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# Quasi-linear fractional differential equations with non-local condition [1]

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## Abstract

In this paper, we study the existence of solutions for quasi-linear fractional differential equations with non-local condition using the Schauder fixed point theorem in Banach space. Later, we discuss a particular example which satisfies all the existence conditions.

**Keywords:** Quasi-linear fractional differential equations; Schauder fixed point; non-local condition.

## 1. Introduction and preliminaries

In recent years, the theory of linear and nonlinear fractional differential equations have attracted the attention of many authors, because fractional differential equations describe many phenomenons in several fields of engineering and scientific disciplines such as physics, biophysics, chemistry, biology, economics, control theory, signal and image processing, aerodynamics, viscose-elasticity, electromagnetic and archeology (see [1,4,13-23]), and a considerable number of results have been obtained.

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This paper is devoted to study the existence of the solution of a fractional differential problem with non-local condition

$$^cD^{\alpha}\left(x(t)+\sigma\left(t,x(t)\right)\right)=A(t,y)x(t)+f\left(t,x(t)\right), \tag{1}$$

$$x(0)+g\left(x\right)=x_0, \tag{2}$$

where $^cD^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0<\alpha<1)$, $(f,\sigma): J\times X\to X$ are given functions, $J=[0,\infty)$, $A(t,y): J\times X\to B\left(X\right)$ is linear bounded operator. $B\left(X\right)$ is a Banach space of linear bounded operator in the Banach space $X$.

An outline of this paper is as follows: in Section 2, we prove the main result using Schauder fixed point theorem, in Section 3, an example is illustrated for better understanding.

Consider the space, $C_\alpha=\left\{x\in C\left(J,X\right),\lim\limits_{t\mapsto\infty}\frac{x\left(t\right)}{1+t^{\alpha+1}}=0\right\}$, equipped with the norm $\|x\|_{C_\alpha}=\sup_{t\geqslant0}\frac{\|x\left(t\right)\|}{1+t^{\alpha+1}}$, $(C_\alpha,\|.\|_{C_\alpha})$ is a Banach space.

**Definition 1.** *The Riemann-Liouville fractional integral of order* $\alpha>0$ *of a function* $f:(0,\infty)\ 7\!\to \mathrm{R}$ *is given by*

$$I^{\alpha}f\left(t\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_0^t\left(t-s\right)^{\alpha-1}f\left(s\right)ds, \tag{3}$$

*provided that the right-hand side is defined point wise, where* $\Gamma(.)$ *is the Gamma function.*

**Definition 2.** *For a function* $f:(0,\infty)\to \mathrm{R}$, *the Caputo derivative of fractional order* $\alpha>0$ *is defined as*

$$^cD^{\alpha}f\left(t\right)=\frac{1}{\Gamma\left(n-\alpha\right)}\int_0^t\left(t-s\right)^{n-\alpha-1}f^{(n)}\left(s\right)ds, \tag{4}$$

*where* $n-1<\alpha<n$, $n=[\alpha]+1$, $[\alpha]$ *denotes the integer part of the real number* $\alpha$.

**Theorem 1 (Schauder fixed point theorem).** *Let* $(X,d)$ *be a complete metric space and E be a part convex and closed of X, and A*: $E\to E$ *application such as the unit* $\{Au,\ u\in E\}$ *is relatively compact in X. Then A has at least a fixed point.*

**Lemma 1.** *Let M be a subset of* $C_\alpha$. *So M is relatively compact if and only if the following conditions are met:*

1. $\left\{\frac{x\left(t\right)}{1+t^{\alpha+1}},x\in M\right\}$ *is uniformly bounded.*
2. $\left\{\frac{x\left(t\right)}{1+t^{\alpha+1}},x\in M\right\}$ *is equicontinued for* $t\in[0,\infty)$.

