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Existence and uniqueness results of solution for the initial value problem of Hadamard fractional sequential differential systems[1]

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Abstract

In this paper, we study the existence and uniqueness of solutions for Hadamard fractional sequential differential systems involving the Hadamard fractional derivative with orders $\alpha \in (1,2]$ and $\beta \in (2,3]$. The main tools in our study are Banach fixed point theorem and schauder fixed point theorem. An example is provided to illustrate our main results.

Keywords: Hadamard fractional sequential differential system, Banach fixed point theorem, schauder fixed point theorem, existence and uniqness of solution.

1. Introduction

Fractional calculus generalizes the integer order integration and differentiation concepts to an arbitrary(real or complex) order. Fractional calculus is the most well known and valuable branch of

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mathematics which gives a good framework for biological and physical phenomena, mathematical modeling of engineering etc. To get a couple of developments about the theory of fractional differential equations, one can allude to the monographs of Kilbas, Srivastava and Trujillo [1], Lakshmikanthem and Devi [2], Miller and Ross [3], Pudlubny [4], Tarasov [5] and the references therein.

Fixed point theory is an important tool in nonlinear analysis, in particular, in obtaining exis- tence results for a variety of mathematical problems. For instance, techniques of nonlinear analysis, such as, Banach fixed point theorem, Schauders fixed point theorem and Leray-Schauder nonlinear alternative have been used by several authors to establish existence results for non- linear initial and nonlocal boundary value problem[2]s [6, 7]. We also refer the readers to the papers [8-15] and references therein for more literature on some classes of hybrid fractional differential equations in Banach algebras using the well-known fixed point theorem of Dhage.[3]

However, if compactness and Lipschitz condition are not satisfied, these results cannot be used.

As we know, Hadamard fractional derivative is also a famous fractional derivative given by Hadamard [16] in 1892, and we can find this kind of derivative in the literature. The key of this definition involves a log- arithmic function of arbitrary exponent. In the past decades, there were more studies on Hadamard fractional differential equations under different boundary conditions, see [17-32].

Coupled systems of fractional order differential equations have also been investigated by many authors. Such systems appear naturally in many real world situations. For example, see [33]. Some recent results on the topic can be found in a series of papers [34-39] and the refrences cited therein. More recently, in [40], the authors discussed a coupled system of Hadamard type fractional differential equations with Hadamard integral boundary conditions of the form:

$$D^{\alpha}u(t) = f(t,u(t),v(t))$$

$$D^{\beta}v(t) = g(t,u(t),v(t))$$

with Hadamard integral boundary conditions

$$u(1) = 0, u(e) = \frac{1}{\Gamma(\gamma)} \int_1^{\sigma_1} \left(\log \frac{\sigma_1}{s}\right)^{\gamma-1} \frac{u(s)}{s} ds,$$
$$v(1) = 0, v(e) = \frac{1}{\Gamma(\gamma)} \int_1^{\sigma_2} \left(\log \frac{\sigma_2}{s}\right)^{\gamma-1} \frac{v(s)}{s} ds,$$

where $1 \leq \alpha, \beta \leq 2$, and $t,\sigma_1,\sigma_2 \in]1,e[$ and $f,g : [1,e] \times \mathrm{R} \times \mathrm{R} \to \mathrm{R}$ are given continuous functions.

In this paper, we consider an important problem from the point of view of application in sciences and engineering, we study the existence and uniqueness of solution for Hadamard fractional sequential differential systems given by

$$\left(D_a^{\alpha}+\gamma D_a^{\alpha-1}\right)x(t)=f(t,x(t),y(t)), \quad (1)$$

$$\left(D_a^{\beta}+\lambda D_a^{\beta-1}+\left(\tfrac{\lambda^2-1}{4}\right)D_a^{\beta-2}\right)y(t)=g(t,x(t),y(t)) \quad (2)$$

$$x(a)=x'(a)=0,\, y(a)=y'(a)=y''(a)=0 \quad (3)$$

where $1<\alpha<2$, $2<\beta<3$, $t\in J=[a,T]$, $1\leq a<T$, and $f,g : J\times \mathbb{R}\to\mathbb{R}$ are given continuous functions, and γ and λ are real numbers.

The rest of paper is organized as follows. In section 2, we recall some useful preliminaries. Section 3 contains the main result which is obtained by means of Banach fixed point theorem and schauder fixed point theorem. Section 4 contains an example to illestrait our main result

2. Preliminaries

We firstly, present the definitions and some properties of the Hadamard fractional itegral and derivatives.

