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On Randomness

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Abstract

Using a primitive formulation of the circle method together with a random number generator $\theta(n)$, which is based on the naive notion of randomness, we prove there exists an infinite number of twin primes. Then we show that a deduction, crucial to the proof, which is immediately implied by $\theta(n)$, is false.

Keywords: Goldbach Conjecture; Twin Primes Conjecture; circle method; generalized Riemann Hypothesis

Introduction

The Merriam-Webster Collegiate Dictionary defines random: being or related to a set or to an element of a set each of whose elements has equal probability of occurrence. Jacob Bernoulli: Two contingent events are considered equally probable if after taking into consideration all relevant evidence, one of them cannot be expected in preference to the other.

We define a random number generator $\theta(n) = \left\{ \begin{matrix} n \\ n+1 \end{matrix} \text{ randomly} \right\}$ where "randomly" is defined naively as above. By embedding $\theta(n)$ into Statement 2 below we derive a proof of the existence of an infinite number of twin primes. We then show that Statement 3 below, crucial to the proof, which is immediately implied by that embedding of $\theta(n)$ into Statement 2, is false.

The notation in this paper is based on that found in [1].

Let

$$f(x,v) = \sum_{p \le v} \epsilon(px)\text{, where } \epsilon(z) = e^{2\pi iz} (v \ge 0);$$

$$g(x,v) = \sum_{2 \le m \le v} \frac{\epsilon(mx)}{\log m} \ (v \ge 2), g(x,v) = 0 (v < 2);$$

$$Ls[v] = \sum_{2 \le m \le v} \frac{1}{\log m} = g(0,v) \sim \left(\frac{v}{\log v}\right), Ls[0] = Ls[1] = 0;$$

$$x_0 = \left(\frac{\log^{2+\epsilon} n}{n}\right)$$

$$r(n,m) = \text{number of } (p_i, p_j), p_i \le n, p_j \le n, p_i - p_j = m.$$

In what follows *n* and *m* are assumed to be natural numbers.

We consider two statements:

1. $r\,(n,2) < \frac{n}{\log^{2+\epsilon} n}$ for $n > N_1$

2. For all *n* for all *m* such that

$$\frac{n}{\log^{2+} n} \le m \le n \text{ we have}$$

$$(-1)^{\theta(n)}\, r(n,m) \le \frac{n}{\log^{2+\epsilon} n}$$

$$\text{or}$$

$$(-1)^{\theta(n)+1} r(n,m) \le \frac{n}{\log^{2+\epsilon} n}$$

Statement 2 is introduced as a vehicle, in which to embed $\theta(n)$. Clearly, for all *n*, Statement 2 is true.

The crux of the proof is that the definition of $\theta(n)$, based on the naive definition of random, combined with Bernoulli's observation immediately implies (Statement 3) there exists an infinite sequence of *n*, say $a(n)$, for all *m* such that

$$\frac{n}{\log^{2+\epsilon} n} \le m \le n \text{ we have } r(n,m) \le \frac{n}{\log^{2+\epsilon} n}.$$

Then we are able to prove for n in that infinite sequence, $I_{4(n)} = o\left(\frac{n}{\log^{2+\epsilon} n}\right)$ (cf. page 7), which, from our primitive formulation of the circle method construction, immediately implies that for n in that sequence $r\ (n, 2)\ \geq \frac{n}{\log^2 n}$ for $n\ >\ N_2$.

But that fact contradicts Statement 1; so that, by elementary logic, either its negation is true (which implies there exists an infinite number of twin primes) or the negation of Statement 2 is true or the negation of Statement 3 is true.

But the negation of Statement 2 must be false; since Statement 2 is true, and the negation of Statement 3 must be false; since by the naive definition of randomness and the definition of $\theta(n)$ it is most reasonable to assume Statement 3 is true.

Proof

We assume $n > C_0$. For each n, let $E_n = [x_0, 1 - x_0]$.

Clearly,

$$r(n, 2) = \int_{x_0}^{1+x_0} \left|f_{x,n}\right|^2 \epsilon(-2x)dx.$$

We decompose this integral $r(n) = m(n) + T(1,1)$, where

$$m(n) = \int_{E_n} |f(x, n)|^2\, \epsilon(-2x)dx,\ \ T(1,1) = \int_{1-x_0}^{1+x_0} |f(x, n)|^2\, \epsilon(-2x)dx.$$

Theorem 1

[1] Let $m \geq 3$, $k \leq \log^u n$ and $(k, l) = 1$

$$\left|\pi(m, k, l) - \frac{1_k m}{\phi(k)}\right| \leq C\, m \exp\left(\frac{-\log^{1/2} n}{200}\right)$$

This is proved by standard analytic number theory techniques.

Corollary 1

$$\left|\pi([v]; q, l) - \frac{\text{ls}[v]}{\phi(q)}\right| < n \log^{-100} n \text{ for } (0 \leq v \leq n), (q, n) = 1, n > C_0.$$

This follows immediately from Theorem 1 by considering case 1 $0 \leq v \leq n^{1/2}$ and case 2 $n^{1/2} < v \leq n$.