3. $\forall\epsilon > 0, \exists T > 0$ *such that* $\forall x \in M$ *and* $t > T$

$$\left\| \frac{x(t)}{1+t^{\alpha+1}} \right\| \leqslant \epsilon.$$

## 2. Result of existence

In order to establish the existence of the solution of the problem (1)-(2) we propose the following hypotheses:

$(H_1)$ : For all $(t,y) \in J \times X$, $A(t,y)$ is a bounded linear operator, the mapping $(t,y) \to A(t,y)$ is continuous and there are two continuous, bounded and non negative functions $\phi(t)$ and $\psi(t)$ defined in $J$ such that

$$\|A(t,y)\| \leqslant \frac{\varphi(t)}{1+t^{\alpha+1}} \psi\left(\frac{\|y\|}{1+t^{\alpha+1}}\right).$$

$(H_2)$ : The function $f : J \times X \to X$ is continuous and there are two continuous, bounded and non negative functions $a(t)$ and $b(t)$ defined in $J$ such that

$$\|f(t,x)\| \leqslant \frac{a(t)}{1+t^{\alpha+1}} \|x\| + b(t), \quad t \geqslant 0, \quad x \in X. \tag{5}$$

$(H_3)$ : The function $\sigma : J \times X \to X$ is continuous and there exists a constant $c > 0$ such that

$$\| \sigma(t,u) - \sigma(t,v) \| \leq c \| u - v \|, \qquad u,v \in X, \tag{6}$$

and

$$\delta = sup \| \sigma(t,0) \| \leq \infty, \qquad \sigma_0 = \sigma(0,x(0)).$$

$(H_4)$ : The function $g : X \, 7\!\!\to X$ is continuous and there exists a constant $G > 0$ such that

$$\| g(u) - g(v) \| \leq G \| u - v \|, \qquad u,v \in X.$$

$(H_5)$ : There exists $r > 0$ such that

$$c_0(r) + c_1(r) < r, \tag{7}$$

where $c_0(r) = \|x_0\| + \| g(0) \| + \| \sigma(0) \| + \delta + (G + c)r$, and $c_1(r) = \frac{1}{\Gamma(\alpha+1)} \left( \|\varphi\|_\infty r^2 + \|a\|_\infty r + \|b\|_\infty \right)$.

**Theorem 2.** *Under the assumptions (H1)-(H5) the problem (1)-(2) has at least one solution* $x(t)$ *in the space* $C_\alpha$.

To proof this theorem, we will use the fixed point method. Initially, by transforming the problem (1)-(2) into an integral equation of Volterra (Lemma 2), then we will prove the existence of the solution of the integral equation. Thus, we obtained the results by using Schauder's fixed point theorem.

**Lemma 2.** *A function $x(t) \in C_\alpha$ is solution of the problem (1)-(2) if and only if it satisfies the integral equation of Volterra*

$$\begin{aligned} x(t) &= x_0 + \sigma_0 - g(x) - \sigma(t, x(t)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A(s,y)\, x(s)\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s))\, ds. \end{aligned} \tag{8}$$

**Proof**. Suppose that $x(t)$ satisfies the problem (1)-(2), then we have

$$^{c}D^{\alpha}(x(t) + \sigma(t,x(t))) = A(t,y)x(t) + f(t,x(t)). \tag{9}$$

Applying the fractional integration operator $I^\alpha$ to both sides of the above equation, we get

$$x(t) + \sigma(t, x(t)) + k_1 = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (A(s,y)\, x(s) + f(s, x(s)))\, ds. \tag{10}$$

Now, using the non-local condition $x(0) + g(x) = x_0$, we find

$$k_1 = -x(0) - \sigma_0,$$

substituting in (12) we obtained equation (10).

Conversely, if there is a solution of the Volterra equation (10), then we can write this equation in the following form

$$x(t) + \sigma(t,x(t)) = x_0 + \sigma_0 - g(x) + I^\alpha (A(s,y)x(s) + f(s,x(s))).$$

We apply the differential operator $^{c}D^{\alpha}$ on the both sides of this equality, then we assume that $x(t)$ is also a solution of the differential equation (11).

On the other hand, it is obvious that $x(t)$ satisfies the non-local condition

$$x(0) + g(x) = x_0.$$

Therefore, $x(t)$ is the solution of the problem (1) - (2).

Using the above result we can prove Theorem 2.