Definition 1. *The Hadamard fractional integral of order α for a function f is defined as*

$$I_a^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f(s)}{s}ds,\alpha>0 .$$

Provided the integral exists.

Definition 2. *The Hadamard fractional derivative of fractional order α for a function $f:[a,+\infty)\to\mathbb{R}$ is defined as*

$$D_a^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n\int_a^t\left(\log\frac{t}{s}\right)^{n-\alpha-1}\frac{f(s)}{s}ds,$$

where $n-1<\alpha<n$, $n=[\alpha]-1$.

Lemma 1. 1. *The equality $D_a^{\alpha}x(t)=0$ is valid if, and only if,*

$$x(t)=\sum_{j=1}^{n}c_j\left(\log\frac{t}{s}\right)^{\alpha-j},$$

where $c_j\in\mathbb{R}$, $j=1,2,..,n$, are arbitrary constants, and $x\in C(J,\mathbb{R})$.

2. *Let* $x \in C\,(J,\mathbb{R})$

$$I_a^{\alpha}\left(I_a^{\beta}x\right) = I_a^{\alpha+\beta}x,$$

3. *Let* $x,y \in C\,(J,\mathbb{R})$ *and* $c_1,c_2 \in \mathbb{R}$

$$I_a^{\alpha}\left(c_1 x\,(t) + c_2 y\,(t)\right) = c_1 I_a^{\alpha}x(t) + c_2 I_a^{\alpha}y(t),$$

4. *Let* $x \in C_\delta^{n}\,(J,\mathbb{R})$ *and* $c_j \in \mathbb{R}$, $(j = 1,2,...,n)$

$$I_a^{\alpha}\left(D_a^{\alpha}x(t)\right) = x(t) - \sum_{j=1}^{n} c_j(\log\frac{t}{a})^{\alpha-j}.$$

Theorem 2.1 (Ascoli-Arzela). *A be a subset of* $C\,(J,E)$, *A is relatively compact in* $C\,(J,E)$ *if and only if the following conditions are checked:*

(i) The unit A is limited.

$\exists k > 0$ *such that* $\|f\,(x)\|_E \leq k$ *for* $x \in J$ *and* $f \in A$.

(ii) Unit A is equicontinuous.

$\forall \varepsilon > 0, \exists \delta > 0$ *and for evry* $t_1,t_2 \in J$ *we have* $|t_1,t_2| < \delta \Rightarrow \|f\,(t_1) - f\,(t_2)\|_E < \varepsilon$.

(iii) For any $x \in J$ *the unit* $\{f\,(x), f \in A\} \subset E$ *is relatively compact.*

Theorem 2.2 (Banach fixed point theorem). *E a space of Banach and* $A: E \mapsto E$ *a contracting operator. then A admits a singel fixed point.*

$$\exists! u \in E: Au = u$$

3. Main Results

The first result is obtaining the solution of the corresponding linear system of (1)(2)(3).

Theorem 3.1. *Let* $f,g \in C\,(J,\mathbb{R})$ *and* $x \in C_\delta^{2}\,(J,\mathbb{R})$, $y \in C_\delta^{3}\,(J,\mathbb{R})$. *The Hadamard fractional linear differential system*

$$\left(D_a^{\alpha} + \gamma D_a^{\alpha-1}\right)x\,(t) = f\,(t)\,, \tag{4}$$

$$\left(D_a^{\beta} + \lambda D_a^{\beta-1} + \left(\frac{\lambda^2-1}{4}\right)D_a^{\beta-2}\right)y\,(t) = g\,(t) \tag{5}$$

$$x\,(a) = x'\,(a) = 0, y\,(a) = y'\,(a) = y''\,(a) = 0, \tag{6}$$

where $1 < \alpha < 2$, $2 < \beta < 3$ *has a solution given by*

$$x(t) = t^{-\gamma} \int_a^t s^{\gamma-1}(I_a^{\alpha-1} f(s))ds, \tag{7}$$

$$y(t) = t^{\frac{-(1+\lambda)}{2}} \int_a^t (t-s)s^{\frac{1+\lambda}{2}-2}(I_a^{\beta-2} g(s))ds \tag{8}$$

