SCIREA Journal of Mathematics
ISSN: 2995-5823
http://www.scirea.org/journal/Mathematics
October 9, 2024
Volume 9, Issue 4, August 2024
https://doi.org/10.54647/mathematics110485

Knowledge Reasoning Based on the Generalized Syllogism *FMO-3*

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Abstract

This paper focuses on the validity and reducibility of the non-trivial generalized syllogisms with the quantifiers in Square{*some*} and Square{*fewer than half of the*}. To this end, this paper firstly presents knowledge representations of generalized syllogisms, and then proves the validity of the syllogism *FMO-3*, and subsequently deduces the other 21 valid non-trivial generalized syllogisms from the validity of the syllogism *FMO-3*. That is to say that there are reducible relations between/among these 22 valid non-trivial generalized syllogisms. This is so because the deduced syllogisms in this paper only involve the following 8 quantifiers: *no*, *all*, *not all*, *some, fewer than half of the, most, at most half of the*, and *at least half of the,* and because any of the first four quantifiers can define the other three quantifiers, as well as because the same goes for the last four quantifiers. This formal deductive reasoning not only provides a theoretical basis for English language information processing, but also inspiration for studying other kinds of syllogisms.

Keywords: generalized quantifiers; generalized syllogisms; validity; reducibility; knowledge reasoning

1. Introduction

Syllogistic reasoning occupies an important position in human thinking, which involves two types of quantifiers: Aristotelian and generalized quantifiers. There are only four Aristotelian quantifiers, namely *all*, *not all*, *some* and *no* (Hao, 2023). There are infinitely many generalized quantifiers in natural language (Zhang and Wu, 2021), such as, *most*, *fewer than half of the, at least two-thirds*, *many*, *few*, etc. Aristotelian quantifiers are trivial generalized ones (Peters and Westerståhl, 2006). A syllogism that only contains Aristotelian quantifiers is an Aristotelian one which is a trivial generalized one (Zhang, 2018). A syllogism that contains at least one and at most three non-trivial generalized quantifiers is a non-trivial generalized one (Hao, 2024). There are many works on Aristotelian syllogisms both domestically and internationally, for example, Moss (2008), Zhang (2022), and Yu and Zhang (2024), and so on. While there are few works on non-trivial generalized syllogisms which are the focus of this paper.

Syllogistic reasoning is a common form of reasoning in scientific language and natural language (Wei and Zhang, 2023), and belongs to knowledge reasoning in artificial intelligence. There are a large number of generalized quantifiers in English language (Peters and Westerståhl, 2006). The generalized syllogisms studied in this paper only involve the four Aristotelian quantifiers in Square{*some*}={*some*, *not all*, *all* and *no*}, and the non-trivial generalized quantifiers in Square{*fewer than half of the*}={*fewer than half of the, most*, *at least half of the,* and *at most half of the*}. Just as any of the quantifiers in Square{*some*} can define the other three, and any of the quantifiers in Square{*fewer than half of the*} can define the other three (Hao, 2024). The above eight quantifiers are common quantifiers in the English language.

2. Knowledge Representation of Generalized Syllogisms

In the following, let *t*, *g*, and *w* be lexical variables, and *D* be their domain. The sets formed of *t*, *g*, and *w* are respectively *T, G*, and *W*. Let α, μ, ϕ and ξ be well-formed formulas (shortened as wff). Let *Q* be a quantifier, $\neg Q$ and $Q\neg$ be its outer and inner quantifier, respectively. '$\vdash\xi$' shows that ξ is provable, and '$\phi =_{def} \xi$' that ϕ can be defined by ξ. The others are similar.

The non-trivial generalized syllogisms discussed in this paper only involve the following eight propositions: *all(t, w)*, *some(t, w), no(t, w)*, *not all(t, w), fewer than half of the(t, w), at* *most half of the(t, w), most(t, w),* and *at least half of the(t, w),* and they are respectively abbreviated as Proposition *A*, *I*, *E*, *O*, *F*, *H*, *M*, and *S*, in which '*all(t, w)'* means that 'all *t*s *are w*s', and '*fewer than half of the(t, w)* means that 'fewer than half of the *t*s *are w*s', the others are similar. The third figure generalized syllogism '*fewer than half of the(g, w)* $\wedge$ *most(g, t)*$\rightarrow$*not all(t, w)*' is shortened as *FMO-3,* which is the foundation of knowledge reasoning in this paper. An instance of the syllogism is as follows:

Major premise: Fewer than half of the dogs in this farm eat vegetables.

