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Bifurcations of a discrete-time Phytoplankton-Zooplankton model

Vimbiso Shiri, Xianyi Li*

Department of Big Data Science, School of Science, Zhejiang University of Science and Technology, Hangzhou 310023, China

Email: vimbieshiri@hotmail.com, mathxyli@zust.edu.cn

*Corresponding author: Xianyi Li. Email: mathxyli@zust.edu.cn

Abstract: This work mainly investigates the bifurcation problems and topological classifications of a discrete phytoplankton-zooplankton model, whose continuous version was proposed by Truscott and Brindley in 1994. The model is derived by using the semi-discretization technique. Within this study, trivial and semi-trivial fixed points are identified, as well as an interior fixed point that emerges based on specific parametric conditions. Subsequently, an exploration of their topological classifications is undertaken, employing linear stability theory in the vicinity of the trivial, semi-trivial, and interior fixed points. Leveraging the center manifold theorem and bifurcation theory, it is feasible to derive conditions under which the flip and Neimark-Sacker bifurcations are expected to transpire. To validate these findings and draw conclusions, numerical simulations are conducted which produce proof of a Neimark-Sacker bifurcation. Through these comprehensive analyses and simulations, the main aim of this research is to effectively augment the understanding of the dynamics of the model and affirm the validity of our results.

Keywords: Phytoplankton-zooplankton model, semi-discretization, flip bifurcation, Neimark-Sacker bifurcation.

1. Establishment of the model

1.1. Phytoplankton population

Over the past few decades, academics have developed an interest in phytoplankton due to its relevance in the maritime ecosystem, as the basis of its food chain and the effects of hazardous algal blooms, such as red tides, [1-12]. The mathematical formulas and parameters utilized to simulate the populations vary among the models which involve systems of differential equations. Despite several models having different biological, chemical, or physical structures, there are generally two groups of models: those with just two or three differential equations and those that contain several non-coupled differential equations. Only numerical exploration will elicit a response from the first class of models, [13-14]. However, instead of adapting the model output to specific data, the second class primarily analyzes the flaws in observational data and displays expected qualitative behaviors. The phytoplankton-zooplankton models, [1] and [2], are examples of the second class.

Truscott and Brindley, [1], were the first to develop a model that considers the predator-prey dynamics of phytoplankton and zooplankton as a nonlinear "excitable system" to explain red tide dynamics. Excitability means that a system, which is normally in equilibrium, is able to develop a huge response in the form of a pulse when disturbed by a particular perturbation. The Hollings type III has often been used in predator-prey models and phytoplankton-zooplankton models. It is derived from the paradox of enrichment [15].

Truscott and Brindley, [1], described the evolution of phytoplankton biomass, P, and zooplankton biomass, Z, where the growth of P takes a logistic form with a carrying capacity K, the predation of Z on P follows a Holling Type III, and the external mortality of Z is assumed to be linear. It is also mentioned that a phytoplankton-zooplankton model, illustrated by using ordinary differential equations, depicts the red tide environment as a constant system with population emerging through time.

The model by Truscott and Brindley is a two-component, Phytoplankton-Zooplankton, model which only compares the relationship between phytoplankton and zooplankton assuming ceteris paribus for the remaining components.

The hypothesis that phytoplankton blooms are caused by abrupt temperature changes was inspired by the excitability model developed by Truscott and Brindley, [1]. A fast-growing phytoplankton population and a slow-reproducing zooplankton population are both influenced by rising temperatures, and because the latter is unable to keep up with the former, a bloom results. The model exhibits various transient behaviors depending on the rate of temperature increase, v. In this instance, the temperature is a slow variable, similar to zooplankton, because it is an environmental factor that fluctuates over time at a specific rate, [16].

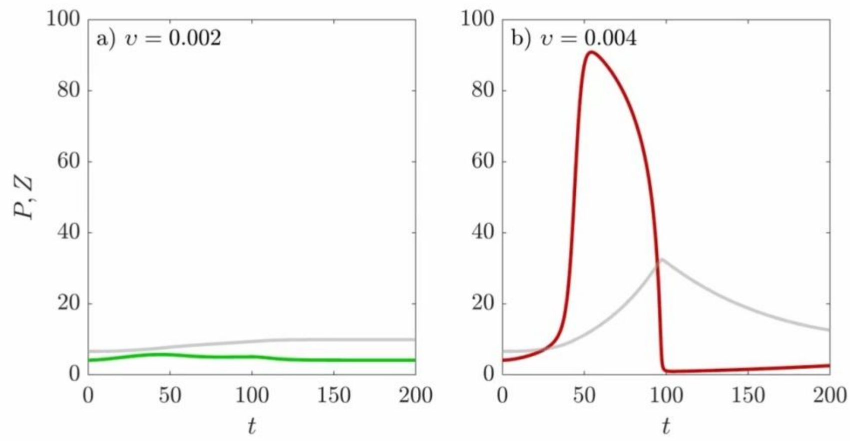


Figure 1. 1. Truscott and Brindley Phytoplankton (P) and Zooplankton (Z) model at different rates, ν, of environmental change, [16].

In Figure 1.1 (a) the rate is lower than $\boldsymbol{\nu \approx 0.003 day^{-1}}$ and there are low densities of the phytoplankton and zooplankton as shown by the green and grey lines respectively. In (b) the rate is higher and we see a sharp spike in the population density of the phytoplankton represented by the red line and a shorter spike for that of the zooplankton, shown by the grey line. This is an indication of a phytoplankton bloom after a rapid environmental change, [16]. Technically as the rate of environment change increases, it bypasses a threshold which can cause a nonlinear change in the dynamics of the system which we call bifurcation.

With reference to the Truscott and Brindley P-Z model, the breakdown of our parameters is as follows:

Table 1.1 Meaning of parameters in our model .

Parameter	Meaning
P	Population of phytoplankton
Z	Population of zooplankton
r	Gross rate of production of phytoplankton
R_m	Maximum specific rate of predation
K	Environmental carrying capacity of the phytoplankton
α	Used to determine how quickly the maximum of the carrying capacity is attained as prey densities increase
γ	Ratio of biomass phytoplankton consumed to biomass of new predators
μ	The rate of elimination of phytoplankton by death from other causes not including predation

According to [1], the continuous P-Z model is as follows:

$$\begin{cases} \frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - R_m Z \frac{P^2}{a^2+P^2}, \\ \frac{dZ}{dt} = \gamma R_m Z \frac{P^2}{a^2+P^2} - \mu Z. \end{cases} \tag{1.1}$$

where $\frac{P^2}{a^2+P^2}$ is derived from the Hollings type III grazing function. Ecological observations justify the use of a predation function which saturates for high prey densities, [8]. The excitability of the system arises because, γ, the efficiency of the zooplankton's conversion of food into biomass, is small. If γ<<1, then P evolves on a fast time scale relative to Z. γ covers a wide range of processes in feeding and reproduction. A small percentage of zooplankton is capable of reproduction, only a part of the ingested food is assimilated and only a small proportion is used for reproduction. It is not necessary to calculate these ratios directly, as they can be estimated implicitly from the values of the stable populations [8]. Furthermore, α is << K so that in any stable equilibrium of the P-Z subsystem that is not a bloom, P has a value far below that of a bloom. To derive our model we let,

$$\boldsymbol{P = K\widetilde{P}\,, \qquad Z = K\widetilde{Z}, \qquad t = \frac{\tilde{t}}{R_m}\,.} \tag{1.2}$$

System (1.1) then takes the form,

$$\begin{cases} \boldsymbol{R_m \frac{d\widetilde{P}}{d\tilde{t}} = r\widetilde{P}\left(1-\widetilde{P}\right) - R_m\widetilde{Z}\left(\frac{K^2\widetilde{P}^2}{a^2+K^2\widetilde{P}^2}\right)}, \\ \boldsymbol{R_m \frac{d\widetilde{Z}}{d\tilde{t}} = \gamma R_m\widetilde{Z}\left(\frac{K^2\widetilde{P}^2}{a^2+K^2\widetilde{P}^2}\right) - \mu Z.} \end{cases} \tag{1.3}$$

Using the transformation,

$$\boldsymbol{v = \frac{a}{K},\ \beta = \frac{r}{R_m},\ \omega = \frac{\mu}{\gamma R_m}}\,, \tag{1.4}$$

system (1.3) is modified into,

$$\begin{cases} \frac{d\tilde{P}}{d\tilde{t}} = \beta\tilde{P}(1-\tilde{P}) - \tilde{Z}\left(\frac{\tilde{P}^2}{v^2+\tilde{P}^2}\right), \\ \frac{d\tilde{Z}}{d\tilde{t}} = \gamma\tilde{Z}\left(\frac{\tilde{P}^2}{v^2+\tilde{P}^2} - \omega\right). \end{cases} \quad (1.5)$$

Finally our continuous-time phytoplankton-zooplankton model, (1.5), takes the following form after dropping the tildes,

$$\begin{cases} \frac{dP}{dt} = \beta P(1-P) - Z\left(\frac{P^2}{v^2+P^2}\right), \\ \frac{dZ}{dt} = \gamma Z\left(\frac{P^2}{v^2+P^2} - \omega\right). \end{cases} \quad (1.6)$$

Because solving a complicate system, generally speaking, one needs to use computer. So, one naturally considers its discrete model. We next study the corresponding discrete model of system (1.6), which we will derive using the semi-discretization method.

1.2. Piecewise constant arguments

First we will make an assumption that allows $[t]$ to be the greatest integer not exceeding t, also known as the greater-integer function.

Consider the avarage change rate of model (1.6) at the integer point. Our model (1.6) then becomes,

$$\begin{cases} \frac{1}{P(t)}\frac{dP(t)}{dt} = \beta(P([t]))(1-P([t])) - Z([t])\frac{(P([t]))^2}{v^2+(P([t]))^2}, \\ \frac{1}{Z(t)}\frac{dZ(t)}{dt} = \gamma(Z[t])\left(\frac{(P([t]))^2}{v^2+(P([t]))^2} - \omega\right). \end{cases} \quad (1.7)$$

We can see that piecewise constant arguments occur in system (1.7) and that any solution, $(P(t), Z(t))$, of the system for $t\epsilon(0, +\infty)$, will have the following three characteristics [17]:

i. $P(t)$ and $Z(t)$ are continuous on $(0, \infty)$.

ii. The derivatives $P'(t)$ and $Z'(t)$ exist at each point $t\epsilon(0,\infty)$ with the possible exception of the points $t\epsilon(0,\infty)$ where one-sided derivatives exist.

iii. The following (1.8) and (1.9) are satisfied and true in each interval $[n, n+1) \subset (0,\infty)$.

By integrating system (1.7) over the interval, $[n,t]$ for any $t\epsilon[n,n+1)$, where n=0, 1, 2, 3…, we get the following system,

$$\begin{cases} \boldsymbol{P(t) = P_n e^{\beta(1-P_n)-Z\frac{P_n}{v^2+P_n^2}}(t-n)}, \\ \boldsymbol{Z(t) = Z_n e^{\gamma\left(\frac{P_n^2}{v^2+P_n^2}-\omega\right)}(t-n)}. \end{cases} \tag{1.8}$$

We use P_n as $P(n)$ and Z_n for $Z(n)$.

As $t \to (n+1)^-$ in system (1.8), the following system is formed

$$\begin{cases} \boldsymbol{P_{n+1} = P_n e^{\beta(1-P_n)-Zn\frac{P_n}{v^2+P_n^2}}}, \\ \boldsymbol{Z_{n+1} = Z_n e^{\gamma\left(\frac{P_n^2}{v^2+P_n^2}-\omega\right)}}, \end{cases} \tag{1.9}$$

where all parameters $\beta, v, \omega, \gamma > 0$.

In this paper, our main contribution is the consideration of the dynamical properties of system (1.9), primarily for its stability and bifurcation. This includes topological classifications around the fixed points of the P-Z model (1.9), bifurcation analysis around the fixed points of the system using bifurcation theory and numerical simulation. We assume the space of parameters $\Omega = \{(\gamma, \beta, \omega, v) \in R_+^4\}$ with R_+ to be $(0,\infty)$.

The next section will be about the study of the fixed points and the topological classifications around them. Section 3, is dedicated to the comprehensive bifurcation analysis around the fixed points and Section 4, focuses on the verification of the theoretical results and finally the conclusion of the paper is found in Section 5.