Proof 3.1. *Taking the Hadamard fractional integrals* I_a^{α} *and* I_a^{β} *to the both sides of (4) and (5) respectively, we obtain*

$$I_a^{\alpha}\left(D_a^{\alpha}(x(t))\right) + \gamma I_a^1\left(I_a^{\alpha-1}\left(D_a^{\alpha-1}(x(t))\right)\right) = I_a^{\alpha}(f(t)),$$

and

$$I_a^{\beta}\left(D_a^{\beta}(y(t))\right) + \lambda I_a^1\left(I_a^{\beta-1}\left(D_a^{\beta-1}(y(t))\right)\right) + \frac{\lambda^2-1}{4}\left(I_a^2\left(I_a^{\beta-2}\left(D_a^{\beta-2}(y(t))\right)\right)\right) = I_a^{\beta}(g(t)).$$

Using Lemma 2.1, implies that

$$\left(x(t) - a_1\left(\log\frac{t}{a}\right)^{\alpha-1} - a_2\left(\log\frac{t}{a}\right)^{\alpha-2}\right) + \gamma\left({}^H I_a^1\left(x(t) - c_1\left(\log\frac{t}{a}\right)^{\alpha-2}\right)\right) = {}^H I_a^{\alpha}(f(t)),$$

and

$$\begin{aligned}&\left(y(t) - b_1\left(\log\frac{t}{a}\right)^{\beta-1} - b_2\left(\log\frac{t}{a}\right)^{\beta-2} - b_3\left(\log\frac{t}{a}\right)^{\beta-3}\right)\\ &+\lambda\left({}^H I_a^1\left(y(t) - d_1\left(\log\frac{t}{a}\right)^{\beta-2} - d_2\left(\log\frac{t}{a}\right)^{\beta-3}\right)\right)\\ &+\frac{\lambda^2-1}{4}\left({}^H I_a^2\left(y(t) - e_1\left(\log\frac{t}{a}\right)^{\beta-3}\right)\right) = I_a^{\beta}(g(t)).\end{aligned}$$

Which implies that

$$x(t) - a_1\left(\log\frac{t}{a}\right)^{\alpha-1} - a_2\left(\log\frac{t}{a}\right)^{\alpha-2} + \frac{\gamma}{\Gamma(1)}\int_a^t \frac{1}{s}\left(x(s) - c_1\left(\log\frac{s}{a}\right)^{\alpha-2}\right)ds = I_a^{\alpha}(f(t)), \tag{9}$$

and

$$\begin{aligned}&y(t) - b_1\left(\log\frac{t}{a}\right)^{\beta-1} - b_2\left(\log\frac{t}{a}\right)^{\beta-2} - b_3\left(\log\frac{t}{a}\right)^{\beta-3}\\ &+\frac{\lambda}{\Gamma(1)}\int_a^t \frac{1}{s}\left(y(s) - d_1\left(\log\frac{s}{a}\right)^{\beta-2} - d_2\left(\log\frac{s}{a}\right)^{\beta-3}\right)ds\\ &+\frac{\lambda^2-1}{4}\int_a^t\left(\log\frac{t}{s}\right)\frac{1}{s}\left(y(s) - e_1\left(\log\frac{s}{a}\right)^{\beta-3}\right)ds = I_a^{\beta}(g(t)).\end{aligned} \tag{10}$$

The initial conditions $x(a) = 0$, $y(a) = 0$ *leads* $a_2 = 0$, $b_3 = 0$.

Now, taking the first ordinary derivatives for equations (9) and (10), we obtain

$$x'(t) + \gamma\frac{x(t)}{t} = (a_1(\alpha-1)+\gamma c_1)\frac{1}{t}\left(\log\frac{t}{a}\right)^{\alpha-2} + \frac{1}{t}I_a^{\alpha-1}(f(t)), \tag{11}$$

and

$$y'(t) + b_1\frac{\beta-1}{t}\left(\log\frac{t}{a}\right)^{\beta-2} - b_2\frac{\beta-2}{t}\left(\log\frac{t}{a}\right)^{\beta-3} + \lambda\left(\frac{y(t)}{t} - \frac{d_1}{t}\left(\log\frac{t}{a}\right)^{\beta-2} - \frac{d_2}{t}\left(\log\frac{t}{a}\right)^{\beta-3}\right)$$

$$+\frac{\lambda^2-1}{4t}\int_a^t \frac{1}{s}\left(y(s) - e_1\left(\log\frac{s}{a}\right)^{\beta-3}\right)ds = \frac{1}{t}I_a^{\beta-1}(g(t)). \tag{12}$$