**Proof**. In order to demonstrate the existence of the solution of the integral equation (10) we introduce the subset $B_r = \{x \in C_\alpha : \| x \|_{C_\alpha} \leq r\}$, where $r$ is the constant as introduced in ($H_5$). It is clear that $B_r$ is a closed and convex part from space $C_\alpha$ equipped with the norm $\| . \|_{C_\alpha}$. We define the operator $F$ on $B_r$ as

$$(Fx)(t) = x_0 + \sigma_0 - g(x) - \sigma(t, x(t)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi(s, x(s))\, ds,$$

where $\varphi(s,x(s)) = A(s,y)x(s) + f(s,x(s))$. Then the integral equation (10) is reduced to

$$x = Fx,$$

and in order to establish our existence result, we must show that $F$ has a fixed point.

**Step 1.** *F maps bounded sets into bounded sets in X. For all $x \in B_r$, we have*

$$\begin{aligned}
\left\| \frac{(Fx)(t)}{1+t^{\alpha+1}} \right\| &= \left\| \frac{x_0 + \sigma_0 - g(x) - \sigma(t, x(t))}{1+t^{\alpha+1}} + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha+1}} \phi(s, x(s))\, ds \right\| \\
&\leqslant \frac{1}{1+t^{\alpha+1}} \left( \|x_0\| + \|\sigma_0\| + \|g(0)\| + \|\sigma(t,0)\| + \|g(x) - g(0)\| + \|\sigma(t, x(t) - \sigma(t,0))\| \right) \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha+1}} \|\phi(s, x(s))\| ds \\
&\leqslant \frac{1}{1+t^{\alpha+1}} \left( \|x_0\| + \|\sigma_0\| + \|g(0)\| + \delta + G\|x\| + c\|x(t)\| \right) \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha+1}} \left( \|A(s,y) x(s)\| + \|f(s, x(s))\| \right) ds.
\end{aligned}$$

*By using hypotheses* ($H_1$),($H_2$) *and* ($H_3$), *we obtain the following estimate*

$$\left\| \frac{(Fx)(t)}{1+t^{\alpha+1}} \right\| \leqslant \frac{c_0(r)}{1+t^{\alpha+1}} + c_1(r) \frac{t^\alpha}{1+t^{\alpha+1}}.$$

*Therefore,*

$$\lim_{t \mapsto \infty} \frac{(Fx)(t)}{1+t^{\alpha+1}} = 0,$$

*and*

$$\| Fx \|_{C_\alpha} \leq c_0(r) + c_1(r) \leq r, \tag{11}$$

*which proves that $Fx \in B_r$ if $x \in B_r$. The operator F is bounded.*

**Step 2.** *F is continuous.*

*Let $\{x_n\}$ be a sequence such that $x_n \rightarrow x$ in $B_r$. For all $\epsilon > 0$, there exists $N > 0$ such that for all* $n > N$, $\|Fx_n - Fx\|_{C_\alpha} < \epsilon$.

*We have*

$$\begin{aligned}\|Fx_n - Fx\| &\leqslant \|g(x_n) - g(x)\| + \|\sigma(t, x_n(t)) - \sigma(t, x(t))\| \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha+1}} \|A(s,y)x_n(s) - A(s,y)x(s)\| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha+1}} \|f(s, x_n(s)) - f(s, x(s))\| ds.\end{aligned}$$

*Using the hypotheses (H3) and (H4), we obtain the estimate*

$$\begin{aligned}\|Fx_n - Fx\|_{C_\alpha} &\leqslant (G+c)\|x_n - x\| \\ &+ \sup_{t\geqslant 0} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha+1}} \|A(s,y)x_n(s) - A(s,y)x(s)\| ds \\ &+ \sup_{t\geqslant 0} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha+1}} \|f(s, x_n(s)) - f(s, x(s))\| ds.\end{aligned}$$

*As* $\lim_{n\to\infty} \| x_n - x \|_{C_\alpha} = 0$*, then there exists a constant* $\mu > 0$ *such that*

$$\| x_n \|_{C_\alpha} \leq \mu, \quad \text{and} \quad \| x \|_{C_\alpha} \leq \mu.$$

