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AN OVERVIEW ON INTEGERS OF THE FORM $6^n + 1$

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Abstract

We pose various congruences on the integers of form $6^n + 1$, $n \in Z_+$, which may encourage younger number theorist to do research in number theory and settled new dimension in this field. We saw that there are only three prime numbers, namely $7, 37$, and 1297 of form $6^n + 1$, whenever $n \in Z_+ - \{2^k m, k \geq 6, m \equiv 1 (mod\ 2)\}$, and no one Fermat numbers represent in this form. Moreover, these integers end with seven, like Fermat numbers $F_n, n \geq 2$. Also, we discussed some congruences with number theoretic functions σ, φ and Möbious function μ, and generates various families of integers with $\mu(n) = 0$.

Keywords: Congruences, Fermat Number, Number Theoretic Functions, Prime Number

AMSC: 11A07, 11A41, 11A51, 11A67

1. Introduction

Theory of numbers grows with properties of integers and especially with the positive integers $1, 2, 3, \ldots \ldots \ldots$ (also known as the natural numbers). Primes behaves like the nucleus in number theory. Primes and forms of integers have been studied for over two thousand years at the time of Euclid. The Euclid theorem posed various consequences, like that there are an infinite number of primes of form $4k + 3$ [3]. Multitude problems on primes and forms of integers are still open until now. The famous Goldbach conjecture ([4], [10]) for even integers, initially tells that every even integer $n > 2$ can be expressed as a sum of two primes. The twin prime conjecture ([1], [11], [16]), asserts that there are infinitely many primes that differ by 2. In the published paper [9] the author proved that all the integers of the form $p^6 + 6^p, p$ with prime $p \geq 2$ are composite. After the studies of above cited work and various literature referenced in ([2], [5], [6], [7], [8], [9], [10], [12], [13], [14], [15]) on various conjectures concerned with primes, the form of integers, numbers of special forms like Fermat numbers and number theoretic functions have great importance in the field of number theory. This study aims to provide some congruences on the integers of the form $6^n + 1$, whenever $n \in Z_+ - \{2^k m, k \geq 6, m \equiv 1 (mod\, 2)\}$, with number theoretic functions σ, φ, Möbious function μ and established various forms of primes. Prominently we have that all numbers $6^n + 1, n \in Z_+$, end with 7 like the Fermat numbers $F_n = 2^{2^n} + 1, n \geq 2$. For the study of primality of numbers $6^n + 1, n \in Z_+$, we consider equivalence representation of number n.

Preliminary we consider $n \in Z_+$ as the member of the following two families $f(1), \& f(2)$ of integers respectively where these families defined as

$f(1) = \{n | n = 2^k m, k \in Z_+, m \equiv 1 (mod\, 2)\}$,

$f(2) = \{n | n = 2k + 1, k \in Z_0\}$

2. $\boldsymbol{n \in f(1) = \{n | n = 2^k m, k \in Z_+, m \equiv 1 (mod\, 2)\}}$ **.** Firstly, we consider $n \in f(1) = \{n | n = 2^k m, k \in Z_+, m \equiv 1 (mod\, 2)\}$, and deduce the following results.

Theorem 2.1. If the integer $n = 2m$, where m is an odd integer, then congruences $6^n + 1 \equiv 0(37)$, and $6^n + 1 \not\equiv 0(mod\, (37)^2)$ always holds.

Proof. Since $6^2 \equiv -1(mod\, 37)$, and $6^2 \not\equiv -1(mod\, (37)^2)$. Consider for $n = 2m$, where m is odd, then clearly, the congruences $6^n + 1 \equiv 0(37), \& 6^n + 1 \not\equiv 0(mod\, (37)^2)$holds.

Theorem 2.2. If the integer $n = 2^2 m$, where m is an odd integer, then congruences $6^n + 1 \equiv 0(1297)$, and $6^n + 1 \not\equiv 0(mod\ (1297)^2)$ always holds.

Proof. Since $6^4 \equiv -1(mod\ 1297)$, and $6^4 \not\equiv -1(mod\ (1297)^2)$. Consider for $n = 2^2 m$, where m is odd, then clearly, the congruences $6^n + 1 \equiv 0(1297)$, & $6^n + 1 \not\equiv 0(mod\ (1297)^2)$ holds.