The conditions $x'(a) = 0$ *and* $y'(a) = y(a) = 0$ *implies that*

$$a_1(\alpha-1)+\gamma c_1 = 0, \text{ and } b_2(\beta-2)+\lambda d_2 = 0.$$

Multiplying (12) by t, and taking the derivative again, we have

$$ty''(t) + (1+\lambda)y'(t) + \left(\frac{\lambda^2-1}{4}\right)\frac{y(t)}{t} = \left(b_1(\beta-1)(\beta-2) + \lambda d_1(\beta-2) + \frac{\lambda^2-1}{4}e_1\right)\frac{1}{t}\left(\log\frac{t}{a}\right)^{\beta-3}$$

$$+\frac{1}{t}\ {}^{H}I_a^{\beta-2}(g(t)). \tag{13}$$

Since $y''(a) = y'(a) = y(a) = 0$*, it follows that*

$$b_1(\beta-1)(\beta-2) + \lambda d_1(\beta-2) + \tfrac{\lambda^2-1}{4}e_1 = 0.$$

Multilying the equation (13) by t, we obtain

$$t^2y''(t) + (1+\lambda)ty'(t) + \left(\frac{\lambda^2-1}{4}\right)y(t) = I_a^{\beta-2}(g(t)). \tag{14}$$

Let $x(t) = t^{-\gamma}u(t)$*, and* $y(t) = t^{\frac{-(1+\lambda)}{2}}v(t)$.

Then

$$\begin{aligned} x'(t) &= t^{-\gamma}u'(t) - \gamma t^{-\gamma-1}u(t) \\ y'(t) &= t^{\frac{-(1+\lambda)}{2}}v'(t) - \frac{1+\lambda}{2}t^{\frac{-(3+\lambda)}{2}}v(t) \\ y''(t) &= t^{\frac{-(1+\lambda)}{2}}v''(t) - (1+\lambda)t^{\frac{-(3+\lambda)}{2}}v'(t) + \frac{(1+\lambda)(3+\lambda)}{4}t^{\frac{-(5+\lambda)}{2}}v(t). \end{aligned}$$

Substiting the transformation into (11) and (14) and then simplify, we obtain

$$t^{-\gamma}u'(t) = \frac{1}{t}I_a^{\alpha-1}(f(t)),$$
$$t^{\frac{-(1+\lambda)}{2}+2}v''(t) = I_a^{\beta-2}(g(t)).$$

Therefore,

$$u^{'}(t) = t^{\gamma-1} I_a^{\alpha-1}(f(t)), \tag{15}$$
$$v^{"}(t) = t^{\frac{(1+\lambda)}{2}-2} I_a^{\beta-2}(g(t)) \tag{16}$$

Integrating equation (15), it follows that

$$u(t) = u(a) + \int_a^t s^{\gamma-1} I_a^{\alpha-1} f(s) ds.$$

The condition $x(a) = 0$ *implies* $u(a) = 0$ *hence*

$$x(t) = t^{-\gamma} \int_a^t s^{\gamma-1} I_a^{\alpha-1} f(s) ds.$$

Twice integration for equation (16),

$$v(t) - v^{'}(a)(t-a) - v(a) = \int_a^t (t-s) s^{\frac{1+\lambda}{2}-2} I_a^{\beta-2} g(s) ds.$$

The condition $y(a) = y'(a) = 0$ *implies that* $v(a) = v'(a) = 0$ *hence*

$$y(t) = t^{\frac{-(1+\lambda)}{2}} \int_a^t (t-s) s^{\frac{1+\lambda}{2}-2} I_a^{\beta-2} g(s) ds.$$

This finish the proof.

In view of Theorem 3.1, we define the operators A and B on $C(J,\mathrm{R})$ as,

$$Ax(t) = t^{-\gamma} \int_a^t s^{\gamma-1} I_a^{\alpha-1} f(s, x(s)) ds, \tag{17}$$

and

$$By(t) = t^{\frac{-(1+\lambda)}{2}} \int_a^t (t-s) s^{\frac{1+\lambda}{2}-2} I_a^{\beta-2} g(s, y(s)) ds. \tag{18}$$

Theorem 3.2. *The operators A and B are completely continuous.*

Proof 3.2. *The continuity of the operators A and B follows respectively by he continuity of the functions f and g.*