*In view of the hypothesis* $(H_1)$*, we have*

$$\frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha+1}} \|A(s,y)x_n(s) - A(s,y)x(s)\| ds \leqslant \left(\frac{2\mu^2\|\varphi\|_\infty}{\Gamma(\alpha+1)}\right) \frac{t^\alpha}{1+t^{\alpha+1}}.$$

*So for all* $\epsilon > 0$*, there exists* $T_1 > 0$ *and* $\delta_1 > 0$ *such that*

$$\frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha+1}} \|A(s,y)x_n(s) - A(s,y)x(s)\| ds \leqslant \epsilon, \qquad t > T_1,$$

*and*

$$\frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha+1}} \|A(s,y)x_n(s) - A(s,y)x(s)\| ds \leqslant \epsilon, \qquad 0 < t < \delta_1.$$

*Therefore, since A is continuous then*

$$\sup_{t\in[\delta_1, T_1]} \| A(s,y)x_n(s) - A(s,y)x(s) \| \to 0, \quad n \to \infty.$$

*It follows that there exists N > 0 such that for all n > N,*

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)} \int_{\delta_1}^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha+1}} \|A(s,y)x_n(s) - A(s,y)x(s)\| ds \\ \leqslant\ &\frac{T_1^\alpha}{\Gamma(\alpha+1)} \sup_{s\in[\delta_1,T_1]} (\|A(s,y)\|\mu + \|x_n - x\|_{C_\alpha}\|A(s,y)x(s)\|) \\ \longrightarrow\ &0, n \longrightarrow \infty.\end{aligned}$$

*On the other hand, using the hypothesis* $(H_2)$ *we get for all n > 1*

$$\frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha+1}}\|f(s,x_n(s))-f(s,x(s))\|ds \leqslant \left(\frac{2\mu\|a\|_\infty+2\|b\|_\infty}{\Gamma(\alpha+1)}\right)\frac{t^\alpha}{1+t^{\alpha+1}}.$$

*So for all* $\epsilon > 0$*, there exists* $T_2 > 0$ *and* $\delta_2 > 0$ *such that*

$$\frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha+1}}\|f(s,x_n(s))-f(s,x(s))\|ds \leqslant \epsilon, \qquad t \geqslant T_2.$$

*and*

$$\frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha+1}}\|f(s,x_n(s))-f(s,x(s))\|ds \leqslant \epsilon, \qquad 0 \leqslant t \leqslant \delta_2.$$

*In addition, the continuity of f*

$$\sup_{s\in[\delta_2,T_2]} \| f(s,x_n(s)) - f(s,x(s)) \| \to 0, \quad n \to \infty.$$

*So, there exists N > 0 such that for all n > N we have*

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha+1}}\|f(s,x_n(s))-f(s,x(s))\|ds \\
\leqslant\ & \sup_{s\in[\delta_2,T_2]}\|f(s,x_n(s))-f(s,x(s))\| \sup_{t\in[\delta_2,T_2]}\frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha+1}}ds \\
\leqslant\ & \frac{T_2}{\Gamma(\alpha+1)}\sup_{s\in[\delta_2,T_2]}\|f(s,x_n(s))-f(s,x(s))\| \longrightarrow 0, n \longrightarrow \infty.
\end{aligned}$$

*The above estimates show that the operator F is continuous.*

**Step 3.** *It remains only to show that* $F(B_r) = \{Fx, x \in B_r\}$ *is a relatively compact to be able to apply Schauder's fixed point theorem. For all* $x \in B_r$ *and* $t_1, t_2 \in J$ *with* $t_1 < t_2$*, we have*

$$\begin{aligned}
\left\|\frac{(Fx)(t_2)}{1+t_2^{\alpha+1}} - \frac{(Fx)(t_1)}{1+t_1^{\alpha+1}}\right\| \leqslant\ & \left\|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left(\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha+1}} - \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha+1}}\right)A(s,y)x(s)ds\right\| \\
& + \left\|\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha+1}}A(s,y)x(s)ds\right\| \\
& + \left\|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left(\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha+1}} - \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha+1}}\right)f(s,x(s))ds\right\| \\
& + \left\|\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha+1}}f(s,x(s))ds\right\|.
\end{aligned}$$

