A(a_1, ..., a_n) \] exist for \( i = 1, ..., n \) and
\((R^n(t_s, s_{n_s}, \ldots, s_{n_n}))^A(a_{1_s}, a_{n_s}), \ldots, (R^n(t_s, s_{n_s}, \ldots, s_{n_n}))^A(a_{1_n}, a_{n_n})\) \(\hat{r}^A\)

\(\langle a_{1_s}, \ldots, a_{n_n} \rangle \hat{\langle} \langle R^n(t_s, s_{n_s}, \ldots, s_{n_n}) \rangle^A, \ldots, \langle R^n(t_s, s_{n_s}, \ldots, s_{n_n}) \rangle^A \hat{\rangle}\)

By Definition 2.3(iii).

Hence \(R^n(A)(r(t_s, \ldots, t_n), s_{1_s}, \ldots, s_{n_n}) = (R^n(r(t_s, \ldots, t_n), s_{1_s}, \ldots, s_{n_n}))^A\).

(iii) Let \(\hat{C}F \hat{\langle} \langle CF(t_{n_s}, n_s) \rangle \hat{\rangle}(X_{n_s})\) and assume that \(R^n(A)(\emptyset CF, s_{1_s}, \ldots, s_{n_s}) = (R^n(CF, s_{1_s}, \ldots, s_{n_s}))^A\).

Then we have \(\langle a_{1_s}, \ldots, a_{n_n} \rangle \hat{\langle} \langle R^n(A)(\emptyset CF), s_{1_s}, \ldots, s_{n_s} \rangle \hat{\rangle}\)

\(\langle a_{1_s}, \ldots, a_{n_n} \rangle \hat{\langle} \langle R^n(A)(CF, s_{1_s}, \ldots, s_{n_s}) \rangle \hat{\rangle}\)

\(\langle a_{1_s}, \ldots, a_{n_n} \rangle \hat{\langle} \langle R^n(A)(\emptyset CF), s_{1_s}, \ldots, s_{n_s} \rangle \hat{\rangle}\)

(iv) Let \(\hat{C}F \hat{\langle} \langle CF(t_{n_s}, n_s) \rangle \hat{\rangle}(X_{n_s})\) and assume that \(R^n(A)(\emptyset CF, s_{1_s}, \ldots, s_{n_s}) = (R^n(CF, s_{1_s}, \ldots, s_{n_s}))^A\).

Then we have \(\langle a_{1_s}, \ldots, a_{n_n} \rangle \hat{\langle} \langle R^n(A)(\emptyset CF), s_{1_s}, \ldots, s_{n_s} \rangle \hat{\rangle}\) by Definition 2.3(iv).

\(\langle a_{1_s}, \ldots, a_{n_n} \rangle \hat{\langle} \langle R^n(\emptyset CF), s_{1_s}, \ldots, s_{n_s} \rangle \hat{\rangle}\)
and or (by Definition 2.5(iv))

or by Definition 2.3(v).

The operation $R^{n,A}$ satisfies the following equation (see also (C1),(C2),(C3)):

**Theorem 2.8** The algebra

\[
(W^C_{t_n}(X_n))^{A^s} \hat{E} \ (CF_{(t_n, n)}(X_n))^{A^s} ; R^{n,A}, x_1^{A^s}, \ldots, x_n^{A^s})
\]

satisfies the following identities:

\[\text{(CFC1)} \quad R^{n,A}(b^{A^s}, R^{n,A}(t_1^{A^s}, s_1^{A^s}, \ldots, s_n^{A^s}), \ldots, R^{n,A}(t_n^{A^s}, s_1^{A^s}, \ldots, s_n^{A^s})) \]

\[= R^{n,A}(R^{n,A}(b^{A^s}, t_1^{A^s}, \ldots, t_n^{A^s}), s_1^{A^s}, \ldots, s_n^{A^s}), \]

\[\text{(CFC2)} \quad R^{n,A}(x_i^{A^s}, s_1^{A^s}, \ldots, s_n^{A^s}) = t_i^{A^s} \text{ for } 1 \leq i \leq n, \]

\[\text{(CFC3)} \quad R^{n,A}(b^{A^s}, x_1^{A^s}, \ldots, x_n^{A^s}) = b^{A^s}\]
**Proof** Using Lemma 2.7, we can prove following way:

for (CFC 1), if \( b \overset{C}{\vDash} W_{t_n}^C (X_n) \) is clear by C 1. Let now \( b \overset{C}{\vDash} CF(t_n(n)) (X_n) \).