Let Ω *be a bounded proper subset af* $C(J,\mathrm{R})$*, then, there exist positive real numbers* k_f *and* k_g *such that* $|f(t,x)| \leq k_f$ *and* $|g(t,y)| \leq k_g$ *for any order pairs* $(t,x) \in J \times \Omega$ *and* $(t,y) \in J \times \Omega$*.*

Therefore

$$|Ax(t)| = |t^{-\gamma}\int_a^t s^{\gamma-1} I_a^{\alpha-1} f(s,x(s))ds|$$

$$\leqslant \frac{k_f\left(\log\frac{t}{a}\right)^{\alpha-1}}{\Gamma(\alpha)}\left|\frac{1-\left(\frac{a}{t}\right)^{\gamma}}{\gamma}\right|,$$

and

$$|By(t)| = |t^{\frac{-(1+\lambda)}{2}}\int_a^t (t-s)s^{\frac{1+\lambda}{2}-2} I_a^{\beta-2} g(s,y(s))ds|$$

$$\leqslant \frac{k_g\left(\log\frac{t}{a}\right)^{\beta-2}}{\Gamma(\beta-1)}\left|\frac{\left(1-\frac{a}{t}\right)\left(1-\left(\frac{a}{t}\right)^{\frac{1+\lambda}{2}-1}\right)}{\frac{(\lambda-1)}{2}}\right|,$$

where $\gamma \leq 0$ *and* $\lambda \leq 1$.

Taking the maximum over J*, we deduce that the operators* A *and* B *are bounded on* $C(J,\mathbb{R})$.

Next, we show the equicontinuity of A *and* B*. For this, let* $a \leq t_1 \leq t_2 \leq T$*, then*

$$|Ax(t_2) - Ax(t_1)| = |t_2^{-\gamma}\int_a^{t_2} s^{\gamma-1} I_a^{\alpha-1} f(s,x(s))ds - t_1^{-\gamma}\int_a^{t_1} s^{\gamma-1} I_a^{\alpha-1} f(s,x(s))ds|$$

$$\begin{aligned}
&= \left|t_2^{-\gamma}\int_a^{t_2} s^{\gamma-1} I_a^{\alpha-1} f(s,x(s))ds - t_1^{-\gamma}\left[\int_a^{t_1} s^{\gamma-1} I_a^{\alpha-1} f(s,x(s))ds - \int_{t_1}^{t_2} s^{\gamma-1} I_a^{\alpha-1} f(s,x(s))ds\right]\right| \\
&\leqslant |t_1^{-\gamma} - t_2^{-\gamma}|\left|\int_a^{t_1} s^{\gamma-1} I_a^{\alpha-1} f(s,x(s))ds\right| + \left|t_2^{-\gamma}\int_{t_1}^{t_2} s^{\gamma-1} I_a^{\alpha-1} f(s,x(s))ds\right| \\
&\leqslant \frac{k_f\left(\log\frac{t_2}{a}\right)^{\alpha-1}}{\Gamma(\alpha)}\left|t_2^{-\gamma} - t_1^{-\gamma}\right|\left|\frac{t_1^{\gamma} - a^{\gamma}}{\gamma}\right| + \frac{k_f\left(\log\frac{t_2}{t_1}\right)^{\alpha-1}}{\Gamma(\alpha)}\left|\frac{1-\left(\frac{t_1}{t_2}\right)^{\gamma}}{\gamma}\right|.
\end{aligned}$$

As $|t_2 - t_1| \to 0$*, then* $|Ax(t_2) - Ax(t_1)| \to 0$*, this imply that* A *is equicontinuous operator on* J.