Minor premise: Most dogs in this farm are chihuahuas.

Conclusion: Not all chihuahuas in this farm eat vegetables.

Let *g* be a dog in this farm, *w* be an animal eats vegetables, and *t* be a chihuahuas in this farm. Then this syllogism can be symbolized as '*fewer than half of the(g, w)*$\wedge$*most(g, t)*$\rightarrow$*not all(t, w)*', which is abbreviated as *FMO-3*. Similar knowledge representations can be made for other generalized syllogisms.

3. Generalized Syllogism System with the Quantifiers in Square{*some*} and Square{*fewer than half of the*}

This system includes the following: primitive symbols, formation and deductive rules, and basic axioms, etc.

3.1 Primitive Symbols

(1) lexical variables: *t*, *g*, *w*

(2) quantifiers: *some, fewer than half of the*

(3) operators: $\neg$, $\rightarrow$

(4) brackets: (,)

3.2 Formation Rules

(1) If *Q* is a quantifier, *t* and *w* are lexical variables, then *Q(t, w)* is a wff.

(2) If ξ is a wff, then so is $\neg\xi$.

(3) If ϕ and ξ are wffs, then so is $\phi\rightarrow\xi$.

(4) Only the formulas formed from the above rules are wffs.

3.3 Basic Axioms

A1: If ξ is a valid formula in proposition logic, then $\vdash\xi$.

A2: $\vdash$ *fewer than half of the(g, w)*$\wedge$*most(g, t)*$\rightarrow$*not all(t, w)* (i.e. the syllogism *FMO-3*).

3.4 Deductive Rules

Rule 1 (subsequent weakening): $\vdash(\mu\wedge\phi\rightarrow\alpha)$ can be deduced from $\vdash(\mu\wedge\phi\rightarrow\xi)$ and $\vdash(\xi\rightarrow\alpha)$.

Rule 2 (anti-syllogism): $\vdash(\neg\xi\wedge\mu\rightarrow\neg\phi)$ can be deduced from $\vdash(\mu\wedge\phi\rightarrow\xi)$.

Rule 3 (anti-syllogism): $\vdash(\neg\xi\wedge\phi\rightarrow\neg\mu)$ can be deduced from $\vdash(\mu\wedge\phi\rightarrow\xi)$.

3.5 Relevant Definitions

D1 (conjunction): $(\mu\wedge\phi)=_{\text{def}}\neg(\mu\rightarrow\neg\phi)$

D2 (bicondition): $(\mu\leftrightarrow\phi)=_{\text{def}}(\mu\rightarrow\phi)\wedge(\phi\rightarrow\mu)$

D3 (inner negation): $(Q\neg)(t, w)=_{\text{def}}Q(t, D-w)$

D4 (outer negation): $(\neg Q)(t, w)=_{\text{def}}$ It is not that $Q(t, w)$

D5 (truth value): *all(t, w)*$=_{\text{def}}T\subseteq W$;

D6 (truth value): *some(t, w)*$=_{\text{def}}T\cap W\neq\varnothing$;

D7 (truth value): *no(t, w)*$=_{\text{def}}T\cap W=\varnothing$;

D8 (truth value): *not all(t, w)*$=_{\text{def}}T\not\subseteq W$;

D9 (truth value): *fewer than half of the(t, w)* is true iff $|T\cap W|<0.5|T|$ is true;

D10 (truth value): *most(t, w)* is true iff $|T\cap W|>0.5|T|$ is true;

D11 (truth value): *at most half of the(t, w)* is true iff $|T\cap W|\leq 0.5|T|$ is true;

D12 (truth value): *at least half of the(t, w)* is true iff $|T\cap W|\geq 0.5|T|$ is true.