2. Existence and topological classifications around fixed points.

In this section we will focus on the existence of fixed points and each point's stability. The fixed points of system (1.9) satisfy the following:

$$\boldsymbol{P = Pe^{\beta(1-P)-Z\frac{P}{v^2+P^2}}}, \qquad \boldsymbol{Z = Ze^{\gamma(\frac{P^2}{v^2+P^2}-\omega)}}. \tag{2.1}$$

We then find the fixed points which, due to the biological nature of the system, are only non-negative. Thus the system only has the fixed points $E_0(0,0), E_1(1,0)\ and\ E^*(P^*, Z^*)$,

where $P^* = \sqrt{\frac{\omega v^2}{1-\omega}}, \quad Z^* = \frac{\beta v(\sqrt{1-\omega}-v\sqrt{\omega})}{(1-\omega)\sqrt{\omega}}, \quad for\ \ 0 < \omega < \frac{1}{v^2+1}$.

The Jacobian matrix at any fixed point in system (1.9) takes on the following form

$$J(E) = \begin{pmatrix} \left(1 + P\left(\frac{Z(P^2-v^2)}{(v^2+P^2)^2} - \beta\right)\right) e^{\beta(1-P)-\frac{ZP}{v^2+P^2}} & \frac{-P^2}{v^2+P^2} e^{\beta(1-P)-\frac{ZP}{v^2+P^2}} \\ \frac{2\gamma PZv^2}{(v^2+P^2)^2} e^{\gamma\left(\frac{P^2}{v^2+P^2}-\omega\right)} & e^{\gamma\left(\frac{P^2}{v^2+P^2}-\omega\right)} \end{pmatrix}.$$

The characteristic polynomial of $J(E)$ is in the form,

$F(\lambda) = \lambda^2 - A\lambda + \mathrm{B}$,

where $A = tr(J(E))\ and\ B = \det(J(E))$, namely,

$$A = \left(\left(1 + P\left(\frac{Z(P^2-v^2)}{(v^2+P^2)^2} - \beta\right)\right) e^{\beta(1-P)-\frac{ZP}{v^2+P^2}} + e^{\gamma\left(\frac{P^2}{v^2+P^2}-\omega\right)}\right), \quad B =$$

$$\left(\left(1 + P\left(\frac{Z(P^2-v^2)}{(v^2+P^2)^2} - \beta\right)\right) + \frac{2\gamma P^3 Z v^2}{(v^2+P^2)^3}\right) e^{\beta(1-P)-\frac{PZ}{v^2+P^2}+\gamma\left(\frac{P^2}{v^2+P^2}-\omega\right)}.$$

To analyze the fixed points we need to use the following definition and lemma [18-20, 23].

Definition 2.1. *Let $E(x, y)$ be a fixed point of a 2D discrete system with multipliers λ_1 and λ_2.*

(i) *If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, $E(x, y)$ is called sink, so a sink is locally asymptotically stable.*

(ii) *If $|\lambda_1| > 1$ and $|\lambda_2| > 1$, $E(x, y)$ is called source, so a source is locally asymptotically unstable.*

(iii) *If $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$), $E(x, y)$ is called saddle.*

(iv) *If either $|\lambda_1| = 1$ or $|\lambda_2| = 1$, $E(x, y)$ is called to be non − hyperbolic.*

Lemma 2.1

Let

$$\boldsymbol{F(\lambda) = \lambda^2 - A\lambda + B}, \tag{2.2}$$

where A and B are two real constants.

Suppose λ_1 and λ_2 are two roots of $F(\lambda) = 0$, then the following statements hold.

a) *if $F(1) > 0$, then*

$(a.1)$ $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) > 0$ *and* $B < 1$;

$(a.2)$ $\lambda_1 = -1$ *and* $\lambda_2 \neq -1$ *if and only if* $F(-1) = 0$ *and* $A \neq 2$;

$(a.3)$ $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) < 0$;

$(a.4)$ $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and $only$ if $F(-1) > 0$ and $B > 1$;

$(a.5)$ λ_1 and λ_2 are a $pair$ of $conjugate$ $complex$ $roots$ $with$ $|\lambda_1| = |\lambda_2| = 1$, if and $only$ if $-2 < A < 2$ and $B = 1$;

$(a.6)$ $\lambda_1 = \lambda_2 = -1$ if and $only$ if $F(-1) = 0$ and $A = 2$.

b) If $F(1) = 0$, $namely$, 1 is one $root$ of $F(\lambda) = 0$, $then$ the $other$ $root$ λ satisfies

$|\lambda| = (<, >)1$ if and $only$ if $|B| = (<, >)1$.

c) If $F(1) < 0$, $then$ $F(\lambda) = 0$ has one $root$ $lying$ in $(1, \infty)$. $Moreover$,

$(c.1)$ the $other$ $root$ λ $satisfies$ $\lambda < (=) -1$ if and $only$ if $F(-1) < (=)0$;

$(c.2)$ the $other$ $root$ $-1 < \lambda < 1$ if and $only$ if $F(-1) > 0$.

For the stability of fixed points $E_0(0,0)$, $E_1(1,0)$, $E^*(P^*, Z^*)$ we can get the following theorems 2.1, 2.2, 2.3., respectively.

Theorem 2.1

The $fixed$ $point$ $E_0(0,0)$ in $system$ (1.9) is a $saddle$.

Proof:

$$J(E_0) = \begin{pmatrix} e^{\beta} & 0 \\ 0 & e^{-\omega\gamma} \end{pmatrix}.$$

For the fixed point $E_0(0,0)$, we have,

$\lambda_1 = e^{\beta} > 1, \quad \lambda_2 = e^{-\omega\gamma} < 1.$

So according to Lemma 2.1, we see that E_0 is a $saddle$. **(Proven)**

Theorem 2.2

The $following$ $statements$ $about$ the $fixed$ $point$ $E_1(1,0)$ in $system$ (1.9) are $true$.

$If \quad \omega > \dfrac{1}{v^2+1} \; then,$

1. $For \; 0 < \beta < 2 \,, E_1 \; is \; a \; sink$ (stable node).
2. $For \;\; \beta = 2, \; E_1 \; is \; non - hyperbolic.$
3. $For \;\; \beta > 2 \,, E_1 \; is \; a \; saddle.$

$If \; \omega < \dfrac{1}{v^2+1} \;\; then,$

1. $For \; 0 < \beta < 2, \; E_1 \; is \; a \; saddle.$
2. $For \;\; \beta = 2, \; E_1 \; is \; non - hyperbolic.$
3. $For \;\; \beta > 2, \; E_1 \; is \; a \; source$ (unstable node).

$If \;\; \omega = \frac{1}{v^2+1} \; then \,, E_1 \; is \; non - hyperbolic.$

Proof: For the fixed point $E_1(1,0)$, we have

$$J(E_1) = \begin{pmatrix} 1-\beta & -\frac{1}{v^2+1} \\ 0 & e^{\gamma\left(\frac{1}{v^2+1}-\omega\right)} \end{pmatrix}.$$

So the eigenvalues are $\lambda_1 = 1-\beta$ and $\lambda_2 = e^{\gamma\left(\frac{1}{v^2+1}-\omega\right)}$.

By applying Lemma 2.1 we consider the following three cases.

(i) Case (a): $\boldsymbol{\omega > \frac{1}{v^2+1}}$.

$If \; 0 < \beta < 2, then, \; from \;\; Lemma \; 2.1 \; (a.1), \; E_0 \; is \; a \; sink \; because$

$$F(1) = \beta\left(1 - e^{\gamma\left(\frac{1}{v^2+1}-\omega\right)}\right) > 0,$$

$$F(-1) = (2-\beta)\left(1 + e^{\gamma\left(\frac{1}{v^2+1}-\omega\right)}\right) > 0,$$

$$\mathrm{B} = (1-\beta)\left(e^{\gamma\left(\frac{1}{v^2+1}-\omega\right)}\right) \le |1-\beta|\left(e^{\gamma\left(\frac{1}{v^2+1}-\omega\right)}\right) < |1-\beta| < 1.$$

$If\ \ \beta > 2\ , then, E_0\ is\ a\ saddle\ because$

$$F(1) = \beta\left(1 - e^{\gamma\left(\frac{1}{v^2+1} - \omega\right)}\right) > 0,$$

$$F(-1) = (2 - \beta)\left(1 + e^{\gamma\left(\frac{1}{v^2+1} - \omega\right)}\right) < 0.$$

$If\ \ \beta = 2\ then, E_0\ is\ non - hyperbolic\ because$

$F(1) = \beta\left(1 - e^{\gamma\left(\frac{1}{v^2+1} - \omega\right)}\right) < 0,$

$F(-1) = (2 - \beta)\left(1 + e^{\gamma\left(\frac{1}{v^2+1} - \omega\right)}\right) = 0.$

(ii) Case (b): $\boldsymbol{\omega < \frac{1}{v^2+1}}$.

$If\ 0 < \beta < 2\ then\ \ E_0\ is\ a\ saddle\ because,$

$$F(1) = \beta\left(1 - e^{\gamma\left(\frac{1}{v^2+1} - \omega\right)}\right) < 0,$$

$F(-1) = (2 - \beta)\left(1 + e^{\gamma\left(\frac{1}{v^2+1} - \omega\right)}\right) > 0.$

$If\ \ \beta > 2\ then\ E_0\ is\ a\ source\ because,$

$$F(1) = \beta\left(1 - e^{\gamma\left(\frac{1}{v^2+1} - \omega\right)}\right) < 0,$$

$F(-1) = (2 - \beta)\left(1 + e^{\gamma\left(\frac{1}{v^2+1} - \omega\right)}\right) < 0.$

$If\ \ \beta = 2\ \ then\ E_0\ is\ non - hyperbolic\ because,$

$$F(1) = \beta\left(1 - e^{\gamma\left(\frac{1}{v^2+1} - \omega\right)}\right) < 0,$$

$F(-1) = (2 - \beta)\left(1 + e^{\gamma\left(\frac{1}{v^2+1} - \omega\right)}\right) = 0.$

(iii) **Case (c):** $\boldsymbol{\omega = \frac{1}{v^2+1}}$, E_0 *is non − hyperbolic in all cases because*

$F(1) = \beta\left(1 - e^{\gamma\left(\frac{1}{v^2+1}-\omega\right)}\right) = 0.$ **(Proven)**

Theorem 2.3

For $0 < \omega < \frac{1}{v^2+1}$, $E^*(P^*, Z^*)$ *is the unique positive fixed point of system* (1.9),

where $P^* = \sqrt{\frac{\omega v^2}{1-\omega}}$ and $Z^* = \frac{\beta v(\sqrt{1-\omega}-v\sqrt{\omega})}{(1-\omega)\sqrt{\omega}}$. The following consequences *are true*.

We have

$$J(E^*) = \begin{pmatrix} \beta\left(2\omega - 1 - 2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right) + 1 & -\omega \\ 2\gamma\beta(1-\omega)(1 - \sqrt{\frac{\omega v^2}{1-\omega}}) & 1 \end{pmatrix},$$

and

$F(\lambda) = \lambda^2 - A\lambda + B,$

$$= \lambda^2 - \left(2 + \beta\left(2\omega - 1 - 2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right)\right)\lambda + \beta\left(2\omega - 1 - 2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right) + 1 + 2\gamma\beta\omega(1-\omega)(1 - \sqrt{\frac{\omega v^2}{1-\omega}})$$

The roots of the equation are,

$$\lambda_{1,2} = (2 + \beta\left(-1 + 2\omega - 2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right) \pm \sqrt{\Delta},$$

where,

$$\Delta = (2 + \beta\left(2\omega - 1 - 2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right))^2 - 4\left(\beta\left(2\omega - 1 - 2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right) + 1 + 2\gamma\beta\omega(1-\omega)(1 - \sqrt{\frac{\omega v^2}{1-\omega}})\right).$$

$$=(\beta\left(2\omega - 1 - 2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right))^2 - \left(8\gamma\beta\omega(1-\omega)(1 - \sqrt{\frac{\omega v^2}{1-\omega}})\right)$$

Lemma 2.2

Given $\Delta = (2 + \beta\left(-1 + 2\omega - 2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right)^2 - 4\left(\beta\left(-1 + 2\omega - 2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right) + 1 + \frac{2\gamma v^2\beta\omega\left(1-\sqrt{\frac{\omega v^2}{1-\omega}}\right)}{\left(v^2+\frac{\omega v^2}{1-\omega}\right)}\right) \geq$

0, *then the following statements about* $E^*(P^*, Z^*)$, *are true. Let* $\gamma = \gamma_0$.