And,

$$
\begin{aligned}
|By(t_2)-By(t_1)| &= |t_2^{\frac{-(1+\lambda)}{2}}\int_a^{t_2}(t_2-s)s^{\frac{1+\lambda}{2}-2}I_a^{\beta-2}g(s)ds - t_1^{\frac{-(1+\lambda)}{2}}\int_a^{t_1}(t_1-s)s^{\frac{1+\lambda}{2}-2}I_a^{\beta-2}g(s)ds| \\
&\leqslant t_2^{\frac{-(1+\lambda)}{2}}|t_2-t_1|\int_a^{t_2}(t_2-s)s^{\frac{1+\lambda}{2}-2}\left|I_a^{\beta-2}g(s,y(s))\right|ds \\
&\quad + \left|t_2^{\frac{-(1+\lambda)}{2}} - t_1^{\frac{-(1+\lambda)}{2}}\right|\int_a^{t_1}(t_1-s)s^{\frac{1+\lambda}{2}-2}\left|I_a^{\beta-2}g(s,y(s))\right|ds \\
&\quad + t_2^{\frac{-(1+\lambda)}{2}}\int_{t_1}^{t_2}(t_1-s)s^{\frac{1+\lambda}{2}-2}\left|I_a^{\beta-2}g(s,y(s))\right|ds \\
&\leqslant \frac{k_g\left(\log\frac{t_2}{a}\right)^{\beta-2}}{\Gamma(\beta-1)}\left|1-\frac{t_1}{t_2}\right|\left|\frac{\left(1-\left(\frac{a}{t_2}\right)^{\frac{1+\lambda}{2}-1}\right)}{\frac{(\lambda-1)}{2}}\right| \\
&\quad + \frac{k_g\left(\log\frac{t_1}{a}\right)^{\beta-2}(t_1-a)}{\Gamma(\beta-1)}\left|t_2^{\frac{-(1+\lambda)}{2}} - t_1^{\frac{-(1+\lambda)}{2}}\right|\left|\frac{t_1^{\frac{1+\lambda}{2}-1} - a^{\frac{1+\lambda}{2}-1}}{\frac{(\lambda-1)}{2}}\right| \\
&\quad + \frac{k_g\left(\log\frac{t_2}{t_1}\right)^{\beta-2}}{\Gamma(\beta-1)}\left|1-\frac{t_1}{t_2}\right|\left|\frac{\left(1-\left(\frac{t_1}{t_2}\right)^{\frac{1+\lambda}{2}-1}\right)}{\frac{(\lambda-1)}{2}}\right|
\end{aligned}
$$

As $|t_2 - t_1| \to 0$, *then* $|By(t_2) - By(t_1)| \to 0$, *this imply that* B *is equicontinuous operator on* J. *In consequence, by the Areola-Ascoli theorem the operators A and B are completely continuous.*

We state next the so-called Schauder's fixed point theorem.

Theorem 3.3. *If F is closed, bounded, convex subset of a Banach space X and the mapping* $\Delta : F \to F$ *is completely continuous, then* Δ *has a fixed point in F.*

Accordingly, if we define a closed, bounded, convex subset F of $C(J,\mathrm{R})$ on which the operators A and B, as defined by (17) − (18), are completely continuous, then the problem (1)(2)(3) have the respective solution.

Theorem 3.4. *Let* L_f *and* L_g *be positive constants such that*

$$\lim_{x\longrightarrow 0}\frac{f(t,x)}{x}\leqslant L_f<\infty, \text{ and } \lim_{x\longrightarrow 0}\frac{g(t,x)}{x}\leqslant L_g<\infty,$$

then, the problem (1)(2)(3) *has a solution.*

Proof 3.3. *The given conditions imply that there exist positive constants* ρ_f *and* ρ_g *such that*

$|f(t,x(t))| \le (1 + L_f)\rho_f$ *and* $|g(t,y(t))| \le (1 + L_g)\rho_g$. *Therefore, define the subsets F and G of* $C(J,\mathrm{R})$ *as*

$$F_f = \{x \in C(J,\mathrm{R}) : |x(t)| \le \rho_f, t \in J\}, \text{ and } F_g = \{x \in C(J,\mathrm{R}) : |x(t)| \le \rho_g,\ t \in J\}.$$

Hence, F_f *and* F_g *are closed, bounded, and convex subsets of* $C(J,\mathrm{R})$. *By Theorem 3.3, the operators A and B are completely continuous, by the schauder s fixed point theorem the problem (1)(2)(3) has a solution.*

This finishes the proof.

Now, we show the existence and uniqueness of solution for the problem (1)(2)(3) by using the contraction principal (The Banach fixed point theorem).

Theorem 3.5. *Let f and g be Lipshitz functions*

$$|f(t,x) - f(t,z)| \ 6 \ C_f|x - z|,$$

$$|g(t,y) - g(t,z)| \ 6 \ C_g|y - z|,$$

where $t \in J$, $x,y,z \in C(J,\mathrm{R})$ and $C_f, C_g > 0$.