We will give a proof by induction on the complexity of an \( n \)-ary \( C \)-formula \( CF \).

(i) If \( CF \) has the form \( s \rightarrow t \), then

\[
R^n.(R^n.(s \rightarrow t, t_1^s, ..., t_n^s, s_1^s, ..., s_n^s))
\]

\[
= R^n.(R^n((s \rightarrow t), t_1^s, ..., t_n^s, s_1^s, ..., s_n^s))
\]

\[
= (R^n((s \rightarrow t), t_1^s, ..., t_n^s, s_1^s, ..., s_n^s))
\]

the operation \( R^n \) satisfies (CFC 1), hence the previous line yields

\[
= (R^n(s \rightarrow t), R^n(t_1, s_1, ..., s_n), ..., R^n(t_n, s_1, ..., s_n))
\]

\[
= R^n.(R^n.(s \rightarrow t, t_1^s, ..., t_n^s, s_1^s, ..., s_n^s))
\]

(ii) If \( CF \) has the form \( r(l_1, ..., l_n) \), then

\[
R^n.(R^n.(r(l_1, ..., l_n)), t_1^s, ..., t_n^s, s_1^s, ..., s_n^s)
\]

\[
= R^n.(R^n(r(l_1, ..., l_n), t_1, ..., t_n))
\]

\[
= (R^n(r(l_1, ..., l_n), t_1, ..., t_n))
\]

\[
= (R^n(r(l_1, ..., l_n), t_1, ..., t_n, s_1, ..., s_n))
\]

\[
= (R^n(r(l_1, ..., l_n), t_1, ..., t_n, s_1, ..., s_n))
\]

\[
= (R^n(r(l_1, ..., l_n), t_1, ..., t_n, s_1, ..., s_n))
\]

\[
= R^n.(R^n.(r(l_1, ..., l_n), t_1^s, ..., t_n^s, s_1^s, ..., s_n^s))
\]

(iii) We have to show that if \((CFC1)\) is satisfied for an \( n \)-ary \( C \)-formula \( CF \), then it is also satisfied for \( \emptyset CF \). In fact,

\[
R^n.(R^n.(\emptyset CF), t_1^s, ..., t_n^s, s_1^s, ..., s_n^s)
\]
\[ R^{n,A}( (R^n(\emptyset CF, t_1, ..., t_n))^{A^s}, s_1^{A_s}, ..., s_n^{A_s}) \]
\[ = (R^n(R^n(\emptyset CF, t_1, ..., t_n), s_1, ..., s_n))^{A_s} \]
\[ = (R^n(\emptyset CF, R^n(t_1, s_1, ..., s_n), ..., R^n(t_n, s_1, ..., s_n)))^{A_s} \]
\[ = R^{n,A}( (\emptyset CF)^{A_s}, R^{n,A}(t_1^{A_s}, s_1^{A_s}, ..., s_n^{A_s}), ..., R^{n,A}(t_n^{A_s}, s_1^{A_s}, ..., s_n^{A_s})) \].

(iv) Here we have to show that if \( CF_1, CF_2 \) satisfy (CFC 1), then it is also satisfied for \( CF_1 \uplus CF_2 \).
In fact
\[ R^{n,A}(R^n(\emptyset CF, t_1, ..., t_n), s_1, ..., s_n) \]
\[ = R^{n,A}(R^n(CF_1 \uplus CF_2, t_1, ..., t_n), s_1, ..., s_n) \]
\[ = (R^n(R^n(CF_1 \uplus CF_2, t_1, ..., t_n), s_1, ..., s_n))^{A_s} \]
\[ = (R^n(CF_1 \uplus CF_2, R^n(t_1, s_1, ..., s_n), ..., R^n(t_n, s_1, ..., s_n)))^{A_s} \]
\[ = R^{n,A}( (\emptyset CF)^{A_s}, R^{n,A}(t_1^{A_s}, s_1^{A_s}, ..., s_n^{A_s}), ..., R^{n,A}(t_n^{A_s}, s_1^{A_s}, ..., s_n^{A_s})) \].