3.5 Relevant Facts

Fact 1 (inner negation):

(1.1) *all(t, w)=no¬(t, w)*;

(1.2) *no(t, w)=all¬(t, w)*;

(1.3) *some(t, w)=not all¬(t, w)*;

(1.4) *not all(t, w)=some¬(t, w)*;

(1.5) *most(t, w)=fewer than half of the¬(t, w)*;

(1.6) *fewer than half of the(t, w)=most¬(t, w)*;

(1.7) *at least half of the(t, w)=at most half of the¬(t, w)*;

(1.8) *at most half of the(t, w)=at least half of the¬(t, w)*.

Fact 2 (outer negation):

(2.1) *¬all(t, w)=not all(t, w)*;

(2.2) *¬not all(t, w)=all(t, w)*;

(2.3) *¬no(t, w)=some(t, w)*;

(2.4) *¬some(t, w)=no(t, w)*;

(2.5) *¬most(t, w)=at most half of the(t, w)*;

(2.6) *¬at most half of the(t, w)=most(t, w)*;

(2.7) *¬fewer than half of the(t, w)=at least half of the(t, w)*;

(2.8) *¬at least half of the(t, w)=fewer than half of the(t, w).*

Fact 3 (symmetry):

(3.1) *some(t, w)↔some(w, t)*;   (3.2) *no(t, w)↔no(w, t)*.

Fact 4 (Subordination):

(4.1) ⊢*all(t, w)→some(t, w)*;

(4.2) ⊢*no(t, w)→not all(t, w)*;

(4.3) ⊢*all(t, w)→most(t, w)*;

(4.4) ⊢*most(t, w)→some(t, w)*;

(4.5) ⊢*at least half of the(t, w)→some(t, w)*;

(4.6) ⊢*all(t, w)→at least half of the(t, w)*;

(4.7) ⊢*at most half of the(t, w)→not all(t, w)*;

(4.8) ⊢*fewer than half of the(t, w)→not all(t, w)*.

The above facts are basic knowledge in proposition logic (Hamilton, 1978) and generalized quantifier theory (Westerståhl, 2007), then their proofs are not given.

4. Knowledge Reasoning about the Generalized Syllogisms with the Quantifiers in Square{*some*} and Square{*fewer than half of the*}

The following Theorem 1 proves that the generalized syllogism *FMO-3* is valid. Theorem 2 shows that other 21 valid generalized syllogisms can be inferred from the syllogism *FMO-3*. That is to say that there are reducible relations between/among the 22 valid generalized syllogisms.

Theorem 1 (*FMO-3*): The generalized syllogism *fewer than half of the(g, w)*$\wedge$*most(g, t)*$\rightarrow$*not all(t, w)* is valid.

Proof: Suppose that *fewer than half of the(g, w)* and *most(g, t)* are true, then $|G\cap W|<0.5|G|$ and $|G\cap T|>0.5|G|$ are true according to Definition D9 and D10, respectively. Thus, it is easy to see that $|T\cap W|\leq 0.5|T|$ is true. Thus, *at most half of the(t, w)* is true in line with Definition D11. It follows that *not all(t, w)* by means of Fact (4.7), just as required.

Theorem 2: The following 21 valid generalized syllogisms can be deduced from the validity of the syllogism *FMO-3*:

(2.1) $\vdash$*FMO-3*$\rightarrow$*AFH-2*

(2.2) $\vdash$*FMO-3*$\rightarrow$*AFH-2*$\rightarrow$*AFO-2*

(2.3) $\vdash$*FMO-3*$\rightarrow$*AMS-1*

(2.4) $\vdash$*FMO-3*$\rightarrow$*AMS-1*$\rightarrow$*AMI-1*

(2.5) $\vdash$*FMO-3*$\rightarrow$*AMS-1*$\rightarrow$*AMI-1*$\rightarrow$*MAI-4*

(2.6) $\vdash$*FMO-3*$\rightarrow$*MMI-3*

(2.7) $\vdash$*FMO-3*$\rightarrow$*AFH-2*$\rightarrow$*EMH-2*

(2.8) $\vdash$*FMO-3*$\rightarrow$*AFH-2*$\rightarrow$*EMH-2*$\rightarrow$*EMH-1*

(2.9) $\vdash$*FMO-3*$\rightarrow$*AFH-2*$\rightarrow$*EMH-2*$\rightarrow$*EMO-2*