Then, the problem (1)(2)(3) has a unique solution whenever

$$v_f = \frac{C_f \left(\log \frac{T}{a}\right)^{\alpha-1}}{|\gamma|\Gamma(\alpha)} \max_{t\in J} \left|1 - \left(\frac{a}{t}\right)^{\gamma}\right| < 1$$

$$v_g = \frac{2C_g \left(\log \frac{T}{a}\right)^{\beta-2}}{|\lambda - 1|\Gamma(\beta - 1)} \max_{t\in J} \left|1 - \left(\frac{a}{t}\right)^{\frac{1+\lambda}{2}-1}\right| < 1$$

Proof 3.4. *The continuity of f and g implies that there exist positive constants D_f and D_g such that* $\max_{t\in J} f(t,0) \leq D_f$ *and* $\max_{t\in J} g(t,0) \leq D_g$.

First, we see that $A\Omega_r \subset \Omega_r$ and $B\Omega_r \subset \Omega_r$ where Ω_r is defined by $\Omega_r = \{z \in C(J,\mathrm{R}) : \|z\| \leq r\}$ such that r is given by $r = \max(r_f, r_g)$,

where

$$r_f \geqslant \frac{D_f \left(\log \frac{T}{a}\right)^{\alpha-1}}{|\gamma|\Gamma(\alpha)} \max_{t\in J} \left|1 - \left(\frac{a}{t}\right)^{\gamma}\right| (1 - v_f)^{-1}$$

$$r_g \geqslant \frac{2D_g \left(\log \frac{T}{a}\right)^{\beta-2}}{|\lambda - 1|\Gamma(\beta - 1)} \max_{t\in J} \left|1 - \left(\frac{a}{t}\right)^{\frac{1+\lambda}{2}-1}\right| (1 - v_g)^{-1}$$

Let $x,y \in \Omega_r$, then

$$\begin{aligned}
|Ax(t)| &= |t^{-\gamma} \int_a^t s^{\gamma-1} I_a^{\alpha-1} f(s,x(s))ds| \\
&\leqslant t^{-\gamma} \int_a^t s^{\gamma-1} I_a^{\alpha-1} \left(|f(s,x(s)) - f(s,0)| + |f(s,0)|\right) ds \\
&\leqslant t^{-\gamma} \int_a^t s^{\gamma-1} I_a^{\alpha-1} \left(|f(s,x(s)) - f(s,0)|\right) ds + \int_a^t s^{\gamma-1} I_a^{\alpha-1} \left(|f(s,0)|\right) ds \\
&\leqslant \frac{\left(C_f \|x\| + D_f\right) \left(\log \frac{T}{a}\right)^{\alpha-1}}{\Gamma(\alpha)} \left|\frac{1 - \left(\frac{a}{t}\right)^{\gamma}}{\gamma}\right| \\
&\leqslant \frac{D_f \left(\log \frac{T}{a}\right)^{\alpha-1}}{|\gamma|\Gamma(\alpha)} \max_{t\in J} \left|1 - \left(\frac{a}{t}\right)^{\gamma}\right| + r_f v_f \\
&\leqslant r
\end{aligned}$$

$$
\begin{aligned}
|By(t)| &= |t^{\frac{-(1+\lambda)}{2}}\int_a^t (t-s)s^{\frac{1+\lambda}{2}-2} I_a^{\beta-2} g(s,y(s))ds| \\
&\leqslant t^{\frac{-(1+\lambda)}{2}}\int_a^t (t-s)s^{\frac{1+\lambda}{2}-2} I_a^{\beta-2}\left(|g(s,y(s))-g(s,0)|+|g(s,0)|\right)ds \\
&\leqslant t^{\frac{-(1+\lambda)}{2}}\left[\int_a^t (t-s)s^{\frac{1+\lambda}{2}-2} I_a^{\beta-2}\left(|g(s,y(s))-g(s,0)|\right)ds+\int_a^t (t-s)s^{\frac{1+\lambda}{2}-2} I_a^{\beta-2}\left(|g(s,0)|\right)ds\right] \\
&\leqslant \frac{\left(C_g\|y\|+D_g\right)\left(\log\frac{t}{a}\right)^{\beta-2}}{\Gamma(\beta-1)}\left|\frac{\left(1-\frac{a}{t}\right)\left(1-\left(\frac{a}{t}\right)^{\frac{1+\lambda}{2}-1}\right)}{\frac{(\lambda-1)}{2}}\right| \\
&\leqslant (1-v_g)r_g+v_g r_g \\
&\leqslant r
\end{aligned}
$$