(i) $E^*(P^*, Z^*)$ *is a stable node if*,

$$\mathbf{0} < \boldsymbol{\gamma_0} < \frac{\left(\boldsymbol{\beta}\left(\mathbf{1}-\mathbf{2}\boldsymbol{\omega}+\mathbf{2}\boldsymbol{\omega}\sqrt{\frac{\boldsymbol{\omega v^2}}{\mathbf{1}-\boldsymbol{\omega}}}\right)-\mathbf{2}\right)\left(\mathbf{1}+\frac{\boldsymbol{\omega}}{\mathbf{1}-\boldsymbol{\omega}}\right)}{\boldsymbol{\beta\omega}(\mathbf{1}-\sqrt{\frac{\boldsymbol{\omega v^2}}{\mathbf{1}-\boldsymbol{\omega}}})}, \tag{2.11}$$

with

$$\boldsymbol{\beta} > -\frac{\mathbf{2}}{\left(\mathbf{1}-\mathbf{2}\boldsymbol{\omega}+\mathbf{2}\boldsymbol{\omega}\sqrt{\frac{\boldsymbol{\omega v^2}}{\mathbf{1}-\boldsymbol{\omega}}}\right)}; \tag{2.12}$$

(ii) $E^*(P^*, Z^*)$ *is an unstable node if* (2.3) *holds and if*

$$\gamma_0 > \frac{\left(\beta\left(1-2\omega+2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right)-2\right)\left(1+\frac{\omega}{1-\omega}\right)}{\beta\omega(1-\sqrt{\frac{\omega v^2}{1-\omega}})}; \tag{2.13}$$

(iii) $E^*(P^*, Z^*)$ is *non − hyperbolic if* (2.3) *holds and if*

$$\gamma_0 := \frac{\left(\beta\left(1-2\omega+2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right)-2\right)\left(1+\frac{\omega}{1-\omega}\right)}{\beta\omega(1-\sqrt{\frac{\omega v^2}{1-\omega}})}. \tag{2.14}$$

Lemma 2.3

$If\ \Delta = (2 + \beta\left(-1+2\omega-2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right)^2 - 4\left(\beta\left(-1+2\omega-2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right)+1+\frac{2\gamma v^2\beta\omega\left(1-\sqrt{\frac{\omega v^2}{1-\omega}}\right)}{\left(v^2+\frac{\omega v^2}{1-\omega}\right)}\right) <$ 0, *then the following statements are true. Let* $\gamma = \gamma_1$.

$E^*(P^*, Z^*)$ *is a stable focus if*

$$\mathbf{0} < \boldsymbol{\gamma_1} < \frac{\left(\left(\mathbf{1}-\mathbf{2}\boldsymbol{\omega}+\mathbf{2}\boldsymbol{\omega}\sqrt{\frac{\boldsymbol{\omega v^2}}{\mathbf{1}-\boldsymbol{\omega}}}\right)\right)\left(\mathbf{1}+\frac{\boldsymbol{\omega}}{\mathbf{1}-\boldsymbol{\omega}}\right)}{\mathbf{2}\boldsymbol{\omega}(\mathbf{1}-\sqrt{\frac{\boldsymbol{\omega v^2}}{\mathbf{1}-\boldsymbol{\omega}}})}, \tag{2.15}$$

with

$$\boldsymbol{v} < \sqrt{\frac{\mathbf{1}-\boldsymbol{\omega}}{\boldsymbol{\omega}}}; \tag{2.16}$$

$E^*(P^*, Z^*)$ is an $unstable\ focus\ if\ (2.3)\ holds\ and\ if$

$$\gamma_1 > \frac{\left(\left(1-2\omega+2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right)\right)\left(1+\frac{\omega}{1-\omega}\right)}{2\omega(1-\sqrt{\frac{\omega v^2}{1-\omega}})}; \tag{2.17}$$

$E^*(P^*, Z^*)$ is $non-hyperbolic\ if\ (2.7)\ holds\ and\ if$

$$\gamma_1 := \frac{\left(\left(1-2\omega+2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right)\right)\left(1+\frac{\omega}{1-\omega}\right)}{2\omega(1-\sqrt{\frac{\omega v^2}{1-\omega}})}. \tag{2.18}$$

3. Bifurcation analysis

In this section we will use the center manifold theorem and bifurcation theorem to navigate problems of the fixed points of the biological system (1.9), $E_1(1,0)$ and $E^*(P^*, Z^*)$.

We refer to previous biological work that used bifurcation analysis [17-22] to analyze the fixed points.

3.1. Fixed point $E_1(1,0)$

Theorem 2.2 shows that a bifurcation of system (1.9) may occur at the fixed point $E_1(1,0)$ in the space of the parameters

$\Omega_1 = \left\{(\beta, \omega, \gamma, \nu) \in \Omega | \ \ \omega > \frac{1}{\nu^2+1}\right\}$,

or

$\Omega_2 = \left\{(\beta, \omega, \gamma, \nu) \in \Omega | \ \ \omega < \frac{1}{\nu^2+1}\right\}$.

We now consider the following cases.

3.1.1. Case 1: $\boldsymbol{\beta_0 = 2}$, $\boldsymbol{\omega > \frac{1}{\nu^2+1}}$.

Our result is as follows.

Theorem 3.1.

Suppose the parameters $(\beta, \omega, \gamma, \nu) \in \Omega_1$, *where* $\beta = 2$. *If the parameter* β *varies closely in the neighorhood of the critical point* β_0 *system* (1.9) *may undergo a flip bifurcation at the fixed point,* $E_1(1,0)$.

Proof:

We first transform the fixed point, E_1 to the origin, $O(0,0)$.

Let $x_n = P_n - P_0$, $s_n = Z_n - Z_0$. We also give a small perturbation $\beta^* = \beta - \beta_0$ and letting ${\beta_{n+1}}^* = {\beta_n}^* = \beta^*$, system (1.9) becomes;

$$\begin{cases} \boldsymbol{x_{n+1} = (x_n + 1)e^{({\beta_n}^*+2)(1-(x_n+1))-(s_n)\frac{(x_n+1)}{\nu^2+(x_n+1)^2}} - 1}, \\ \boldsymbol{s_{n+1} = (s_n)e^{\gamma\left(\frac{(x_n+1)^2}{\nu^2+(x_n+1)^2}-\omega\right)}}, \\ \boldsymbol{\beta^*_{n+1} = {\beta_n}^*}. \end{cases} \tag{3.19}$$

By Taylor expansion, the system (3.3) at $(x_n, s_n, \beta^*_n) = (0,0,0)$ becomes,

$$\begin{cases} \boldsymbol{x_{n+1} = c_{100}x_n + c_{010}s_n + c_{001}\beta_n^* + c_{110}x_ns_n + c_{011}s_n\beta_n^* + c_{101}x_n\beta_n^* + c_{111}x_ns_n\beta_n^* + c_{200}{x_n}^2 + c_{020}{s_n}^2} \\ \boldsymbol{+ c_{002}(\beta_n^*)^2 + c_{120}x_n{s_n}^2 + c_{210}{x_n}^2s_n + c_{102}x_n(\beta_n^*)^2 + c_{201}{x_n}^2\beta_n^* + c_{021}{s_n}^2\beta_n^* + c_{012}s_n(\beta_n^*)^2 +} \\ \boldsymbol{c_{300}{x_n}^3 + c_{030}{s_n}^3 + c_{003}(\beta_n^*)^3 + o(\rho_{11}^3),} \\ \boldsymbol{s_{n+1} = d_{100}x_n + d_{010}s_n + d_{110}x_ns_n + d_{200}{x_n}^2 + d_{020}{s_n}^2 + d_{210}{x_n}^2s_n + d_{120}x_n{s_n}^2 + d_{300}{x_n}^3 + d_{030}{s_n}^3} \\ \boldsymbol{+o(\rho_{11}^3),} \\ \boldsymbol{\beta_{n+1}^* = \beta_n^*,} \end{cases} \tag{3.20}$$

where $\rho_{11} = \sqrt{{x_n}^2 + {s_n}^2 + (\beta_n^*)^2}$.

$c_{100} =-1,\ c_{101} =-\frac{1}{2},\ \ c_{201} = \frac{1}{3},\ c_{300} = \frac{2}{3},$

$c_{001} = c_{002} = c_{003} = c_{200} = \ \ c_{011} = c_{102} = c_{012} = c_{021} = 0,$

$c_{010} = \frac{-1}{v^2+1}\ \ ,\ \ \ c_{020} = \frac{1}{2(v^2+1)^2},\ \ \ c_{030} = \frac{-1}{6(v^2+1)^3},\ \ \ c_{110} = \frac{1}{(v^2+1)^2},\ \ \ \ c_{111} = \frac{1}{6(v^2+1)},$

$c_{120} = \frac{(v^2-3)}{6(v^2+1)^3},\ c_{210} = \frac{1}{3(v^2+1)} + \frac{1}{3(v^2+1)^2}\ \ \ -\frac{4}{3(v^2+1)^3},$

$d_{100} = d_{200} = d_{300} = d_{020} = d_{030} = d_{120} = 0,$

$d_{010} = e^{\gamma\left(\frac{1}{v^2+1}-\omega\right)},\ \ \ \ d_{110} = \frac{2v^2\gamma}{(v^2+1)^2}e^{\gamma\left(\frac{1}{v^2+1}-\omega\right)},\ \ \ \ d_{210} = \frac{2v^2\gamma(2v^2\gamma+v^4-2v^2-3)}{(v^2+1)^4}e^{\gamma\left(\frac{1}{v^2+1}-\omega\right)}.$

Let

$$M(E_1) = \begin{pmatrix} c_{100} & c_{010} & 0 \\ d_{100} & d_{010} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & \frac{-1}{v^2+1} & 0 \\ 0 & e^{\gamma\left(\frac{1}{v^2+1}-\omega\right)} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues of $M(E_1)$ are $\lambda_1 = \frac{-v^2-1}{v^2+1} =-1,\ \lambda_2 = e^{\gamma\left(\frac{1}{v^2+1}-\omega\right)}$ and $\lambda_3 = 1$.

Given $\frac{1}{v^2+1} \neq \omega$, a flip bifurcation may occur at the fixed point, (E_1), when β varies in the neighborhood of the critical point, β_0.

3.1.2. Case 2: $\boldsymbol{\beta_0 = 2, \omega < \frac{1}{v^2+1}}$

We expect the same result as in Case 1 hence the proof has been omitted.

3.2. Fixed point $\boldsymbol{E^*(P^*, Z^*)}$.

Lemmas (2.2) and (2.3) show that when $\gamma = \gamma_0$ and $\gamma = \gamma_1$, the fixed point $E^*(P^*, Z^*)$ is non-hyperbolic. Moreover the dimensional numbers for the stable and unstable manifold of $E^*(P^*, Z^*)$ vary when the parameter, γ, goes through these values. This shows that a bifurcation may occur in each case.

3.2.1. Flip bifurcation

When (2.5) holds , $F(-1) = 0$, which is one of the conditions for a flip bifurcation to occur. The following analysis is proof of the occurrence of a flip bifurcation.

Theorem 3.2.1.

Suppose the parameters $(\beta, \omega, \gamma, \nu) \in \Omega_3$. *If the parameter* γ *varies closely in the neighorhood of the critical point* γ_0 *system* (1.9) *may undergo a flip bifurcation at the fixed point,* $E^*(P^*, Z^*)$ *where,*

$$P^* = \sqrt{\frac{\omega v^2}{(1-\omega)}} \quad \text{and} \quad Z^* = \frac{\beta v(\sqrt{1-\omega} - v\sqrt{\omega})}{(1-\omega)\sqrt{\omega}} \; for \;\; \omega < \min\{1, \frac{1}{v^2+1}\}.$$

Proof: We first transform the fixed point, $E^*(P^*, Z^*)$to the origin, $O(0,0)$.