The proof of (CFC 2) is clear by (CFC 2) and the fact that
\[ R^{n,A}(x_i^{A_s}, t_1^{A_s}, ..., t_n^{A_s}) = (R^n(x_i, t_1, ..., t_n))^{A_s}. \]

For (CFC 3), we can calculate formula as follows. If \( CF \uplus CF_{(t_n(n))}(X_n) \), then
\[ R^{n,A}(CF^{A_s}, x_1^{A_s}, ..., x_n^{A_s}) \]
\[ = \{ (a_1, ..., a_n) \} \uplus A^n \mid \{ (a_1, ..., a_n) \} \uplus \{ \text{dom} x_i^{A_s} \text{ and } (x_i^{A_s}(a_1, ..., a_n), ..., x_n^{A_s}(a_1, ..., a_n)) \} \uplus CF^{A_s} \}
\[ = \{ (a_1, ..., a_n) \} \uplus A^n \mid (a_1, ..., a_n) \uplus CF^{A_s} \} \]
3. Monoid of $C$ -Hypersubstitutions for Algebraic Systems of Type $(t_n, (n))$

In 1991, K. Denecke, D. Lau, R. Pöschel and D. Schweigert [8] introduced the concept of a hypersubstitution algebras. A hypersubstitution for algebraic systems was first introduced by K. Denecke and D. Phusanga [10]. It is a mapping that maps operation symbols to terms and relation symbols to formulas preserving arities. They defined a binary operation on the set of all hypersubstitutions for algebraic systems and then proved that this set the binary operation and an identity element forms a monoid. So, in this section, we would like to form the new structure of the so-called "Monoid of $C$ -Hypersubstitution for Algebraic Systems of Type $(t_n, (n))$". The way to approach this, we first define the based set.

**Definition 3.1** Let $n \hat{1} X^*$. A $C$ -hypersubstitution for algebraic systems of type $(t_n, (n))$ is a mapping $s : \{f_i \mid i \hat{1} I\} \ni r \mapsto CF_{(t_n, (n))}(X_n)$, which maps each $n$ -ary operation symbols of type $t_n$ to $n$ -ary $C$ -term of type $t_n$ and maps an $n$ -ary relation symbol $r$ to an $n$ -ary $C$ -formula of type $(t_n, (n))$.

We denote the set of all $C$ -hypersubstitutions for algebraic systems of type $(t_n, (n))$ by $Hyp^C (t_n, (n))$.

For every $C$ -hypersubstitution for algebraic systems of type $(t_n, (n))$, we can define an extension which maps an $n$ -ary $C$ -term to an $n$ -ary $C$ -term and maps an $n$ -ary $C$ -formula to an $n$ -ary $C$ -formula as follows:

**Definition 3.2** Let $n \hat{1} Y^*$ and $s \hat{1} Hyp^C (t_n, (n))$. Then we define a mapping $s : CF_{(t_n, (n))}(X_n) \circ CF_{(t_n, (n))}(X_n) \circ CF_{(t_n, (n))}(X_n)$ inductively as follows:
(i) \( \hat{s}[x_i] := x_i \) for every \( x_i \in X_n \),

(ii) \( \hat{s}[e_i(s_1, ..., s_k)] := e_i(\hat{s}[s_1], ..., \hat{s}[s_k]) \), where \( s_1, ..., s_k \in W_{t_n}^c(X_n) \),

(iii) \( \hat{s}[f(t_1, ..., t_n)] := R^n(s(f)), \hat{s}[t_1], ..., \hat{s}[t_n] \), where \( t_1, ..., t_n \in W_{t_n}^c(X_n) \),

(iv) \( \hat{s}[s \circ t] := \hat{s}[s] \circ \hat{s}[t] \) for every \( s, t \in W_{t_n}^c(X_n) \),