Next, we show the contraction principle. Let

$x,y,z \in C(J,\mathrm{R})$, then

$$
\begin{aligned}
|Ax(t)-Az(t)| &= \left|t^{-\gamma}\int_a^t s^{\gamma-1} I_a^{\alpha-1} f(s,x(s))ds - t^{-\gamma}\int_a^t s^{\gamma-1} I_a^{\alpha-1} f(s,z(s))ds\right| \\
&\leqslant t^{-\gamma}\int_a^t s^{\gamma-1}\left(I_a^{\alpha-1}|f(s,x(s))|ds - I_a^{\alpha-1}|f(s,z(s))|ds\right) \\
&\leqslant t^{-\gamma}\int_a^t s^{\gamma-1} I_a^{\alpha-1}\left(|f(s,x(s))-f(s,z(s))|\right)ds \\
&\leqslant \frac{C_f\left(\log\frac{T}{a}\right)^{\alpha-1}}{|\gamma|\Gamma(\alpha)}\max_{t\in J}\left|1-\left(\frac{a}{t}\right)^{\gamma}\right|\|x-z\| \\
&\leqslant v_f\|x-z\|,
\end{aligned}
$$

and

$$
\begin{aligned}
|By(t)-Bz(t)| &= \left|t^{\frac{-(1+\lambda)}{2}}\int_a^t (t-s)s^{\frac{1+\lambda}{2}-2} I_a^{\beta-2} g(s,y(s))ds - t^{\frac{-(1+\lambda)}{2}}\int_a^t (t-s)s^{\frac{1+\lambda}{2}-2} I_a^{\beta-2} g(s,z(s))ds\right| \\
&\leqslant t^{\frac{-(1+\lambda)}{2}}\left(\int_a^t (t-s)s^{\frac{1+\lambda}{2}-2} I_a^{\beta-2}|g(s,y(s))|ds - \int_a^t (t-s)s^{\frac{1+\lambda}{2}-2} I_a^{\beta-2}|g(s,z(s))|ds\right) \\
&\leqslant t^{\frac{-(1+\lambda)}{2}}\int_a^t (t-s)s^{\frac{1+\lambda}{2}-2} I_a^{\beta-2}\left(|g(s,y(s))-g(s,z(s))|\right)ds \\
&\leqslant \frac{2C_g\left(\log\frac{T}{a}\right)^{\beta-2}}{|\lambda-1|\Gamma(\beta-1)}\max_{t\in J}\left|1-\left(\frac{a}{t}\right)^{\frac{1+\lambda}{2}-1}\right|\|y-z\| \\
&\leqslant v_g\|y-z\|
\end{aligned}
$$

As $v_f, v_g < 1$, the contraction principle is satisfied, by the Banach fixed point theorem the problem

(1)(2)(3) has a unique solution. The

proof is completed.

Now we see an example to illustrate our results

4. Example

Consider the following Hadamard fractional sequential system

$$(D_a^{\frac{3}{2}} - D_a^{\frac{1}{2}})x(t) = C_1 t \sin x(t \tag{19}$$

$$(D_a^{\frac{14}{5}} + 3D_a^{\frac{9}{5}} + 2D_a^{\frac{4}{5}})y(t) = \frac{C_2 t|y(t)|}{1+|y(t)|} \tag{20}$$

$$x\left(1\right) = x^{'}\left(1\right) = 0, y\left(1\right) = y^{'}\left(1\right) = y^{"}\left(1\right) = 0 \tag{21}$$

Here $\alpha = \frac{3}{2}$, $\beta = \frac{14}{5}$, $\gamma = -1$, λ= 3,

and

$$f(t, x(t)) = C_1 t \sin x(t)$$
$$g(t, y(t)) = \frac{C_2 t|y(t)|}{1+|y(t)|}.$$

We notice that

$$\lim_{x \longrightarrow 0} \frac{C_1 t \sin x}{x} = C_1 t \leqslant C_1 e,$$
$$\lim_{y \longrightarrow 0} \frac{C_2 t}{1+|y|} = C_2 t \leqslant C_2 e,$$

amd

$$v_f = \frac{C_f}{\Gamma(\frac{3}{2})} \max_{t \in J} |1-t| \leqslant 1.96 C_1,$$
$$v_g = \frac{C_g}{\Gamma(\frac{14}{5})} \max_{t \in J} \left|1 - \frac{1}{t}\right| \leqslant 1.85 C_2.$$

Therefore, choosing any real numbers $0 < C_1 \leq 0.5$ and $0 < C_2 \leq 0.54$, the theorems can be applied. Hence the problem (19)(20)(21) has a unique solution in $C([1,e],\mathrm{R})$

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