Theorem 2.3. If the integer $n = 2^3 m$, where m is an odd integer, then congruences $6^n + 1 \equiv 0(37)$, and $6^n + 1 \not\equiv 0(mod\ (37)^2)$ always holds.

Proof. Since $6^8 \equiv -1(mod\ 17)$, and $6^8 \not\equiv -1(mod\ (17)^2)$. Consider for $n = 2^3 m$, where m is odd, then clearly, the congruences $6^n + 1 \equiv 0(17)$, & $6^n + 1 \not\equiv 0(mod\ (17)^2)$holds.

Theorem 2.4. If the integer $n = 2^3 m$, where m is an odd integer, then congruences $6^n + 1 \equiv 0(98801)$, and $6^n + 1 \not\equiv 0(mod\ (98801)^2)$ always holds.

Proof. Since $6^8 \equiv -1(mod\ 98801)$, and $6^8 \not\equiv -1(mod\ (98801)^2)$. Consider for $n = 2^3 m$, where m is odd, then clearly, the congruences $6^n + 1 \equiv 0(98801)$, & $6^n + 1 \not\equiv 0(mod\ (17)^2)$holds.

Theorem 2.5. If the integer $n = 2^4 m$, where m is an odd integer, then congruences $6^n + 1 \equiv 0(353)$, and $6^n + 1 \not\equiv 0(mod\ (353)^2)$ always holds.

Proof. Since $6^{16} \equiv -1(mod\ 353)$, and $6^{16} \not\equiv -1(mod\ (353)^2)$. Consider for $n = 2^{16} m$, where m is odd, then clearly, the congruences $6^n + 1 \equiv 0(98801)$, & $6^n + 1 \not\equiv 0(mod\ (353)^2)$ holds.

Theorem 2.6. If the integer $n = 2^4 m$, where m is an odd integer, then congruences $6^n + 1 \equiv 0(1697)$, and $6^n + 1 \not\equiv 0(mod\ (1697)^2)$ always holds.

Proof. Since $6^{16} \equiv -1(mod\ 1697)$, and $6^{16} \not\equiv -1(mod\ (1697)^2)$. Consider for $n = 2^{16} m$, where m is odd, then clearly, the congruences $6^n + 1 \equiv 0(1697)$, & $6^n + 1 \not\equiv 0(mod\ (1697)^2)$ holds.

Theorem 2.7. If the integer $n = 2^4 m$, where m is an odd integer, then congruences $6^n + 1 \equiv 0(4709377)$, and $6^n + 1 \not\equiv 0(mod\ (4709377)^2)$ always holds.

Proof. Since $6^{16} \equiv -1(mod\ 4709377)$, and $6^{16} \not\equiv -1(mod\ (4709377)^2)$. Consider for $n = 2^{16} m$, where m is odd, then clearly, the congruences $6^n + 1 \equiv 0(4709377)$, & $6^n + 1 \not\equiv 0(mod\ (4709377)^2)$ holds.

Theorem 2.8. If the integer $n = 2^5 m$, where m is an odd integer, then congruences $6^n + 1 \equiv 0(2753)$, and $6^n + 1 \not\equiv 0(mod\,(2753)^2)$ always holds.

Proof. Since $6^{32} \equiv -1(mod\,2753)$, and $6^{32} \not\equiv -1(mod\,(2753)^2)$. Consider for $n = 2^{32} m$, where m is odd, then clearly, the congruences $6^n + 1 \equiv 0(98801)$, & $6^n + 1 \not\equiv 0(mod\,(2753)^2)$ holds.

Theorem 2.9. If the integer $n = 2^5 m$, where m is an odd integer, then congruences $6^n + 1 \equiv 0(145601)$, and $6^n + 1 \not\equiv 0(mod\,(145601)^2)$ always holds.

Proof. Since $6^{32} \equiv -1(mod\,145601)$, and $6^{32} \not\equiv -1(mod\,(145601)^2)$. Consider for $n = 2^{32} m$, where m is odd, then clearly, the congruences $6^n + 1 \equiv 0(145601)$, & $6^n + 1 \not\equiv 0(mod\,(145601)^2)$ holds.