(v) \( \hat{s}[r(s_1, ..., s_n)] := R^n(s(r)), \hat{s}[s_1], ..., \hat{s}[s_n] \), where \( s_1, ..., s_n \in W_{t_n}^c(X_n) \),

(vi) \( \hat{s}[\emptyset CFG] := \emptyset(\hat{s}[CFG]) \) for \( CF \hat{\in} \overline{CF}_{t_n(n)}(X_n) \),

(vii) \( \hat{s}[CF_1 \cup CFG_2] := \hat{s}[CF_1] \cup \hat{s}[CFG_2] \) for \( CF_1, CFG_2 \hat{\in} \overline{CF}_{t_n(n)}(X_n) \).

Then \( \hat{s} \) is called the extension of \( s \).

Now, we define a binary operation \( \circ_r \) on \( Hyp^c(t_n(n)) \) as follows:

**Definition 3.3** Let \( s_1, s_2 \in Hyp^c(t_n(n)) \) and \( \circ \) be the usual composition of mapping. Then we define a binary operation \( \circ_r \) on \( Hyp^c(t_n(n)) \) by \( s_1 \circ_r s_2 := \hat{s}_1 \circ \hat{s}_2 \).

Next, we prove that the binary operation as we already defined in Definition 3.3 satisfies associative law. To get our result, we need some preparations as follows:

**Lemma 3.4** For \( s \hat{\in} Hyp^c(t_n(n)) \) and \( b \hat{\in} W_{t_n}^c(X_n) \) we have

\( \hat{s}[R^n(b, s_1, ..., s_n)] = R^n(\hat{s}[b], \hat{s}[s_1], ..., \hat{s}[s_n]) \).

**Proof** The proof is straightforward and hence omitted.

As a result of Lemma 3.4, we have the following lemma.

**Lemma 3.5** Let \( s_1, s_2 \in Hyp^c(t_n(n)) \). Then we have \( (s_1 \circ_r s_2)^v = \hat{s}_1 \circ \hat{s}_2 \).
Proof Let \( n \uparrow \psi^* \) and \( t \uparrow W_n^C(X_n) \), we give a proof by induction on the complexity of an \( n \)-ary \( C \)-term \( t \).

If \( t = x_i \) with \( i \uparrow \{1,...,n\} \), then \((s_1 \circ s_2)^\uparrow [x_i] = x_i = \hat{s}_1[x_i] = \hat{s}_2[x_i] = (\hat{s}_1 \circ \hat{s}_2)[x_i] \).

If \( t = e_j^k(w_1,...,w_k) \) and assume that \((s_1 \circ s_2)^\uparrow [w_i] = (\hat{s}_1 \circ \hat{s}_2)[w_i] \) for every \( i \uparrow \{1,...,k\} \), then
\[
(s_1 \circ s_2)^\uparrow [e_j^k(w_1,...,w_k)] = e_j^k((s_1 \circ s_2)^\uparrow [w_1],..., (s_1 \circ s_2)^\uparrow [w_k])
\]
\[
= e_j^k((\hat{s}_1 \circ \hat{s}_2)[w_1],..., (\hat{s}_1 \circ \hat{s}_2)[w_k])
\]
\[
= e_j^k(\hat{s}_1[\hat{s}_2[w_1]],..., \hat{s}_1[\hat{s}_2[w_k]])
\]
\[
= \hat{s}_1[e_j^k(\hat{s}_2[w_1],..., \hat{s}_2[w_k])]
\]
\[
= \hat{s}_1[\hat{s}_2[e_j^k(w_1,...,w_k)]]
\]
\[
= (\hat{s}_1 \circ \hat{s}_2)[e_j^k(w_1,...,w_k)].
\]