(2.10) $\vdash$*FMO-3*$\rightarrow$*AFH-2*$\rightarrow$*EMH-2*$\rightarrow$*EMO-2*$\rightarrow$*EMO-1*

(2.11) $\vdash$*FMO-3*$\rightarrow$*AFH-2*$\rightarrow$*EMH-2*$\rightarrow$*EMH-1*$\rightarrow$*EMO-4*

(2.12) $\vdash$*FMO-3*$\rightarrow$*AFH-2*$\rightarrow$*AFO-2*$\rightarrow$*AAS-1*

(2.13) $\vdash$*FMO-3*$\rightarrow$*AFH-2*$\rightarrow$*AFO-2*$\rightarrow$*FAO-3*

(2.14) $\vdash$*FMO-3*$\rightarrow$*AMS-1*$\rightarrow$*EMH-1*

(2.15) $\vdash$*FMO-3*$\rightarrow$*AMS-1*$\rightarrow$*AMI-1*$\rightarrow$*AEH-2*

(2.16) $\vdash$*FMO-3→AMS-1→AMI-1→AEH-2→AEH-4*

(2.17) $\vdash$*FMO-3→AMS-1→AMI-1→EMO-3*

(2.18) $\vdash$*FMO-3→AFH-2→EMH-2→EMO-2→EAH-1*

(2.19) $\vdash$*FMO-3→AFH-2→EMH-2→EMO-2→EAH-1→EAH-2*

(2.20) $\vdash$*FMO-3→AFH-2→EMH-2→EMO-2→MAI-3*

(2.21) $\vdash$*FMO-3→AFH-2→EMH-2→EMO-2→MAI-3→AMI-3*

Proof:

[1] $\vdash$*fewer than half of the(g, w)∧most(g, t)→not all(t, w)* (i.e.*FMO-3*, Axiom A2)

[2] $\vdash$*¬not all(t, w)∧fewer than half of the(g, w)→¬most(g, t)* (by [1] and Rule 2)

[3] $\vdash$*all(t, w)∧fewer than half of the(g, w)→at most half of the(g, t)*

(i.e.*AFH-2*, by [2], Fact (2.2) and (2.5))

[4] $\vdash$*all(t, w)∧fewer than half of the(g, w)→not all(g, t)* (i.e.*AFO-2*, by [3] and Fact (4.7))

[5] $\vdash$*¬not all(t, w)∧most(g, t)→¬fewer than half of the(g, w)* (by [1] and Rule 3)

[6] $\vdash$*all(t, w)∧most(g, t)→at least half of the(g, w)* (i.e.*AMS-1*, by [5], Fact (2.2) and (2.7))

[7] $\vdash$*all(t, w)∧most(g, t)→some(g, w)* (i.e.*AMI-1*, by [6] and Fact (4.5))

[8] $\vdash$*all(t, w)∧most(g, t)→some(w, g)* (i.e.*MAI-4*, by [7] and Fact (3.1))

[9] $\vdash$*most¬(g, w)∧most(g, t)→some¬(t, w)* (by [1], Fact (1.6) and (1.4))

[10] $\vdash$*most(g, D−w)∧most(g, t)→some(t, D−w)* (i.e.*MMI-3*, by [9] and Definition D3)

[11] $\vdash$*no¬(t, w)∧most¬(g, w)→at most half of the(g, t)* (by [1], Fact (1.1) and (1.6))

[12] $\vdash$*no(t, D−w)∧most(g, D−w)→at most half of the(g, t)* (i.e.*EMH-2*, by [11] and Definition D3)

[13] $\vdash$*no(D−w, t)∧most(g, D−w)→at most half of the(g, t)* (i.e.*EMH-1*, by [12] and Fact (3.2))

[14] $\vdash$*no(t, D−w)∧most(g, D−w)→not all(g, t)* (i.e.*EMO-2*, by [12], Rule 1 and Fact (4.7))

[15] $\vdash$*no(D−w, t)∧most(g, D−w)→not all(g, t)* (i.e.*EMO-1*, by [14], Rule 1 and Fact (3.2))