Let $u_n = P_n - P_0,\ v_n = Z_n - Z_0$,

$$\begin{cases} u_{n+1} = (u_n + P_0)e^{(\beta)(1-(u_n+P_0))-(v_n+Z_0)\frac{(u_n+P_0)}{v^2+(u_n+P_0)^2}} - P_0 \\ \qquad v_{n+1} = (v_n + Z_0)e^{\gamma\left(\frac{(u_n+P_0)^2}{v^2+(u_n+P_0)^2} - \omega\right)} - Z_0, \end{cases} \tag{3.21}$$

We also give a small perturbation, γ^*, of the parameter γ around γ_0 , i.e., $\gamma^* = \gamma - \gamma_0$, with $0 < |\gamma^*| \ll 1$. By letting ${\gamma_{n+1}}^* = {\gamma_n}^* = \gamma^*$, system (3.5) is perturbed into

$$\begin{cases} u_{n+1} = (u_n + P_0)e^{\beta(1-(u_n+P_0))-(v_n+Z_0)\frac{(u_n+P_0)}{v^2+(u_n+P_0)^2}} - P_0, \\ v_{n+1} = (v_n + Z_0)e^{(\gamma+\gamma^*)\left(\frac{(u_n+P_0)^2}{v^2+(u_n+P_0)^2}-\omega\right)} - Z_0, \\ {\gamma_{n+1}}^* = {\gamma_n}^*. \end{cases} \tag{3.22}$$

Using the Taylor Series Expansion, system (3.3) at $(u_n, v_n, \gamma_n^*) = (0,0,0)$ becomes,

$$\begin{cases} u_{n+1} = e_{100}u_n + e_{010}v_n + e_{001}\beta_n^* + e_{110}u_nv_n + e_{011}v_n\beta_n^* + e_{101}u_n\beta_n^* + e_{111}u_nv_n\beta_n^* + e_{200}{u_n}^2 + e_{020}{v_n}^2 \\ \quad + e_{002}(\beta_n^*)^2 + e_{120}u_n{v_n}^2 + e_{210}{u_n}^2v_n + e_{102}u_n(\beta_n^*)^2 + e_{201}{u_n}^2\beta_n^* + e_{021}{v_n}^2\beta_n^* + e_{012}v_n(\beta_n^*)^2 + \\ \quad e_{300}{u_n}^3 + e_{030}{v_n}^3 + e_{003}(\beta_n^*)^3 + o(\rho_{21}^3), \\ v_{n+1} = f_{100}u_n + f_{010}v_n + f_{110}u_nv_n + f_{200}{u_n}^2 + f_{020}{v_n}^2 + f_{210}{u_n}^2v_n + f_{120}u_n{v_n}^2 + f_{300}{u_n}^3 + f_{030}{v_n}^3 \\ \quad + o(\rho_{21}^3), \\ \gamma_{n+1}^* = \gamma_n^*, \end{cases} \tag{3.23}$$

where $\rho_{21} = \sqrt{{u_n}^2 + {v_n}^2 + (\gamma_n^*)^2}$.

$$e_{100} = \frac{e^{\beta-\frac{\beta\sqrt{-\frac{v^2\omega}{\omega-1}}\sqrt{1-\omega}}{v\sqrt{\omega}}}\left(-2v\beta\omega^{\frac{3}{2}}\sqrt{-\frac{v^2\omega}{\omega-1}}+v\sqrt{\omega}+2\beta\omega\sqrt{-\frac{v^2\omega}{\omega-1}}\sqrt{1-\omega}-\beta\sqrt{-\frac{v^2\omega}{\omega-1}}\sqrt{1-\omega}\right)}{v\sqrt{\omega}},$$

$$\boldsymbol{e_{200}} = \frac{e^{-\frac{\beta\left(v\sqrt{\omega}\left(\sqrt{-\frac{v^2\omega}{\omega-1}}-1\right)-\sqrt{-\frac{v^2\omega}{\omega-1}}(v\sqrt{\omega}-\sqrt{1-\omega})\right)}{v\sqrt{\omega}}}\beta\left(-2v^2\omega^2+2v\omega^{\frac{5}{2}}\cdot(4\omega-3)(v\sqrt{\omega}-\sqrt{1-\omega})-4v\omega^{\frac{5}{2}}(v\sqrt{\omega}-\sqrt{1-\omega})\right)}{v^2\omega^2} + \frac{2v\omega^{\frac{3}{2}}(v\sqrt{\omega}-\sqrt{1-\omega})+\beta\omega\sqrt{-\frac{v^2\omega}{\omega-1}}\left(2\omega\left(v\sqrt{\omega}-\sqrt{1-\omega}\right)+\sqrt{1-\omega}\right)^2)}{v^2\omega^2},$$

$\boldsymbol{e_{300}} =$ −

$$\frac{e^{-\frac{\beta\left(v\sqrt{\omega}\left(\sqrt{-\frac{v^2\omega}{\omega-1}}-1\right)-\sqrt{-\frac{v^2\omega}{\omega-1}}(v\sqrt{\omega}-\sqrt{1-\omega})\right)}{v\sqrt{\omega}}}\boldsymbol{\beta\cdot(6v\beta\omega^{\frac{7}{2}}\cdot(4\omega-3)(v\sqrt{\omega}-\sqrt{1-\omega})\cdot(2\omega(v\sqrt{\omega}-\sqrt{1-\omega})+\sqrt{1-\omega})-3v\beta\omega^{\frac{5}{2}}(2\omega(v\sqrt{\omega}-\sqrt{1-\omega})+\sqrt{1-\omega})^2+}}{\boldsymbol{v^3\omega^{\frac{7}{2}}}}$$

$$\frac{\boldsymbol{\beta^2\omega^2\sqrt{-\frac{v^2\omega}{\omega-1}}(2\omega(v\sqrt{\omega}-\sqrt{1-\omega})+\sqrt{1-\omega})^3+6\omega^3\sqrt{-\frac{v^2\omega}{\omega-1}}(\omega-1)(4\omega-3)(v\sqrt{\omega}-\sqrt{1-\omega})-6\omega^3\sqrt{-\frac{v^2\omega}{\omega-1}}(\omega-1)(v\sqrt{\omega}-\sqrt{1-\omega})(8\omega^2-8\omega+1))}}{\boldsymbol{v^3\omega^{\frac{7}{2}}}},$$

$$e_{010}=-\omega,\qquad e_{020}=\frac{(-\frac{v^2\omega}{\omega-1})^{\frac{3}{2}}(\omega^2-2\omega+1)}{v^4},\qquad e_{030}=\frac{\omega^2(\omega-1)}{v^2},$$

$$e_{110}=\frac{\left(2v^2\beta\omega^2-2v\beta\omega^{\frac{3}{2}}\sqrt{1-\omega}+v\beta\sqrt{\omega}\sqrt{1-\omega}-2\omega^2\sqrt{-\frac{v^2\omega}{\omega-1}}+4\omega\sqrt{-\frac{v^2\omega}{\omega-1}}-2\sqrt{-\frac{v^2\omega}{\omega-1}}\right)}{v^2},$$

$$e_{120}=\frac{e^{-\frac{\beta(v\sqrt{\omega}(\sqrt{-\frac{v^2\omega}{\omega-1}}-1)-\sqrt{-\frac{v^2\omega}{\omega-1}}(v\sqrt{\omega}-\sqrt{1-\omega}))}{v\sqrt{\omega}}}\sqrt{\omega}(\omega-1)(4v\omega^{\frac{3}{2}}-3v\sqrt{\omega}+\beta\sqrt{-\frac{v^2\omega}{\omega-1}}\cdot(2\omega(v\sqrt{\omega}-\sqrt{1-\omega})+\sqrt{1-\omega}))}{v^3},$$

$$e_{210}=$$

$$\frac{e^{-\frac{\beta\left(v\sqrt{\omega}\left(\sqrt{-\frac{v^2\omega}{\omega-1}}-1\right)-\sqrt{-\frac{v^2\omega}{\omega-1}}(v\sqrt{\omega}-\sqrt{1-\omega})\right)}{v\sqrt{\omega}}}(2v\omega^2(v\omega^{\frac{3}{2}}(\omega-1)(4\omega-3)-2v\omega^{\frac{3}{2}}(\omega-1)+v\sqrt{\omega}(\omega-1)+\beta_0\sqrt{-\frac{v^2\omega}{\omega-1}}\cdot(2\omega(v\sqrt{\omega}-\sqrt{1-\omega})+\sqrt{1-\omega})(2\omega(\omega-1)-\omega+1))}{v^4\omega^{\frac{5}{2}}}$$

$$\frac{-\beta_0\sqrt{\omega}\sqrt{-\frac{v^2\omega}{\omega-1}}(\omega-1)(2v^2\omega^2-2v\omega^{\frac{5}{2}}\cdot(4\omega-3)(v\sqrt{\omega}-\sqrt{1-\omega})+4v\omega^{\frac{5}{2}}(v\sqrt{\omega}-\sqrt{1-\omega})-2v\omega^{\frac{3}{2}}(v\sqrt{\omega}-\sqrt{1-\omega})-\beta_0\omega\sqrt{-\frac{v^2\omega}{\omega-1}}(2\omega(v\sqrt{\omega}-\sqrt{1-\omega})+\sqrt{1-\omega})^2))}{v^4\omega^{\frac{5}{2}}}$$

$$f_{100}=\frac{2\beta\gamma_0\sqrt{-\frac{v^2\omega}{\omega-1}}(v\sqrt{\omega}-\sqrt{1-\omega})(\omega(\omega-1)-\omega+1)}{v\sqrt{\omega}(\omega-1)},$$

$$f_{200}=-\frac{2\beta\gamma_0(v\sqrt{\omega}-\sqrt{1-\omega})(2\gamma_0\omega(\omega-1)^2+4\omega^2-5\omega+1)}{v\sqrt{\omega}},$$

$$f_{300}=-\frac{4\beta\gamma_0\sqrt{-\frac{v^2\omega}{\omega-1}}(\omega-1)(v\sqrt{\omega}-\sqrt{1-\omega})(2\gamma_0{}^2\omega(\omega-1)^3+3\gamma_0(\omega-1)(4\omega^2-5\omega+1)+12\omega^2-18\omega+6)}{v^3\sqrt{\omega}},$$

$$f_{110}=\frac{2\gamma_0\sqrt{-\frac{v^2\omega}{\omega-1}}(\omega(\omega-1)-\omega+1)}{v^2}\;,$$

$$f_{101}=\frac{2\beta\sqrt{-\frac{v^2\omega}{\omega-1}}(\omega-1)(v\sqrt{\omega}-\sqrt{1-\omega})}{v\sqrt{\omega}},$$

$$f_{111}=\frac{2\sqrt{-\frac{v^2\omega}{\omega-1}}(\omega-1)^2}{v^2},$$

$f_{210} = \frac{2\gamma_0(\omega-1)\,(-2\gamma_0\omega(\omega-1)^2-4\omega^2+5\omega-1)}{v^2},$

$f_{201} =-\frac{2\beta(v\sqrt{\omega}-\sqrt{1-\omega})(4\gamma_0\omega(\omega-1)^2+4\omega^2-5\omega+1)}{v\sqrt{\omega}},$

$f_{010} = 1,\ f_{020} = f_{030} = f_{120} = f_{102} = f_{011} = f_{012} = f_{001} = f_{002} = f_{003} = f_{021} = 0.$

Let $M(E^*) = \begin{pmatrix} e_{100} & e_{010} & 0 \\ f_{100} & f_{010} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{-\frac{2v^2\beta_0\omega^2}{\sqrt{1-\omega}}+2v\beta_0\omega^{\frac{3}{2}}-v\beta_0\sqrt{\omega}+v\sqrt{\omega}}{v\sqrt{\omega}} & -\omega & 0 \\ \frac{2\beta_0\gamma(v\sqrt{\omega}-\sqrt{1-\omega})(\omega(\omega-1)-\omega+1)}{\sqrt{1-\omega}(\omega-1)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$

then using (2.5), we get,

$\lambda_1 =- 1,\ \lambda_2 = \frac{A+B}{2}$ and $\lambda_3 = 1,$

with eigenvectors,

$$N^T = (\xi_{0i}, \eta_{0i}, \varphi_{0i})^T = (C,1,0)^T, (D\ ,1,0)^T, (0,0,1)^T,\ \ i = 1,2,3$$

$N = \begin{pmatrix} C & D & O \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$