If \( t = f_j(t_1,...,t_n) \) for any \( i \uparrow l \) and assume that \((s_1 \circ s_2)^\uparrow [t_k] = (\hat{s}_1 \circ \hat{s}_2)[t_k] \) for every \( k \uparrow \{1,...,n\} \), then
\[
(s_1 \circ s_2)^\uparrow [f_j(t_1,...,t_n)] = R^n((s_1 \circ s_2)(f_j), (s_1 \circ s_2)^\uparrow [t_1],..., (s_1 \circ s_2)^\uparrow [t_n])
\]
\[
= R^n((\hat{s}_1 \circ \hat{s}_2)(f_j), (\hat{s}_1 \circ \hat{s}_2)[t_1],..., (\hat{s}_1 \circ \hat{s}_2)[t_n])
\]
\[
= R^n(\hat{s}_1[s_2(f_j)], \hat{s}_1[\hat{s}_2[t_1]],..., \hat{s}_1[\hat{s}_2[t_n]])
\]
\[
= \hat{s}_1[R^n(s_2(f_j), s_2[t_1],..., s_2[t_n]) \text{ (by Lemma 3.4)}]
\]
\[
= \hat{s}_1[\hat{s}_2[f_j(t_1,...,t_n)]
\]
\[
= (\hat{s}_1 \circ \hat{s}_2)[f_j(t_1,...,t_n)].
\]
For $b \in CF_{(t_{n}^{(n)})}(X_{n})$. We will give a proof by induction on the complexity of an $n$-ary $C$-formula $CF$.

(i) If $b$ has the form $s \triangleright t$, then

$$(s_{1}, s_{2})^{\uparrow}[s \triangleright t] = (s_{1}, s_{2})^{\uparrow}[s \triangleright (s_{1}, s_{2})^{\uparrow}[t]$$

$$= (\hat{s}_{1}, \hat{s}_{2})[s] \triangleright (\hat{s}_{1}, \hat{s}_{2})[t]$$

$$= \hat{s}_{1}[\hat{s}_{2}[s]] \triangleright \hat{s}_{1}[\hat{s}_{2}[t]]$$

$$= \hat{s}_{1}[\hat{s}_{2}[s \triangleright t]]$$

$$= (\hat{s}_{1}, \hat{s}_{2})[s \triangleright t].$$

(ii) If $b$ has the form $r(t_{1}, \ldots, t_{n})$, then

$$(s_{1}, s_{2})^{\uparrow}[r(t_{1}, \ldots, t_{n})] = R^{\uparrow}(r((s_{1}, s_{2})^{\uparrow}(r), (s_{1}, s_{2})^{\uparrow}[t_{1}], \ldots, (s_{1}, s_{2})^{\uparrow}[t_{n}])$$

$$= R^{\uparrow}((\hat{s}_{1}, \hat{s}_{2}[r], (\hat{s}_{1}, \hat{s}_{2})[t_{1}], \ldots, (\hat{s}_{1}, \hat{s}_{2})[t_{n}]))$$

$$= R^{\uparrow}(\hat{s}_{2}[r(t_{1}, \ldots, t_{n})], \hat{s}_{2}[t_{1}], \ldots, \hat{s}_{2}[t_{n}]) \quad \text{(by Lemma 3.4)}$$

$$= \hat{s}_{1}[\hat{s}_{2}[r(t_{1}, \ldots, t_{n})]]$$

$$= (\hat{s}_{1}, \hat{s}_{2})[r(t_{1}, \ldots, t_{n})].$$

(iii) If $b$ has the form $\emptyset CF$ and if we assume that $(s_{1}, s_{2})^{\uparrow}[CF] = (\hat{s}_{1}, \hat{s}_{2})[CF]$, then

$$(s_{1}, s_{2})^{\uparrow}[\emptyset CF] = \emptyset((s_{1}, s_{2})^{\uparrow}[CF])$$

$$= \emptyset((\hat{s}_{1}, \hat{s}_{2})[CF])$$

$$= \emptyset(\hat{s}_{1}[\hat{s}_{2}[CF]])$$

83
\( = \hat{s}_i[\emptyset(\hat{s}_2[CF])] \)

\( = \hat{s}_i[\hat{s}_2[\emptyset CF]] \)

\( = (\hat{s}_1 o \hat{s}_2)[\emptyset CF]. \)

(iv) If \( b \) has the form \( CF_1 \hat{U} CF_2 \) and if we assume that \( (s_1 o r, s_2)^\wedge [CF_i] = (\hat{s}_1 o \hat{s}_2)[CF_i] \)

for all \( i = 1, 2 \), then

\( (s_1 o r, s_2)^\wedge [CF_1 \hat{U} CF_2] = (s_1 o r, s_2)^\wedge [CF_1] \hat{U} (s_1 o r, s_2)^\wedge [CF_2] \)