Theorem 2.10. If the integer $n = 2^5 m$, where m is an odd integer, then congruences $6^n + 1 \equiv 0(19854979505843329)$, and $6^n + 1 \not\equiv 0(mod\,(19854979505843329)^2)$ always holds.

Proof. Since $6^{32} \equiv -1(mod\,19854979505843329)$, and $6^{32} \not\equiv -1(mod\,(19854979505843329)^2)$. Consider for $n = 2^{32} m$, where m is odd, then clearly, the congruences $6^n + 1 \equiv 0(19854979505843329)$, & $6^n + 1 \not\equiv 0(mod\,(19854979505843329)^2)$ holds.

3. $\boldsymbol{n \in f(2) = \{n | n = 2k + 1, k \in Z_0,\}}$.

Secondly, we consider $n \in f(2) = \{n | n = 2k + 1, k \in Z_0,\}$, and deduce the following results.

Theorem 3.1. If the integer $n = 2k + 1$, where $k \in Z_0$, then congruency $6^n + 1 \equiv 0(mod\,7)$ holds but the congruency $6^n + 1 \not\equiv 0(mod\,(7)^2)$ not necessarily holds.

Proof. Since $6^1 \equiv -1(mod\,7)$, and $6^1 \not\equiv -1(mod\,(7)^2)$. Consider for $n = 2k + 1$, where k is an integer, i.e., , $k \in Z_0$, also we have $6^7 \equiv -1(mod\,7^2)$, then clearly, the congruences $6^n + 1 \equiv 0(mod\,7)$ holds but the congruency $6^n + 1 \not\equiv 0(mod\,(7)^2)$ not necessarily holds.

4. Remarks.

Here we established the following remarks.

Remark 4.1. The string of integers of the form $6^{4(2k+1)} + 1, k \in Z_0$ has exactly one prime.

Remark 4.2. The string of integers of the form $6^{8(2k+1)} + 1, k \in Z_0$ has no prime.

Remark 4.3. The strings of integers of the form $6^{4(2k+1)} + 1, k \in Z_0$ and $6^{8(2k+1)} + 1, k \in Z_0$ has no Fermat number, moreover these are composite numbers end with 7.

Remark 4.4. The chain of integers of the form $6^{4k+1} + 1, k \in Z_0$ has exactly one prime.

Remark 4.5. The chain of integers of the form $6^{4k+1} + 1, k \in Z_0$ has no Fermat number, moreover these are composite numbers end with 7.

Remark 4.6. The chain of integers of the form $6^{7(2k+1)} + 1, k \in Z_0$ satisfies the congruence $6^{7(2k+1)} + 1 \equiv 0(mod\,(7)^2)$.

Remark 4.7. The chain of integers of the form $6^{7(2k+1)} + 1, k \in Z_0$satisfies the congruence $6^{7(2k+1)} + 1 \equiv 0(mod\,29)$.

Remark 4.8. The chain of integers of the form $6^{7(2k+1)} + 1, k \in Z_0$satisfies the congruence $6^{7(2k+1)} + 1 \equiv 0(mod\,197)$.

Remark 4.9. The chain of integers of the form $6^{4k+2} + 1, k \in Z_0$ holds exactly one prime.

Remark 4.10. The chain of integers of the form $6^{4k+2} + 1, k \in Z_0$ holds no Fermat number, moreover these are composite numbers end with 7.

Remark 4.11. The chain of integers of the form $6^{4k+3} + 1, k \in Z_0$ holds no prime.

Remark 4.12. The chain of integers of the form $6^{4k+3} + 1, k \in Z_0$ holds no Fermat number, moreover these are composite numbers end with 7.

Remark 4.13. The chain of integers of the form $6^{4k+3} + 1, k \in Z_0$ holds no Fermat number, moreover these are composite numbers end with 7.

Remark 4.14. The chain of integers of the form $6^{6k+3} + 1, k \in Z_0$ satisfies the congruence $6^{\,6k+3} + 1 \equiv 0(mod\,31)$.