$N^{-1} = \begin{pmatrix} E & \frac{B+A-2}{2B} & 0 \\ F & \frac{B-A+2}{2B} & 0 \\ 0 & 0 & 1 \end{pmatrix},$

$$A = \left(2 + \beta\left(-1 + 2\omega - 2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right)\right)$$

$$B = \sqrt{\left(2 + \beta\left(-1 + 2\omega - 2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right)\right)^2 - 4\left(\beta\left(-1 + 2\omega - 2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right) + 1 + \frac{2\gamma v^2\beta\omega\left(1-\sqrt{\frac{\omega v^2}{1-\omega}}\right)}{\left(v^2+\frac{\omega v^2}{1-\omega}\right)}\right)}$$

$$C =- \frac{\left(\beta\left(-1+2\omega-2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right)\right)+\sqrt{\left(2+\beta\left(-1+2\omega-2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right)\right)^2-4\left(\beta\left(-1+2\omega-2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right)+1+\frac{2\gamma v^2\beta\omega\left(1-\sqrt{\frac{\omega v^2}{1-\omega}}\right)}{\left(v^2+\frac{\omega v^2}{1-\omega}\right)}\right)}}{\frac{4\gamma v^2\beta(1-\sqrt{\frac{\omega v^2}{1-\omega}})}{(v^2+\frac{\omega v^2}{1-\omega})}}$$

$$D = -\frac{\left(\beta\left(-1+2\omega-2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right)\right)-\sqrt{\left(2+\beta\left(-1+2\omega-2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right)\right)^2-4\left(\beta\left(-1+2\omega-2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right)+1+\frac{2\gamma v^2\beta\omega\left(1-\sqrt{\frac{\omega v^2}{1-\omega}}\right)}{\left(v^2+\frac{\omega v^2}{1-\omega}\right)}\right)}}{\frac{4\gamma v^2\beta(1-\sqrt{\frac{\omega v^2}{1-\omega}})}{(v^2+\frac{\omega v^2}{1-\omega})}}$$

$$E = -\frac{\frac{2\gamma v^2\beta(1-\sqrt{\frac{\omega v^2}{1-\omega}})}{(v^2+\frac{\omega v^2}{1-\omega})}}{\sqrt{\left(2+\beta\left(-1+2\omega-2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right)\right)^2-4\left(\beta\left(-1+2\omega-2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right)+1+\frac{2\gamma v^2\beta\omega\left(1-\sqrt{\frac{\omega v^2}{1-\omega}}\right)}{\left(v^2+\frac{\omega v^2}{1-\omega}\right)}\right)}}$$

$$F = -E$$

Taking the transformation $(u_n, v_n, \gamma_n^*)^T = (X_n, Y_n, \alpha_n)^T$, system (3.6) is changed into the following form:

$$\begin{cases} X_{n+1} = -X_n + F(X_n, Y_n, \alpha_n) + o(\rho_{22}^3), \\ Y_{n+1} = (A+B)Y_n + G(X_n, Y_n, \alpha_n) + o(\rho_{22}^3), \\ \alpha_{n+1} = \alpha_n, \end{cases} \tag{3.24}$$

where $\rho_{22} = \sqrt{{X_n}^2 + {Y_n}^2 + (\alpha_n)^2}$,

$$\begin{aligned} F(X_n, Y_n, \alpha_n) = {} & l_{100}X_n + l_{010}Y_n + l_{001}\alpha_n + l_{110}X_nY_n + l_{011}Y_n\alpha_n + l_{101}X_n\alpha_n + l_{111}X_nY_n\alpha_n \\ & + l_{200}{X_n}^2 + l_{020}{Y_n}^2 + l_{002}(\alpha_n)^2 + l_{120}X_n{Y_n}^2 + l_{210}{X_n}^2Y_n + l_{102}X_n(\alpha_n)^2 \\ & + l_{201}{X_n}^2\alpha_n + l_{021}{Y_n}^2\alpha_n + l_{012}Y_n(\alpha_n)^2 + l_{300}{X_n}^3 + l_{030}{Y_n}^3 + l_{003}(\alpha_n)^3, \end{aligned}$$

$$\begin{aligned} G(X_n, Y_n, \alpha_n) = {} & m_{100}X_n + m_{010}Y_n + m_{001}\alpha_n + m_{110}X_nY_n + m_{011}Y_n\alpha_n + m_{101}X_n\alpha_n + m_{111}X_nY_n\alpha_n \\ & + m_{200}{X_n}^2 + m_{020}{Y_n}^2 + m_{002}(\alpha_n)^2 + m_{120}X_n{Y_n}^2 + m_{210}{X_n}^2Y_n + m_{102}X_n(\alpha_n)^2 \\ & + m_{201}{X_n}^2\alpha_n + m_{021}{Y_n}^2\alpha_n + m_{012}Y_n(\alpha_n)^2 + m_{300}{X_n}^3 + m_{030}{Y_n}^3 + m_{003}(\alpha_n)^3, \end{aligned}$$

$$l_{200} = C^2(Ee_{200} + \frac{B+A-2}{2B}f_{200}) + (Ee_{020} + \frac{B+A-2}{2B}f_{020}) + C\,(Ee_{110} + \frac{B+A-2}{2B}f_{110}),$$

$$l_{020} = D^2(Ee_{200} + \frac{B+A-2}{2B}f_{200}) + (Ee_{020} + \frac{B+A-2}{2B}f_{020}) + D(Ee_{110} + \frac{B+A-2}{2B}f_{110}),$$

$$l_{110} = 2CD(Ee_{200} + \frac{B+A-2}{2B}f_{200}) + 2(Ee_{020} + \frac{B+A-2}{2B}f_{020}) + (C+D)(Ee_{110} + \frac{B+A-2}{2B}f_{110}),$$

$l_{101} = C\left(\frac{B+A-2}{2B} f_{101})\right) + \left(\frac{B+A-2}{2B}\right) f_{011}$,

$l_{011} = D\left(\frac{B+A-2}{2B} f_{101})\right) + \frac{B+A-2}{2B} f_{011},$

$l_{300} = C^3(Ee_{300} + \frac{B+A-2}{2B} f_{300}) + (Ee_{030} + \frac{B+A-2}{2B} f_{030}) + C(Ee_{120} + \frac{B+A-2}{2B} f_{120} + C^2(Ee_{210} + \frac{B+A-2}{2B} f_{210}),$

$l_{030} = D^3(Ee_{300} + \frac{B+A-2}{2B} f_{300}) + (Ee_{030} + \frac{B+A-2}{2B} f_{030}) + D(Ee_{120} + \frac{B+A-2}{2B} f_{120} + D^2(Ee_{210} + \frac{B+A-2}{2B} f_{210}),$

$l_{120} = 3CD^2(Ee_{300} + \frac{B+A-2}{2B} f_{300}) + 3Ee_{030} + (C + 2D)Ee_{120} + (D^2 + 2CD)(Ee_{210} + \frac{B+A-2}{2B} f_{210}),$

$l_{210} = 3CD^3(Ee_{300} + \frac{B+A-2}{2B} f_{300}) + 3Ee_{030} + (2C + D)Ee_{120} + (C^2 + 2CD)(Ee_{210} + \frac{B+A-2}{2B} f_{210}),$

$l_{201} = C^2 \frac{B+A-2}{2B} f_{201} + C\left(\frac{B+A-2}{2B} f_{111}\right),$

$l_{021} = D^2\left(\frac{B+A-2}{2B} f_{021}\right) + D\left(\frac{B+A-2}{2B} f_{111}\right),$

$l_{111} = 2CD\left(\frac{B+A-2}{2B} f_{201}\right) + (C + D)(\frac{B+A-2}{2B} f_{111}),$

$l_{002} = l_{003} = l_{012} = l_{102} = 0$,

$m_{200} = C^2(Fe_{200} + \frac{B-A+2}{2B} f_{200}) + (Fe_{020}) + C\ (Fe_{110} + \frac{B-A+2}{2B} f_{110}),$

$m_{020} = D^2(Fe_{200} + \frac{B-A+2}{2B} f_{200}) + (Fe_{020}) + D(Fe_{110} + \frac{B-A+2}{2B} f_{110}),$

$m_{110} = 2CD(Fe_{200} + \frac{B-A+2}{2B} f_{200}) + 2(Fe_{020}) + (C + D)(Fe_{110} + \frac{B-A+2}{2B} f_{110}),$

$m_{101} = C\left(\frac{B-A+2}{2B} f_{101})\right) + \frac{B-A+2}{2B} f_{011}$

$m_{011} = D\left(\frac{B-A+2}{2B} f_{101})\right) + \left(\frac{B-A+2}{2B}\right) f_{011}$

$m_{300} = C^3(Fe_{300} + \frac{B-A+2}{2B} f_{300}) + (Fe_{030}) + C(Fe_{120}) + C^2(Fe_{210} + \frac{B-A+2}{2B} f_{210}),$

$m_{030} = D^3(Fe_{300} + \frac{B-A+2}{2B} f_{300}) + (Fe_{030}) + D(Fe_{120}) + D^2(Fe_{210} + \frac{B-A+2}{2B} f_{210}),$

$m_{120} = 3CD^2(Fe_{300} + \frac{B-A+2}{2B} f_{300}) + 3(Fe_{030}) + (C + 2D)(Fe_{120}) + (D^2 + 2CD)(Fe_{210} + \frac{B-A+2}{2B} f_{210}),$

$m_{210} = 3C^2D(Fe_{300} + \frac{B-A+2}{2B} f_{300}) + 3(Fe_{030}) + (2C + D)(Fe_{120}) + (C^2 + 2CD)(Fe_{210} + \frac{B-A+2}{2B} f_{210}),$

$m_{201} = C^2 \frac{B-A+2}{2B} f_{201} + C\left(\frac{B-A+2}{2B} f_{111}\right),$

$m_{021} = D^2\left(\frac{B-A+2}{2B} f_{201}\right) + D\left(\frac{B-A+2}{2B} f_{111}\right),$

$$m_{111} = 2CD\left(\frac{B-A+2}{2B}f_{201}\right) + (C+D)\left(\frac{B-A+2}{2B}f_{111}\right),$$

$$m_{102} = m_{012} = m_{002} = m_{003} = 0.$$

Next we suppose on the center manifold

$$Y_n = h(X_n, \alpha_n) = h_{20}{X_n}^2 + h_{11}X_n\alpha_n + h_{02}{\alpha_n}^2 + o(\rho_{23}^3),$$

where $\rho_{23} = \sqrt{{X_n}^2 + (\alpha_n)^2}$.

Having,

$$h(X_{n+1}, \alpha_{n+1}) = h_{20}(-X_n + F(X_n, h(X_n, \alpha_n), \alpha_n))^2 + h_{11}(-X_n + F(X_n, h(X_n, \alpha_n), \alpha_n))\alpha_n + h_{02}{\alpha_n}^2 + o\left(\rho_{33}^3\right)$$

$$= h_{20}{X_n}^2 - h_{11}X_n\alpha_n + h_{02}{\alpha_n}^2 + o\left(\rho_{23}^3\right),$$

and

$$Y_{n+1} = h(X_{n+1}, \alpha_{n+1}) = \left(\frac{A+B}{2}\right)h(X_n, \alpha_n) + G(X_n, h(X_n, \alpha_n), \alpha_n) + o(\rho_{33}^3)$$

$$= \left(\frac{A+B}{2}\right)(h(X_n, \alpha_n) + G(X_n, h(X_n, \alpha_n), \alpha_n))\alpha_n + h_{02}{\alpha_n}^2 + o\left(\rho_{33}^3\right)$$

$$= \{\left(\frac{A+B}{2}\right)h_{20} + l_{200}\}{X_n}^2 + \left\{\left(\frac{A+B}{2}\right)h_{11} + l_{101}\right\}X_n\alpha_n + \left\{\left(\frac{A+B}{2}\right)h_{02} + l_{002}\right\}{\alpha_n}^2 + o\left(\rho_{23}^3\right),$$

and comparing the corresponding coefficients of terms with the same orders in the above center manifold equation, one gets,

$$h_{20} = \frac{2l_{200}}{2-(A+B)},$$

$$h_{11} = -\frac{2l_{101}}{2+(A+B)},$$

$$h_{02} = 0.$$

Therefore, system (3.8), restricted to the center manifold, takes on the form,

$$X_{n+1} = f(X_n, \alpha_n)$$

$$:= -X_n + \mathrm{F}(X_n, (X_n, \alpha_n), \alpha_n) + o(\rho_{33}^3)$$

$$:= -X_n + m_{200}{X_n}^2 + m_{101}X_n\alpha_n + \{m_{300} + h_{20} * m_{110}\}{X_n}^3 + \{l_{110} * h_{11} + m_{011} * h_{20} + m_{201}\}{X_n}^2\alpha_n + \{m_{102} + l_{110} * h_{02} + m_{011} * h_{11}\}X_n{\alpha_n}^2$$

$$:=- X_n + g_{20}{X_n}^2 + g_{11}X_n\alpha_n + g_{30}{X_n}^3 + g_{21}{X_n}^2\alpha_n + g_{12}X_n{a_n}^2.$$

Where,

$g_{20} = m_{200}$, $g_{11} = m_{101}$, $g_{30} = m_{300} + h_{20} * m_{110}$,

$g_{21} = l_{110} * h_{11} + m_{011} * h_{20} + m_{201}$, $g_{12} = m_{102} + l_{110} * h_{02} + m_{011} * h_{11}$.

and

$f^2(X_n, \alpha_n) = f(f(X_n, \alpha_n), \alpha_n) = X_n - 2g_{11}X_n\alpha_n - 2({g^2}_{20} + g_{30}){X_n}^3 + ({g^2}_{11} - 2g_{21})X_n{a_n}^2 - g_{11}g_{22}{X_n}^2\alpha_n + o(\rho_{23}^4)$.