*By calculating the integrals, we obtain*

$$\|(Fx)(t_2)-(Fx)(t_1)\|_{C_\alpha} \leqslant c_1(r)\left|\frac{t_1^\alpha}{1+t_1^{\alpha+1}} - \frac{t_2^\alpha}{1+t_2^{\alpha+1}} + \frac{2(t_2-t_1)^\alpha}{1+t_2^{\alpha+1}}\right|.$$

*Note that the right-hand term of this expression is independent of $x(t)$, more $t_1 \to t_2$, the side on the right tends to 0, which implies that the set $F(Br)$ is equicontinuous, and therefore the condition 2 of Lemma 1 is satisfied.*

*Finally, for every $x(t) \in B_r$, we have*

$$\left\| \frac{(Fx)(t)}{1+t^{\alpha+1}} \right\| \leqslant \frac{c_0(r)}{1+t^{\alpha+1}} + c_1(r) \frac{t^{\alpha}}{1+t^{\alpha+1}}.$$

*This last estimate shows that the last condition of Lemma 1 is also satisfied. Then, $F(B_r)$ is relatively compact according to Lemma 1. The theorem of fixed point of Schauder ensures the existence of a fixed point of F. Therefore, the problem (1)-(2) has at least one solution $x \in C_\alpha$.*

## 3. Example

Consider the following problem:

$$\begin{aligned} {}^cD^{\frac{1}{2}}\left(x(t) + \frac{1}{5}\sin(t + x(t))\right) &= \frac{\cos(t^3+1)}{15}\left(1+t\sqrt{t}\right)^{\frac{-5}{2}} |y|^{\frac{2}{3}} \ln\left(\left(\frac{|y|}{1+t\sqrt{t}}\right)^{\frac{1}{3}} + 1\right) x(t) \\ &\quad + \frac{|x(t)|}{(19+\exp(2t))\left(1+t\sqrt{t}\right)} + \frac{1}{10}, \\ x(0) + \frac{1}{4}\sin(x(t)) &= 10^{-2}. \end{aligned}$$

We take $(X, \|.\|) = (\mathbb{R}, |.|), \alpha = \frac{1}{2}.$

$$A(t,y) = \frac{\cos(t^3+1)}{15}\left(1+t\sqrt{t}\right)^{\frac{-5}{2}} |y|^{\frac{2}{3}} \ln\left(\left(\frac{|y|}{1+t\sqrt{t}}\right)^{\frac{1}{3}} + 1\right), t \geqslant 0, y \in \mathbb{R}.$$

$$\begin{aligned} \sigma(t, x(t)) &= \frac{1}{5}\sin(t + x(t)). \\ f(t, x(t)) &= \frac{|x(t)|}{(19+\exp(2t))\left(1+t\sqrt{t}\right)} + \frac{1}{10}, t > 0, y \in \mathbb{R} \\ g(x) &= \frac{1}{4}\sin(x(t)). \end{aligned}$$

So, we have

$$|A(t,y)| \leqslant \frac{|\cos(t^3+1)|}{15\left(1+t\sqrt{t}\right)}\left(\frac{|y|}{1+t\sqrt{t}}\right)^{\frac{2}{3}} \ln\left(\left(\frac{|y|}{1+t\sqrt{t}}\right)^{\frac{1}{3}}+1\right) \leqslant \frac{|\cos(t^3+1)|}{15\left(1+t\sqrt{t}\right)}\left(\frac{|y|}{1+t\sqrt{t}}\right)^{\frac{2}{3}}\left(\frac{|y|}{1+t\sqrt{t}}\right)^{\frac{1}{3}}$$

$$\leqslant \frac{|\cos(t^3+1)|}{15\left(1+t\sqrt{t}\right)}.\frac{|y|}{1+t\sqrt{t}}.$$

So, $A(t,y)$ satisfies ($H_1$), with $\varphi(t) = \frac{|\cos(t^3+1)|}{15}$, $\|\varphi\| \leqslant \frac{1}{15}$, and $\psi(t) = t^{\frac{2}{3}} \ln\left(t^{\frac{1}{3}}+1\right)$.