Remark 4.15. If m, and n are distinct co prime odd integers then $d = gcd(6^m + 1, 6^n + 1)$ must hold the congruency $d \equiv 0(mod\,7)$.

5. Conjectures.

Here we established the various conjectures regarding number theoretic functions.

Conjecture 5.1. Each member of the string of integers of the form $6^{2(2k+1)} + 1, k \in Z_0$ must hold the following congruences with respect to number theoretic functions σ, φ and Möbious function μ.

$\sigma(6^{\,2(2k+1)} + 1) \equiv 0(mod\, 2^l),\ k \in Z_+, l \geq 2$ (5.1.1)

$\varphi(6^{2(2k+1)} + 1) \equiv 0(mod\, 2^l), k \in Z_0, l \geq 4$ (5.1.2)

$\varphi[\varphi(6^{2(2k+1)} + 1)] \equiv 0(mod\, 2^l 3^m), k \in Z_0, l, m \geq 2$ (5.1.3)

$\mu\,(\sigma(6^{2(2k+1)} + 1)) = 0, k \in Z_+$ (5.1.4)

$\mu\,(\varphi(6^{2(2k+1)} + 1)) = 0, k \in Z_0$ (5.1.5)

$\mu(\varphi[\varphi(6^{2(2k+1)} + 1)]) = 0, k \in Z_0$ (5.1.6)

Proof. Consider theorem (2.1) and recall the definition of number theoretic functions , and Möbious function . Then we find that the congruences (5.1) to (5.6) must hold.

Conjecture 5.2. Each member of the chain of integers of the form $6^{4(2k+1)} + 1, k \in Z_0$ must hold the following congruences with respect to number theoretic functions σ, φ and Möbious function μ.

$\sigma(6^{4(2k+1)} + 1) \equiv 0(mod\, 2^l), k \in Z_0, l \geq 2$ (5.2.1)

$\varphi(6^{4(2k+1)} + 1) \equiv 0(mod\, 2^l), k \in Z_0, l \geq 4$ (5.2.2)

$\varphi[\varphi(6^{4(2k+1)} + 1)] \equiv 0(mod\, 2^l), k \in Z_0, l \geq 2$ (5.2.3)

$\mu\,(\sigma(6^{4(2k+1)} + 1) + 1)) = 0, k \in Z_0$ (5.2.4)

$\mu\,(\varphi(6^{4(2k+1)} + 1) + 1)) = 0, k \in Z_0$ (5.2.5)

$\mu(\varphi[\varphi(6^{4(2k+1)} + 1) + 1)]) = 0, k \in Z_0$ (5.2.6)

Proof. Consider theorem (2.2) and recall the definition of number theoretic functions , and Möbious function . Then we find that the congruences (5.2.1) to (5.2.6) must hold.

Conjecture 5.3. Each member of the chain of integers of the form $6^{2k+1} + 1, k \in Z_0$ must hold the following congruences with respect to number theoretic functions , and Möbious function .

$\sigma(6^{2k+1} + 1) \equiv 0(mod\, 2^l), k \in Z_0, l \geq 2$ (5.3.1)

$\varphi(6^{2k+1} + 1) \equiv 0(mod\, 2^l 3^m), k \in Z_0, l, m \geq 1$ (5.3.2)

$\varphi[\varphi(6^{2k+1} + 1)] \equiv 0(mod\ 2^l 3^m), k \in Z_0, l \geq 1, m \geq 0$ (5.3.3)

$\mu\,(\sigma(6^{2k+1} + 1)) = 0, k \in Z_0$ (5.3.4)

$\mu\,(\varphi(6^{2k+1} + 1)) = 0, k \in Z_+$ (5.3.5)

$\mu(\varphi[\varphi(6^{2k+1} + 1)]) = 0, k \in Z_0$ (5.3.6)

Proof: Consider theorem (2.3) and recall the definition of number theoretic functions , and Möbious function . Then we find that the congruences (5.3.1) to (5.3.6) must hold.

Conjecture 5.4. Each member of the chain of integers of the form $6^{7(2k+1)} + 1, k \in Z_0$ must hold the following congruences with respect to number theoretic functions , and Möbious function .