Hence we have,

$f(0,0) = 0$, $\frac{\partial f(0,0)}{\partial X_n} =- 1, \frac{\partial f^2(0,0)}{\partial a_n} = 0, \frac{\partial^2 f^2(0,0)}{\partial {X_n}^2} = 0,$

$\frac{\partial^2 f^2(0,0)}{\partial X_n \alpha_n} =- 2g_{11}, \frac{\partial^3 f^2(0,0)}{\partial {X_n}^3} =- 12({g^2}_{20} + g_{30}).$

For a Flip bifurcation to occur the following conditions should be satisfied.

$\frac{\partial^2 f^2(0,0)}{\partial X_n \alpha_n}\frac{\partial^3 f^2(0,0)}{\partial {X_n}^3} \neq 0,$

which is equivalent to,

$24g_{11}({g^2}_{20} + g_{30}) \neq 0.,$

Therefore we can conclude that a flip bifurcation may occur at the fixed point $E^*(P^*, Z^*)$.

3.2.2. Neimark-Sacker bifurcation

In the following, we prove the possibility of an occurrence of a Neimark-Sacker, (N-S), bifurcation around the fixed point around $E^*(P^*, Z^*)$ = $E^*\left(\sqrt{\frac{\omega v^2}{(1-\omega)}}, \frac{\beta v(\sqrt{1-\omega} - v\sqrt{\omega})}{(1-\omega)\sqrt{\omega}}\right)$.

If (2.9) holds, then the Jacobian matrix has conjugate complex eigenvalues with $|\lambda_{1,2}| = 1$, which further shows that around $\boldsymbol{E}^*\left(\sqrt{\frac{\omega v^2}{(1-\omega)}}, \frac{\beta v(\sqrt{1-\omega}-v\sqrt{\omega})}{(1-\omega)\sqrt{\omega}}\right)$ there may exist the N-S bifurcation by choosing γ as a bifurcation parameter if $(\beta, \omega, \gamma, \nu) \in \Omega_4$ [11]

The following theorem guarantees that, around $E^*\left(\sqrt{\frac{\omega v^2}{(1-\omega)}}, \frac{\beta v(\sqrt{1-\omega}-v\sqrt{\omega})}{(1-\omega)\sqrt{\omega}}\right)$, the phytoplankton-zooplankton model must undergo a Neimark-Sacker Bifurcation.

Theorem 3.2.2.

$$\boldsymbol{If}\ (\beta, \omega, \gamma, \nu) \in \Omega_4, then\ around,\ E^*\left(\sqrt{\frac{\omega v^2}{(1-\omega)}}, \frac{\beta v(\sqrt{1-\omega}-v\sqrt{\omega})}{(1-\omega)\sqrt{\omega}}\right), the$$

$Phytoplankton - Zooplankton\ model\ in\ (1.9)\ undergoes\ a\ \ Neimark - Sacker\ Bifurcation$.

Proof: $\boldsymbol{ppose\ the\ parameters}$ $(\beta, \omega, \gamma, \nu) \in \Omega_4$. It is clear that γ is the bifurcation parameter therefore, if γ is in the neighborhood of γ^*, i.e $\gamma = \gamma^* + \gamma_1$ where $\epsilon \ll 1$ then the phytoplankton-zooplankton model takes the following form.

Transform the fixed point $E^*\left(\sqrt{\frac{\omega v^2}{(1-\omega)}}, \frac{\beta v(\sqrt{1-\omega}-v\sqrt{\omega})}{(1-\omega)\sqrt{\omega}}\right)$ to the origin $O(0,0)$, and the system (3.5), given a perturbation $\gamma = \gamma^* + \gamma_1$ becomes,

$$\begin{cases} \boldsymbol{u_{n+1}} = \left(\boldsymbol{u_n} + \sqrt{\frac{\boldsymbol{\omega v^2}}{(\boldsymbol{1-\omega})}}\right)\boldsymbol{e}^{\beta\left(1-\left(u_n+\sqrt{\frac{\omega v^2}{(1-\omega)}}\right)\right)-\left(v_n+\frac{\beta v(\sqrt{1-\omega}-v\sqrt{\omega})}{(1-\omega)\sqrt{\omega}}\right)\frac{\left(u_n+\sqrt{\frac{\omega v^2}{(1-\omega)}}\right)}{v^2+\left(u_n+\sqrt{\frac{\omega v^2}{(1-\omega)}}\right)^2}} - \sqrt{\frac{\boldsymbol{\omega v^2}}{(\boldsymbol{1-\omega})}} \\ \boldsymbol{v_{n+1}} = \left(\boldsymbol{v_n} + \frac{\boldsymbol{\beta v(\sqrt{1-\omega}-v\sqrt{\omega})}}{(\boldsymbol{1-\omega})\sqrt{\boldsymbol{\omega}}}\right)\boldsymbol{e}^{(\gamma^*+\gamma_1)\left(\frac{\left(u_n+\sqrt{\frac{\omega v^2}{(1-\omega)}}\right)^2}{v^2+\left(u_n+\sqrt{\frac{\omega v^2}{(1-\omega)}}\right)^2}-\omega\right)} - \frac{\boldsymbol{\beta v(\sqrt{1-\omega}-v\sqrt{\omega})}}{(\boldsymbol{1-\omega})\sqrt{\boldsymbol{\omega}}} \end{cases} \tag{3.25}$$

The characteristic equation for the linearized equation of the system (3.9) at the fixed point $(0,0)$ is

$$\boldsymbol{F(\lambda) = \lambda^2 - A(\gamma^*)\lambda + \mathrm{B}(\gamma^*) = 0}, \tag{3.26}$$

with two roots,

$$\lambda_{1,2} = \frac{A(\gamma^*) \pm \iota\sqrt{4\mathrm{B}(\gamma^*) - A^2(\gamma^*)}}{2},$$

where

$$A(\gamma^*) = \beta\left(-1 + 2\omega - 2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right) + 2,$$

$$\mathrm{B}(\gamma^*) = 2\beta(\gamma^* + \gamma_1)\omega^2\sqrt{-\frac{v^2\omega}{\omega-1}} - 2\beta(\gamma^* + \gamma_1)\omega^2 - 2\beta(\gamma^* + \gamma_1)\omega\sqrt{-\frac{v^2\omega}{\omega-1}} + 2\beta(\gamma^* + \gamma_1)\omega - 2\beta\omega\sqrt{-\frac{v^2\omega}{\omega-1}} + 2\beta\omega - \beta + 1.$$

It is easy to derive

$$(|\lambda_{1,2}(\gamma^*)|)|_{\gamma^*=0} = \sqrt{\mathrm{B}(\gamma^*)}|_{\gamma^*=0}$$

$$= \sqrt{\begin{array}{c} -2\beta\omega\sqrt{-\frac{v^2\omega}{\omega-1}} + \frac{\beta\omega\sqrt{-\frac{v^2\omega}{\omega-1}}\left(\frac{\omega}{1-\omega}+1\right)\left(2\omega\sqrt{\frac{v^2\omega}{1-\omega}}-2\omega+1\right)}{1-\sqrt{\frac{v^2\omega}{1-\omega}}} + 2\beta\omega - \frac{\beta\omega\left(\frac{\omega}{1-\omega}+1\right)\left(2\omega\sqrt{\frac{v^2\omega}{1-\omega}}-2\omega+1\right)}{1-\sqrt{\frac{v^2\omega}{1-\omega}}} - \frac{\beta\sqrt{-\frac{v^2\omega}{\omega-1}}\left(\frac{\omega}{1-\omega}+1\right)\left(2\omega\sqrt{\frac{v^2\omega}{1-\omega}}-2\omega+1\right)}{1-\sqrt{\frac{v^2\omega}{1-\omega}}} \\ -\beta + \frac{\beta(\frac{\omega}{1-\omega}+1)(2\omega\sqrt{\frac{v^2\omega}{1-\omega}}-2\omega+1)}{1-\sqrt{\frac{v^2\omega}{1-\omega}}} + 1 \end{array}} = \mathbf{1},$$

which implies,

$$\left(\frac{d|\lambda_{1,2}(\gamma^*)|}{d\gamma^*}\right)|_{\gamma^*=0} = 2\beta\omega\left(\omega\sqrt{-\frac{v^2\omega}{\omega-1}} - \omega - \sqrt{-\frac{v^2\omega}{\omega-1}} + 1\right) > 0 \neq 0,$$

Furthermore for the existence of an N-S bifurcation around $E^*\left(\sqrt{\frac{\omega v^2}{(1-\omega)}}, \frac{\beta v(\sqrt{1-\omega}-v\sqrt{\omega})}{(1-\omega)\sqrt{\omega}}\right)$**,**

it is also required that $\lambda_{1,2}{}^m \neq 1, m = 1,2,3,4, if\ \gamma^* = 0$.

So if $(|\lambda_{1,2}(\gamma^*)|)|_{\gamma^*=0} = \sqrt{\mathrm{B}(\gamma^*)}|_{\gamma^*=0} = 1$ and $A(0) = \beta\left(-1 + 2\omega - 2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right) + 2,$

we have,

$$\lambda_{1,2} = \frac{\beta\left(-1+2\omega-2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right)+2 \pm \iota\sqrt{-4\left(\beta\left(-1+2\omega-2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right)\right)^2 + 2\beta(\gamma^*+\gamma_1)\omega^2\sqrt{-\frac{v^2\omega}{\omega-1}} - 2\beta(\gamma^*+\gamma_1)\omega^2 - 2\beta(\gamma^*+\gamma_1)\omega\sqrt{-\frac{v^2\omega}{\omega-1}} + 2\beta(\gamma^*+\gamma_1)\omega - 2\beta\omega\sqrt{-\frac{v^2\omega}{\omega-1}} + 2\beta\omega - \beta + 1}}{2}$$

Hence it is easy to derive that $\lambda_{1,2}{}^m \neq 1$.