On the other hand,

$$|f(t,x(t))| \leqslant \frac{1}{19+\exp(2t)}.\frac{|x(t)|}{1+t\sqrt{t}} + \frac{1}{10}.$$

So, ($H_2$) is satisfied with

$$\begin{aligned} a(t) &= \frac{1}{19+\exp(2t)} \Rightarrow \|a\|_\infty = \frac{1}{20} \\ b(t) &= \frac{1}{10} \Rightarrow \|b\|_\infty = \frac{1}{10}. \end{aligned}$$

As for the hypothesis ($H_3$) we have

$$\begin{aligned} |\sigma(t,x(t)) - \sigma(t,y(t))| &= \frac{1}{5}|\sin(t+x(t)) - \sin(t+y(t))| \\ &\leqslant \frac{1}{5}|x(t)-y(t)| \end{aligned}$$

So ($H_3$) is satisfied with

$$c = \frac{1}{5}, \quad \delta = \frac{1}{5} and \quad \sigma_0 = 0.$$

It follows form the condition ($H_4$) that

$$|g(x)-g(y)| \leqslant \frac{1}{4}|x-y|, \qquad x,y \in \mathbb{R}$$

So,

$$G = \frac{1}{4}$$

Finally, the inequality given in the hypothesis ($H_5$) requires that the constant $r$ satisfies

$$\frac{2}{15\sqrt{\pi}}r^2 + \left(\frac{1}{10\sqrt{\pi}} + \frac{1}{4} + \frac{1}{5} - 1\right)r + \left(10^{-2} + \frac{1}{5} + \frac{1}{5\sqrt{\pi}}\right) < 0$$

which is true for every $r \in [1.34, 3.25]$.

So all the hypotheses of Theorem 2 are satisfied, therefore the problem has at least one solution in $C_{\frac{1}{2}}$ space.

## References


[1] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.

[2] V. Lakshmikantham, S. Leela, J. Vasundhara, Theory of Fractional Dynamic Systems, Cambridge Academic Publishers, Cambridge, 2009.

[3] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equation, Wiley, New York,1993.

[4] I. Podlubny, Fractional Differential Equations, Mathematics in Science and Engineering, Academic Press, New York, 1999.

[5] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives, Theory and Applications, Gordon and Breach, Yverdon, 1993.

[6] D. Baleanu , K. Diethelm , E. Scalas , J.J. Trujillo , Fractional Calculus Models and Numerical Methods, in: Series on Complexity, Nonlinearity and Chaos, World Scientific, Boston, 2012 .

[7] X. Dong, Z. Bai, S. Zhang, Positive solutions to boundary value problems of p-laplacian with fractional derivative, Bound. Value Probl. 2017 (2017) 1-15.

[8] S. Das , Functional Fractional Calculus for System Identification and Controls, Springer, New York, 2008.

[9] D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, 2014.

[10] X. Zhao, F. An, The eigenvalues and sign-changing solutions of a fractional boundary value problem, Adv. Diff. Equa. 2016 (2016) 109.

[11] Z.M. Ge , C.Y. Ou , Chaos synchronization of fractional order modified duffing systems with parameters excited by a chaotic signal, Chaos Solitons Fractals 35 (2008) 705-717.

[12] J. Klafter , S.C. Lim , R. Metzler , Fractional Dynamics in Physics, World Scientific, Singapore, 2011 .

[13] R. Metzler , J. Klafter , The random walks guide to anomalous diffusion: a fractional dynamics approach, Phys. Rep. 339 (2000) 1-77 .

[14] M. Ostoja-Starzewski , Towards thermoelasticity of fractal media, J. Therm. Stress. 30 (2007) 889896.

[15] Y.Z. Povstenko , Fractional Thermoelasticity, Springer, New York, 2015 .

[16] Y. Pu, P. Siarry, J. Zhou, N. Zhang, A fractional partial differential equation based multiscale denoising model for texture image, Math. Methods Appl. Sci. 37 (2014) 1784-1806.