$\sigma(6^{7(2k+1)} + 1) \equiv 0(mod\ 2^l), k \in Z_+, l \geq 2$ (5.4.1)

$\varphi(6^{7(2k+1)} + 1) \equiv 0(mod\ 2^l 3^m), k \in Z_0, l, m \geq 2$ (5.4.2)

$\varphi[\varphi(6^{7(2k+1)} + 1)] \equiv 0(mod\ 2^l), k \in Z_0, l \geq 2$ (5.4.3)

$\mu\big(\sigma 6^{7(2k+1)} + 1\big) = 0, k \in Z_+$ (5.4.4)

$\mu(\varphi(6^{7(2k+1)} + 1)) = 0, k \in Z_0$ (5.4.5)

$\mu(\varphi[\varphi(6^{7(2k+1)} + 1)]) = 0, k \in Z_0$ (5.4.6)

Proof. Consider theorem (3.1), remarks (4.6), (4.7), (4.8), and recall the definition of number theoretic functions , and Möbious function . Then we find that the congruences (5.4.1) to (5.4.6) must hold.

Conjecture 5.5. Each member of the chain of integers of the form $6^{4k+3} + 1, k \in Z_0$ must hold the following congruences with respect to number theoretic functions , and Möbious function .

$\sigma(6\wedge(4k+3) + 1) \equiv 0(mod\ 2^l), k \in Z_0, l \geq 2$ (5.5.1)

$\varphi(6^{4k+3} + 1) \equiv 0(mod\ 2^l 3^m), k \in Z_0, l, m \geq 2$ (5.5.2)

$\varphi[\varphi(6^{4k+3} + 1)] \equiv 0(mod\ 2^l), k \in Z_0, l \geq 2$ (5.5.3)

$\mu(\sigma(6^{4k+3} + 1)) = 0, k \in Z_+$ (5.5.4)

$\mu(\varphi(6^{4k+3} + 1)) = 0, k \in Z_0$ (5.5.5)

$\mu(\varphi[\varphi(6^{4k+3} + 1)]) = 0, k \in Z_0$ (5.5.6)

Proof. Consider theorem (3.1), remark (4.11) and recall the definition of number theoretic functions , and Möbious function . Then we find that the congruences (5.5.1) to (5.5.6) must hold.

Conjecture 5.6. Each member of the chain of integers of the form $6^{6k+3} + 1, k \in Z_0$ must holds the following congruences with respect to number theoretic functions , and Möbious function .

$\sigma(6^{6k+3} + 1) \equiv 0(mod\, 2^l), k \in Z_0, l \geq 8$ (5.6.1)

$\varphi(6^{6k+3} + 1) \equiv 0(mod\, 2^l 3^m), k \in Z_0, l, m \geq 2$ (5.6.2)

$\varphi[\varphi(6^{6k+3} + 1)] \equiv 0(mod\, 2^l), k \in Z_0, l \geq 2$ (5.6.3)

$\mu\left(\sigma\left(6^{6k+3} + 1\right)\right) = 0, k \in Z_+$ (5.6.4)

$\mu(\varphi(6^{6k+3} + 1)) = 0, k \in Z_0$ (5.6.5)

$\mu(\varphi[\varphi(6^{6k+3} + 1)]) = 0, k \in Z_0$ (5.6.6)

Proof: Consider theorem (3.1) remark (4.14) and recall the definition of number theoretic functions , and Möbious function . Then we find that the congruences (5.6.1) to (5.6.6) must hold.

6. Remark (final)

Remark 6.1. After considering the conjectures from (1) to (6) we find that family of integers of the form $6^n + 1, n \in Z_+$ are various families of integers with $\mu(n) = 0$.

7. Conclusion

We see that all numbers of form $6^n + 1, n \in Z_+$, end with 7 like the Fermat numbers $F_n = 2^{2^n} + 1, n \geq 2$, but no Fermat numbers reach such form. Only three primes $7, 37,$ and 1297 exist in this form whenever $n \in Z_+ - \{2^k m, k \geq 6, m \equiv 1(mod\, 2)\}$. Mostly members of the string of the form $6^n + 1, n \in Z_+$, of integers hold beautiful congruences with number theoretic functions , and Möbious function and generates beautiful families of integers with $\mu(n) = 0$.

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