The following conditions for a Neimark-Sacker bifurcation have been satisfied:

I. $\left(\frac{d\left|\lambda_{1,2}(\gamma^*)\right|}{d\gamma^*}\right)|_{\gamma^*=0} \neq 0$

II. $\lambda_{1,2}{}^{m} \neq 1, m = 1,2,3,4$

Now, in order to derive the normal form of system (3.9) we will expand the system to a second order power series around the origin.

$$\begin{cases} \boldsymbol{u_{n+1} = i_{10}u_n + i_{01}v_n + i_{11}u_nv_n + i_{21}u_n{}^2v_n + i_{12}u_nv_n{}^2 + i_{20}u_n{}^2 + i_{02}v_n{}^2 + i_{03}v_n{}^3 + i_{30}u_n{}^3 + o(\rho_{31}^3)} \\ \boldsymbol{v_{n+1} = j_{10}u_n + j_{01}v_n + j_{11}u_nv_n + j_{21}u_n{}^2v_n + j_{12}u_nv_n{}^2 + j_{20}u_n{}^2 + j_{02}v_n{}^2 + j_{30}u_n{}^3 + j_{03}v_n{}^3 + o(\rho_{31}^3)} \end{cases} \quad (3.27)$$

Where $\rho_{31} = \sqrt{u_n{}^2 + v_n{}^2}$

$i_{10} = e_{100},\ i_{01} = e_{010},\ i_{11} = e_{110},\ i_{21} = e_{210},\ i_{12} = e_{120},\ i_{20} = e_{200},\ i_{02} = e_{020}\,,\ i_{30} = e_{300},\ i_{03} = e_{030},$

$j_{10} = \frac{2\beta\gamma_1\sqrt{-\frac{v^2\omega}{\omega-1}}(v\sqrt{\omega}-\sqrt{1-\omega})(\omega(\omega-1)-\omega+1)}{v\sqrt{\omega}(\omega-1)},$

$j_{20} = -\frac{2\beta\gamma_1(v\sqrt{\omega}-\sqrt{1-\omega})(2\gamma_1\omega(\omega-1)^2+4\omega^2-5\omega+1)}{v\sqrt{\omega}},$

$j_{30} = -\frac{4\beta\gamma_1\sqrt{-\frac{v^2\omega}{\omega-1}}(\omega-1)(v\sqrt{\omega}-\sqrt{1-\omega})(2\gamma_1{}^2\omega(\omega-1)^3+3\gamma_0(\omega-1)(4\omega^2-5\omega+1)+12\omega^2-18\omega+6)}{v^3\sqrt{\omega}},$

$j_{11} = \frac{2\gamma_1\sqrt{-\frac{v^2\omega}{\omega-1}}(\omega(\omega-1)-\omega+1)}{v^2},$

$j_{21} = \frac{2\gamma_1(\omega-1)\ (-2\gamma_1\omega(\omega-1)^2-4\omega^2+5\omega-1)}{v^2},$

$j_{01} = 1,\ j_{02} = j_{03} = j_{12} = 0.$

Now, we use the following transformation in order to transform the linear part of system (3.11) above into a canonical form.

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} 0 & i_{01} \\ -\zeta & \eta - i_{10} \end{pmatrix} \begin{pmatrix} P_n \\ Z_n \end{pmatrix}, \quad (3.28)$$

$$T = \begin{pmatrix} 0 & i_{01} \\ -\zeta & \eta - i_{10} \end{pmatrix}, \qquad T^{-1} = \begin{pmatrix} \frac{\eta - i_{10}}{\zeta i_{01}} & -\frac{1}{\zeta} \\ \frac{1}{i_{01}} & 0 \end{pmatrix},$$

with

$$\eta = \beta\left(-1 + 2\omega - 2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right) + 2,$$

$$\zeta = \frac{\sqrt{4\left(2\beta(\gamma^* + \gamma_1)\omega^2\sqrt{-\frac{v^2\omega}{\omega-1}} - 2\beta(\gamma^* + \gamma_1)\omega^2 - 2\beta(\gamma^* + \gamma_1)\omega\sqrt{-\frac{v^2\omega}{\omega-1}} + 2\beta(\gamma^* + \gamma_1)\omega - 2\beta\omega\sqrt{-\frac{v^2\omega}{\omega-1}} + 2\beta\omega - \beta + 1\right) - \left(\beta\left(-1 + 2\omega - 2\omega\sqrt{\frac{\omega v^2}{1-\omega}}\right) + 2\right)^2}}{2}.$$

Due to (3.12), system (3.11) then takes the following form,

$$P_{n+1} = \eta P_n - \zeta Z_n + \overline{N}(P_n, Z_n), \tag{3.29}$$

$$Z_{n+1} = \eta Z_n + \zeta P_n + \overline{O}(P_n, Z_n),$$

where,

$$\overline{N}(P_n, Z_n) = n_{11}P_nZ_n + n_{21}{P_n}^2Z_n + n_{20}{P_n}^2 + n_{30}{P_n}^3 + n_{03}{Z_n}^3 + n_{12}P_n{Z_n}^2 + n_{02}{Z_n}^2,$$

$$\overline{O}(P_n, Z_n) = o_{11}P_nZ_n + o_{21}{P_n}^2Z_n + o_{20}{P_n}^2 + o_{30}{P_n}^3,$$

$$n_{20} = \zeta^2\left(Hi_{02} - \frac{Ij_{02}}{\zeta}\right),$$

$$n_{02} = {i_{01}}^2\left(Hi_{20} - \frac{j_{20}}{\zeta}\right) + (\eta - i_{10})\left(Hi_{11} - \frac{j_{11}}{\zeta}\right) + (\eta - i_{10})^2\left(Hi_{02} - \frac{j_{02}}{\zeta}\right),$$

$$n_{11} =- 2\zeta(\eta - i_{10})\left(Hi_{02} - \frac{j_{02}}{\zeta}\right) - \zeta\left(Hi_{11} - \frac{j_{11}}{\zeta}\right),$$

$$n_{21} = 3\zeta^2(\eta - i_{10})\left(Hi_{03} - \frac{j_{03}}{\zeta}\right) + \zeta^2 i_{10}\left(Hi_{12} - \frac{j_{12}}{\zeta}\right),$$

$$n_{03} = (\eta - i_{10})^3\left(Hi_{03} - \frac{j_{03}}{\zeta}\right) + i_{10}(\eta - i_{10})^2\left(Hi_{12} - \frac{j_{12}}{\zeta}\right) + {i_{01}}^2(\eta - i_{10})\left(Hi_{21} - \frac{j_{21}}{\zeta}\right) + {i_{01}}^3\left(Hi_{30} - \frac{j_{30}}{\zeta}\right),$$

$$n_{12} =- 2\zeta i_{01}(\eta - i_{10})\left(Hi_{12} - \frac{j_{12}}{\zeta}\right) - 3\zeta(\eta - i_{10})^2\left(Hi_{03} - \frac{j_{03}}{\zeta}\right) - \zeta{i_{01}}^2(\eta - i_{10})\left(Hi_{21} - \frac{j_{21}}{\zeta}\right),$$

$$n_{30} =- \zeta^3\left(Hi_{03} - \frac{j_{03}}{\zeta}\right),$$

$o_{20} = \zeta^2(Ii_{02}),$

$o_{02} = {i_{01}}^2(Ii_{20}) + (\eta - i_{10})(Ii_{11}) + (\eta - i_{10})^2(Ii_{02}),$

$o_{11} =- 2\zeta(\eta - i_{10})(Ii_{02}) - \zeta(Ii_{11}),$

$o_{21} = 3\zeta^2(\eta - i_{10})(Ii_{03}) + \zeta^2 i_{10}(Ii_{12}),$

$o_{03} = (\eta - i_{10})^3(Ii_{03}) + i_{10}(\eta - i_{10})^2(Ii_{12}) + {i_{01}}^2(\eta - i_{10})(Ii_{21}) + {i_{01}}^3(Ii_{30}),$

$o_{12} =- 2\zeta i_{01}(\eta - i_{10})(Ii_{12}) - 3\zeta(\eta - i_{10})^2(Ii_{03}) - \zeta {i_{01}}^2(\eta - i_{10})(Ii_{21}),$

$o_{30} =- \zeta^3(Ii_{03}),$

$H = \frac{\eta - i_{10}}{\zeta i_{01}}, \quad I = \frac{1}{i_{01}}$.

One can easily derive,

$\overline{N}_{PP} = 2\zeta^2\left(Hi_{02} - \frac{Ij_{02}}{\zeta}\right),$

$\overline{N}_{ZZ} = 2\left({i_{01}}^2\left(Hi_{20} - \frac{j_{20}}{\zeta}\right) + (\eta - i_{10})\left(Hi_{11} - \frac{j_{11}}{\zeta}\right) + (\eta - i_{10})^2\left(Hi_{02} - \frac{j_{02}}{\zeta}\right)\right),$

$\overline{N}_{PPP} =- 6\zeta^3\left(Hi_{03} - \frac{j_{03}}{\zeta}\right),$

$\overline{N}_{PZ} =- 2\zeta(\eta - i_{10})\left(Hi_{02} - \frac{j_{02}}{\zeta}\right) - \zeta\left(Hi_{11} - \frac{j_{11}}{\zeta}\right),$

$\overline{N}_{ZZZ} = 6\left((\eta - i_{10})^3\left(Hi_{03} - \frac{j_{03}}{\zeta}\right) + i_{10}(\eta - i_{10})^2\left(Hi_{12} - \frac{j_{12}}{\zeta}\right) + {i_{01}}^2(\eta - i_{10})\left(Hi_{21} - \frac{j_{21}}{\zeta}\right) + {i_{01}}^3\left(Hi_{30} - \frac{j_{30}}{\zeta}\right)\right),$

$\overline{N}_{PZZ} = 2\left(-2\zeta i_{01}(\eta - i_{10})\left(Hi_{12} - \frac{j_{12}}{\zeta}\right) - 3\zeta(\eta - i_{10})^2\left(Hi_{03} - \frac{j_{03}}{\zeta}\right) - \zeta {i_{01}}^2(\eta - i_{10})\left(Hi_{21} - \frac{j_{21}}{\zeta}\right)\right),$

$\overline{N}_{PPZ} = 2\left(3\zeta^2(\eta - i_{10})\left(Hi_{03} - \frac{j_{03}}{\zeta}\right) + \zeta^2 i_{10}\left(Hi_{12} - \frac{j_{12}}{\zeta}\right)\right),$

$\overline{O}_{PP} = 2\zeta^2(Ii_{02}),$

$\overline{O}_{ZZ} = 2\left({i_{01}}^2(Ii_{20}) + (\eta - i_{10})(Ii_{11}) + (\eta - i_{10})^2(Ii_{02})\right),$

$\overline{O}_{PZ} =- 2\zeta(\eta - i_{10})(Ii_{02}) - \zeta(Ii_{11}),$

$\overline{O}_{PPZ} = 2\left(3\zeta^2(\eta - i_{10})(Ii_{03}) + \zeta^2 i_{10}(Ii_{12})\right),$

$\overline{O}_{ZZZ} = (\eta - i_{10})^3(Ii_{03}) + i_{10}(\eta - i_{10})^2(Ii_{12}) + {i_{01}}^2(\eta - i_{10})(Ii_{21}) + {i_{01}}^3(Ii_{30}),$

$\overline{O}_{PZZ} =- 2\zeta i_{01}(\eta - i_{10})(Ii_{12}) - 3\zeta(\eta - i_{10})^2(Ii_{03}) - \zeta {i_{01}}^2(\eta - i_{10})(Ii_{21}),$

$\overline{O}_{PPP} =- \zeta^3(Ii_{03}).$

In order to determine that (3.13) undergoes a Neimark-Sacker bifurcation, the following requirement should be met. Where L, is the Lyapunov Exponent.

$$\boldsymbol{L} =- \boldsymbol{Re}\left(\frac{(\mathbf{1}-\mathbf{2}\,\boldsymbol{\lambda}_1)\,{\boldsymbol{\lambda}_2}^2}{\mathbf{1}-\boldsymbol{\lambda}_1}\boldsymbol{\varrho}_{\mathbf{11}}\boldsymbol{\varrho}_{\mathbf{20}}\right) - \frac{\mathbf{1}}{\mathbf{2}}|\boldsymbol{\varrho}_{\mathbf{11}}|^2 - |\boldsymbol{\varrho}_{\mathbf{02}}|^2 + \boldsymbol{Re}(\boldsymbol{\lambda}_2\boldsymbol{\varrho}_{\mathbf{21}}). \tag{3.30}$$

$\varrho_{20} = \frac{1}{8}[\overline{N}_{PP} - \overline{N}_{ZZ} + 2\overline{O}_{PZ} + \iota(\overline{O}_{PP} - \overline{O}_{ZZ} - 2\overline{N}_{PZ})],$

$\varrho_{11} = \frac{1}{4}[\overline{N}_{PP} - \overline{N}_{ZZ} + \iota(\overline{O}_{PP} + \overline{O}_{ZZ})],$

$\varrho_{02} = \frac{1}{8}[\overline{N}_{PP} - \overline{N}_{ZZ} + 2\overline{O}_{PZ} + \iota(\overline{O}_{PP} - \overline{O}_{ZZ} + 2\overline{N}_{PZ})],$