[16] $\vdash$*¬not all(g, t)∧all(t, w)→¬fewer than half of the(g, w)* (by [4] and Rule 2)

[17] $\vdash$*all(g, t)∧all(t, w)→at least half of the(g, w)* (i.e.*AAS-1*, by [16], Fact (2.2) and (2.7))

[18] $\vdash$*¬not all(g, t)∧fewer than half of the(g, w)→¬all(t, w)* (by [4] and Rule 3)

[19] $\vdash$*all(g, t)∧fewer than half of the(g, w)→not all(t, w)* (i.e.*FAO-3*, by [18], Fact (2.2) and (2.1))

[20] $\vdash$*no¬(t, w)∧most(g, t)→at most half of the¬(g, w)* (by [6], Fact (1.1) and (1.7))

[21] $\vdash$*no(t, D−w)∧most(g, t)→at most half of the(g, D−w)* (i.e.*EMH-1*, by [20] and Definition D3)

[22] $\vdash$*¬some(g, w)∧all(t, w)→¬most(g, t)* (by [7] and Rule 2)

[23] $\vdash$*no(g, w)∧all(t, w)→at most half of the(g, t)* (i.e.*AEH-2*, by [22], Fact (2.4) and (2.5))

[24] $\vdash$*no(w, g)∧all(t, w)→at most half of the(g, t)* (i.e.*AEH-4*, by [23] and Fact (3.2))

[25] $\vdash$*¬some(g, w)∧most(g, t)→¬all(t, w)* (by [7] and Rule 3)

[26] $\vdash$*no(g, w)∧most(g, t)→not all(t, w)* (i.e.*EMO-3*, by [25], Fact (2.4) and (2.1))

[27] $\vdash$*¬not all(g, t)∧no(t, D−w)→¬most(g, D−w)* (by [14] and Rule 2)

[28] $\vdash$*all(g, t)∧no(t, D−w)→at most half of the(g, D−w)* (i.e.*EAH-1*, by [27], Fact (2.2) and (2.5))

[29] $\vdash all(g, t)\wedge no(D-w, t)\rightarrow at\ most\ half\ of\ the(g, D-w)$ (i.e.*EAH-2*, by [28] and Fact (3.2))

[30] $\vdash\neg not\ all(g, t)\wedge most(g, D-w)\rightarrow\neg no(t, D-w)$ (by [14] and Rule 3)

[31] $\vdash all(g, t)\wedge most(g, D-w)\rightarrow some(t, D-w)$ (i.e.*MAI-3*, by [30], Fact (2.2) and (2.3))

[32] $\vdash all(g, t)\wedge most(g, D-w)\rightarrow some(D-w, t)$ (i.e.*AMI-3*, by [31] and Fact (3.1))

Theorem 2 has proved that the above 21 non-trivial valid generalized syllogisms can be derived from the validity of the generalized syllogism *FMO-3*.

5. Conclusion and Future Work

This paper focuses on the validity and reducibility of the generalized syllogisms with the quantifiers in Square{*some*} and Square{*fewer than half of the*}. To this end, this paper firstly presents knowledge representations of generalized syllogisms, and then proves the validity of the syllogism *FMO-3* in Theorem 1, and subsequently deduces the other 21 non-trivial valid generalized syllogisms from the validity of the syllogism *FMO-3* in Theorem 2. That is to say that there are reducible relations between/among these 22 valid non-trivial generalized syllogisms. In fact, more valid generalized syllogisms can be inferred by means of the above deductive reasoning. This is so because the deduced syllogisms in this paper only involve the following 8 quantifiers: *no*, *all*, *not all*, *some, fewer than half of the, most, at most half of the*, and *at least half of the,* and because any of the first four quantifiers can define the other three quantifiers, as well as because the same goes for the last four quantifiers.

This formal deductive method not only provides a theoretical basis for English language information processing, but also inspiration for studying other kinds of syllogisms, such as syllogisms with verbs, Aristotelian modal syllogisms, relational syllogisms, generalized modal syllogisms. Does the generalized syllogism fragment system in this paper has meta logical properties, such as soundness and completeness? The question deserves in-depth discussion.

Acknowledgement

This work was supported by the National Social Science Fund of China under Grant No.21BZX100.

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