\( = (\hat{s}_1 o \hat{s}_2)[CF_1] \hat{U} (\hat{s}_1 o \hat{s}_2)[CF_2] \)

\( = \hat{s}_1[\hat{s}_2[CF_1]] \hat{U} \hat{s}_1[\hat{s}_2[CF_2]] \)

\( = \hat{s}_1[\hat{s}_2[CF_1 \hat{U} CF_2]] \)

\( = (\hat{s}_1 o \hat{s}_2)[CF_1 \hat{U} CF_2]. \)

It follows from Lemma 3.5 that the binary operation \( o_r \) satisfies the associative law. We prove this fact in the next lemma.

**Lemma 3.6** Let \( s_1, s_2, s_3 \in Hyp^C(t, r, (n)) \). Then we have \( (s_1 o_r, s_2) o_r, s_3 = s_1 o_r (s_2 o_r, s_3). \)

**Proof** By using Lemma 3.5 and the fact that \( \emptyset \) satisfies associative law, it can be shown that \( o_r \) satisfies associative law. In fact, we have

\( (s_1 o_r, s_2) o_r, s_3 = (s_1 o_r, s_2)^\wedge o s_3 \)

\( = (\hat{s}_1 o \hat{s}_2) o s_3 \)

\( = \hat{s}_1 o(\hat{s}_2 o s_3) \)

\( = \hat{s}_1 o(s_2 o_r, s_3) \)
Let $S_{id}$ be a $C$ -hypersubstitution for algebraic systems of type $(t_n, (n))$ which maps the operation symbol $f_i$ to the $n$-ary $C$-term $f_i(x_1, ..., x_n)$ for all $i \equiv 1$, and maps the relation symbol $r$ to the $n$-ary $C$-formulas $r(x_1, ..., x_n)$, i.e. $S_{id}(f_i) = f_i(x_1, ..., x_n)$ for all $i \equiv 1$ and $S_{id}(r) = r(x_1, ..., x_n)$.

**Lemma 3.7** For any $b \in W_{t_n}^C(X_n) \mathcal{E} CF_{(t_n, (n))}(X_n)$, we have $\hat{s}_{id}[b] = b$.

**Proof** The proof is straightforward and hence omitted.

A $C$-hypersubstitution $S_{id}$ is claimed to be an identity, which we will prove this fact in the next lemma.

**Lemma 3.8** Let $S_{id} \mathcal{E} Hyp^C(t_n, (n))$. Then $S_{id}$ is an identity element with respect to $O_r$.

**Proof** First, we prove that $S_{id}$ is a left identity element by using Lemma 3.7. Let $s \mathcal{E} Hyp^C(t_n, (n))$ and $b \in \{ f_i \mid i \equiv 1 \} \mathcal{E} \{ r \}$. Then we have $(S_{id} \circ s)(b) = (\hat{s}_{id} \circ s)(b) = s(b)$.

Now, we show that $S_{id}$ is a right identity element. Let $s \mathcal{E} Hyp^C(t_n, (n))$. By Theorem 2.4 (CFC3), if $i \equiv 1$, then

$(s \circ s_{id})(f_i) = (\hat{s} \circ s_{id})(f_i) = \hat{s}[s_{id}(f_i)] = \hat{s}[f_i(x_1, ..., x_n)] = R^o(s(f_i), x_1, ..., x_n) = s(f_i)$ and

$(s \circ s_{id})(r) = (\hat{s} \circ s_{id})(r) = \hat{s}[s_{id}(r)] = \hat{s}[r(x_1, ..., x_n)] = R^o(s(r), x_1, ..., x_n) = s(r)$.

Therefore, $S_{id} \circ s_{id} = s = s_{id} \circ s$.

**Theorem 3.9** $yp^C(t_n, (n)) := (Hyp^C(t_n, (n)), o_r, \sigma_{id})$ is a monoid.

**Proof** From Lemma 3.6 and 3.8, the conclusion holds.
Acknowledgements

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References