$\varrho_{21} = \frac{1}{16}[\overline{N}_{PPP} + \overline{N}_{PZZ} + \overline{O}_{PPZ} + \overline{O}_{ZZZ} + \iota(\overline{O}_{PPP} + \overline{O}_{PZZ} - \overline{N}_{PPZ} - \overline{N}_{ZZZ})].$

By substitution we get,

$\varrho_{20} = \frac{1}{8}\Big[2\zeta^2\left(Hi_{02} - \frac{Ij_{02}}{\zeta}\right) - 2\left({i_{01}}^2\left(Hi_{20} - \frac{j_{20}}{\zeta}\right) + (\eta - i_{10})\left(Hi_{11} - \frac{j_{11}}{\zeta}\right) + (\eta - i_{10})^2\left(Hi_{02} - \frac{j_{02}}{\zeta}\right)\right) - 4\zeta(\eta - i_{10})(Ii_{02}) - \zeta(Ii_{11}) + \iota\Big(2\zeta^2(Ii_{02}) - 2\big({i_{01}}^2(Ii_{20}) + (\eta - i_{10})(Ii_{11}) + (\eta - i_{10})^2(Ii_{02})\big) + 4\zeta(\eta - i_{10})\left(Hi_{02} - \frac{j_{02}}{\zeta}\right) - \zeta\left(Hi_{11} - \frac{j_{11}}{\zeta}\right)\Big)\Big],$

$\varrho_{11} = \frac{1}{4}\Big[2\zeta^2\left(Hi_{02} - \frac{Ij_{02}}{\zeta}\right) - 2\left({i_{01}}^2\left(Hi_{20} - \frac{j_{20}}{\zeta}\right) + (\eta - i_{10})\left(Hi_{11} - \frac{j_{11}}{\zeta}\right) + (\eta - i_{10})^2\left(Hi_{02} - \frac{j_{02}}{\zeta}\right)\right) + \iota\left(2\zeta^2(Ii_{02}) + 2\left({i_{01}}^2(Ii_{20}) + (\eta - i_{10})(Ii_{11}) + (\eta - i_{10})^2(Ii_{02})\right)\right)\Big],$

$\varrho_{02} = \frac{1}{8}\Big[2\zeta^2\left(Hi_{02} - \frac{Ij_{02}}{\zeta}\right) - 2\left({i_{01}}^2\left(Hi_{20} - \frac{j_{20}}{\zeta}\right) + (\eta - i_{10})\left(Hi_{11} - \frac{j_{11}}{\zeta}\right) + (\eta - i_{10})^2\left(Hi_{02} - \frac{j_{02}}{\zeta}\right)\right) - 4\zeta(\eta - i_{10})(Ii_{02}) - \zeta(Ii_{11}) + \iota\Big(2\zeta^2(Ii_{02}) - 2\big({i_{01}}^2(Ii_{20}) + (\eta - i_{10})(Ii_{11}) + (\eta - i_{10})^2(Ii_{02})\big) - 4\zeta(\eta - i_{10})\left(Hi_{02} - \frac{j_{02}}{\zeta}\right) - \zeta\left(Hi_{11} - \frac{j_{11}}{\zeta}\right)\Big)\Big],$

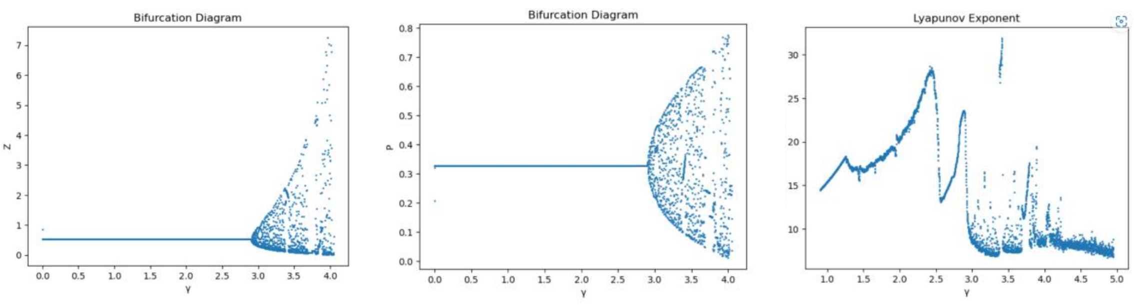
$$\varrho_{21} = \frac{1}{16}\Big[-6\zeta^3\left(Hi_{03} - \frac{j_{03}}{\zeta}\right) + 2\Big(-2\zeta i_{01}(\eta - i_{10})\left(Hi_{12} - \frac{j_{12}}{\zeta}\right) - 3\zeta(\eta - i_{10})^2\left(Hi_{03} - \frac{j_{03}}{\zeta}\right) - \zeta {i_{01}}^2(\eta - i_{10})\left(Hi_{21} - \frac{j_{21}}{\zeta}\right)\Big) + 2\left(3\zeta^2(\eta - i_{10})(Ii_{03}) + \zeta^2 i_{10}(Ii_{12})\right) + (\eta - i_{10})^3(Ii_{03}) + i_{10}(\eta - i_{10})^2(Ii_{12}) + {i_{01}}^2(\eta - i_{10})(Ii_{21}) + {i_{01}}^3(Ii_{30}) + \iota\Big(-\zeta^3(Ii_{03}) - 2\zeta i_{01}(\eta - i_{10})(Ii_{12}) - 3\zeta(\eta - i_{10})^2(Ii_{03}) - \zeta {i_{01}}^2(\eta - i_{10})(Ii_{21}) - 2\left(3\zeta^2(\eta - i_{10})\left(Hi_{03} - \frac{j_{03}}{\zeta}\right) + \zeta^2 i_{10}\left(Hi_{12} - \frac{j_{12}}{\zeta}\right)\right) - 6\Big((\eta - i_{10})^3\left(Hi_{03} - \frac{j_{03}}{\zeta}\right) + i_{10}(\eta - i_{10})^2\left(Hi_{12} - \frac{j_{12}}{\zeta}\right) + {i_{01}}^2(\eta - i_{10})\left(Hi_{21} - \frac{j_{21}}{\zeta}\right) + {i_{01}}^3\left(Hi_{30} - \frac{j_{30}}{\zeta}\right)\Big)\Big)\Big].$$

Finally by substituting into (3.14) one can confirm that a Neimark-Sacker bifurcation of the phytoplankton-zooplankton system may occur around the fixed point E^* since $L \neq 0$ and that the stability of the invariant closed curve bifurcated from the fixed point E^* is determined by the value of L. Furthermore, a supercritical Neimark-Sacker bifurcation will occur if $L < 0$ and a subcritical Neimark-Sacker bifurcation if $L > 0$. [11]. The occurrence of an N-S bifurcation has been proven.

4. Numerical simulation

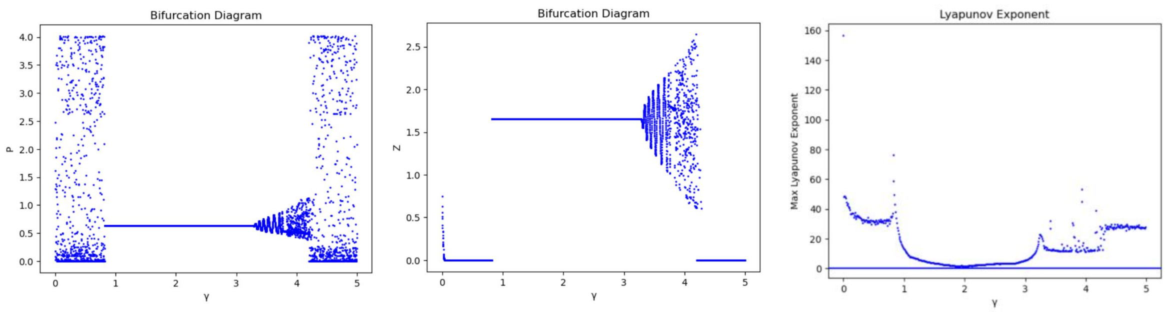
In this section numerical simulations are executed around the fixed point $E^*(P^*, Z^*)$. Through the use of bifurcation diagrams and the Maximum Lyapunov Exponents of the system (1.9) the results above are verified by fixing suitable values of the parameters involved.

$Case$ 1: Vary γ within the range, $(0,4)$ and let $\beta = 0.55, v = 0.6, \omega = 0.23$ with the initial value $(P_0, Z_0) = (0.54, 0.85)$. Then from equation (2.9) one gets $\gamma = 2.90209283439081$ which is a bifurcation parameter value shown in figure 4.1. This is to show that the phytoplankton zooplankton model undergoes a Neimark-Sacker bifurcation. The lyapunov exponents are shown in figure 4.1(c).



(a) (b) (c)

Figure 4.1. Neimark-Sacker bifurcation . 4.1(a) and (b) show the bifurcation diagram of the model around the point E(P,Z). 4.1(c) shows the maximum lyapunov exponents of the system around $E^*(P^*, Z^*)$.



(a) (b) (c)

Figure 4.2. 4.2(a) and (b) show the bifurcation diagram of the model around the point E(P,Z). 4.2 (c) shows the maximum lyapunov exponents of the system around $\boldsymbol{E}^*(\boldsymbol{P}^*, \boldsymbol{Z}^*)$.

$Case$ 2: Vary γ within the range, $(0.5,4.5)$ and fix $\beta = 3.7, v = 0.6, \omega = 0.523$ with the initial value $(P_0, Z_0) = (0.85, 0.75)$. Then from equation (2.5) one gets $\gamma = 0.761558400502631$ which is a bifurcation parameter value shown in figure 4.2. The phytoplankton-zooplankton model undergoes a bifurcation around this point which will discuss in the next section.

From figure 4.1 (a) and (b) we can see that a Neimark-Sacker bifurcation occurs at, $\gamma = 2.90209283439081$, which is the bifurcation parameter. The maximum Lyapunov Exponents figure 4.1(c) shows all exponents are positive meaning that the system is in a chaotic state. Although our maximum Lyapunov exponents are all positive we can see a change in the maximum lyapunov exponents at the value of γ where the bifurcation occurs. Figure 4.2, however, shows a bifurcation which does not seem like a Flip bifurcation as a flip bifurcation would represent a stable equilibrium point losing stability and then branching into a stable limit cycle. The bifurcation occurs at $\gamma = 0.761558400502631$ and its nature can be verified in further research. 4.2(c) we also have positive maximum lyapunov exponents but there are changes that are visible when the bifurcations occur at the γ values displayed in 4.2(a) and (b).

5. Discussion and conclusion

This paper was focusing on the 2-D phytoplankton-zooplankton model by Truscott and Brindley. The system was transformed into (1.9) through the method of semi-discretization. The existence and stability of positive fixed points $E_0(0,0)$, $E_1(1,0)$ and $E^*(P^*,Z^*)$ was determined by existing theory. With the use of the center manifold theorem we explored the existence of a flip bifurcation around the semi-trivial point, $E_1(1,0)$. The fixed point, $E^*(P^*,Z^*)$, theoretically indicated the existence of a Flip bifurcation if parameters go through $\gamma_0 = \frac{\left(\beta\left(1-2\omega+2\omega\sqrt{\frac{\omega v2}{1-\omega}}\right)-2\right)\left(1+\frac{\omega}{1-\omega}\right)}{\beta\omega(1-\sqrt{\frac{\omega v2}{1-\omega}})}$ and a Neimark-Sacker bifurcation if the parameters go through $\gamma_1 = \frac{\left(\left(1-2\omega+2\omega\sqrt{\frac{\omega v2}{1-\omega}}\right)\right)\left(1+\frac{\omega}{1-\omega}\right)}{2\omega(1-\sqrt{\frac{\omega v2}{1-\omega}})}$. However upon numerical simulation only the presence of a Neimark-Sacker bifurcation was confirmed. This difference in the numerical results and theoretical results for the flip bifurcation could be due to the numerical simulation capturing some additional features that may not have been realized in the theory like higher order bifurcations or coexistence of multiple attractors. From a biology standpoint, the presence of a Neimark-Sacker bifurcation suggests that populations of phytoplankton and zooplankton are subject to periodic or quasi-periodic oscillations, [11].

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Competing interests

The authors declare that they have no competing interests.

Author's contribution

All authors contributed significantly to this paper.

All authors read and approved the final manuscript.

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