Lemma 1

$$\sum_{\substack{0<l\leq q\\(l,q)=1\\(h,q)=1}} e\left(\frac{lh}{q}\right) = \mu(q).$$

By Theorem 272 in [2] we have

$$\sum_{\substack{0<l\leq q\\(l,q)=1}} e\left(\frac{lh}{q}\right) = \frac{u(N)\phi(q)}{\phi(N)} \text{ where } q = aN, a = (h,q).$$

But $(h,q) = 1$

Lemma 2

[1]

$$f(x_1 + x_1, v) = e(vx_2)\, f(x_1, v) - 2\pi i x_2 \int_0^v e(ux_2)\, f(x_1, u)\, du\,.$$

$$e(vx_2) - e(px_2) = \int_p^v 2\pi i x_2\, e(ux_2)\, du$$

$$\begin{aligned}
f(x_1 + x_2, v) &= \sum_{p\leq v} e(px_1)\ e(px_2)\\
&= \sum_{p\geq v} \left\{e(vx_2) - 2\pi i x_2 \int_p^v e(ux_2)\, du\right\}\\
&= \sum_{p\leq v} \left\{e(vx_2) - 2\pi i x_2 \int_p^v e(ux_2)\ du\right\}\\
&= e(vx_2) \sum_{p\leq v} e(px_1) - 2\pi i x_2 \int_0^v e(ux_2) \sum_{p\leq v} e(px_1)\, du\\
&= e(vx_2)\, f(x_1, v) - 2\pi i x_2 \int_0^u e(ux_2)\ f(x_1, u).
\end{aligned}$$

Lemma 3

[1]

$$g(x_1 + x_2, v) = e(vx_2)\, g(x_1, v) - 2\pi i x_2 \int_0^v e(ux_2)\, g(x_1, u)\ du.$$

As stated on page 63 in [1] this can be proved in the same way as the proof of Lemma 2.

Lemma 4

[1]

$$f\left(\frac{h}{q}+y,n\right)=e(ny)\,f\left(\frac{h}{q},n\right)-2\pi iy\int_0^n e(uy)\;f\left(\frac{h}{q},u\right)\,du.$$

This follows from Lemma 2 with $x_1=\frac{h}{q}, x_2=y, v=n$.

Lemma 5

[1]

$$g(y,n)=e(ny)\,g(0,n)-2\pi iy\int_0^n e(uy)\;g(0,u)\,du.$$

This follows from Lemma 3 with $x_1=0, x_2=y, v=0$.

Theorem 2

Let [1]

$$\begin{aligned} q &\le \log^{15} n\,, \\ |y| &\le x_0, \\ (q,n) &= 1. \end{aligned}$$

Then

$$\left|f\left(\frac{h}{q}+y,n\right)-\frac{\mu(q)}{\phi(q)}g(y,n)\right|\le \log^{-80+\epsilon} n\,.$$

Proof

$$\left|f\left(\frac{h}{q},v\right)-\sum_{\substack{p\le v\\ p\nmid q}} e\left(\frac{ph}{q}\right)\right|\le\sum_{q\nmid q}1<q\,.$$

Clearly,

$$\sum_{\substack{p\le v\\ p\nmid q}} e\left(\frac{ph}{q}\right)=\sum_{\substack{0<l<q\\ (l,q)=1}} e\left(\frac{lh}{q}\right)\sum_{\substack{p\le q\\ p\equiv l(\operatorname{mod} q)}}1=\sum_{\substack{0<l<q\\ (l,q)=1}} e\left(\frac{lh}{q}\right)\pi([v];q,l).$$

Hence by Lemma 1 and *Corollary 1* we have

$$\left|f\left(\frac{h}{q},v\right)-\frac{\mu(q)}{\phi(q)}g(0,v)\right|<q+\left|\sum_{\substack{p\le q\\ p\nmid q}}e\left(\frac{ph}{q}\right)-\frac{\mu(q)}{\phi(q)}\mathrm{ls}[v]\right|$$

$$=q+\left|\sum_{\substack{0<l\le q\\ (l,q)=1}}e\left(\frac{lh}{q}\right)\left\{\pi([v];q,l)-\frac{\mathrm{ls}[v]}{\phi(q)}\right\}\right|$$

$$=q+qn\log^{-100}n<2n\log^{-85}n\quad(0\le v\le n).$$

By Lemma 4 and Lemma 5 we have

$$\left|f\left(\frac{h}{q}+y,n\right)-\frac{\mu(q)}{\phi(q)}g(y,n)\right|=\left|e(ny)\left\{f\left(\frac{h}{q},n\right)-\frac{\mu(q)}{\phi(q)}g(0,n)\right\}\right.$$

$$\left.-2\pi iy\int_0^n e(vy)\left\{f\left(\frac{h}{q},v\right)-\frac{\mu(q)}{\phi(q)}g(0,v)\right\}dv\right|$$

$$\le\left|f\left(\frac{h}{q},n\right)-\frac{\mu(q)}{\phi(q)}g(0,n)\right|$$

$$+2\pi x_0\int_0^n\left|f\left(\frac{h}{q},v\right)-\frac{\mu(q)}{\phi(q)}g(0,v)\right|dv$$

$$\le 2n\log^{-85}n\,(1+2\pi\,x_0)<n\log^{-80+\epsilon}n\,,$$

By Theorem 2 we have for $|y|\le x_0$

$$|f(1+y,n)-g(y,n)|\le(n\log^{-80+\epsilon}n)$$

By the trivial inequalities $|f(x,n)|\le n$ and $|g(y,n)|\le n$ and the fact that if $|a|\le n$ and $|b|\le n$, then $\left|a^2-b^2\right|\le 2n|a-b|$ with $a-f(1+y,n)$ and $b=g(y,n)$ we have for $|y|\le x_0$

$$\left|f^2(1+y,n)-g^2(y,n)\right|<C_2n^2\log^{-80+\epsilon}n\,.$$

Immediately we have

$$\left||f(1+y,n)|^2-|g(y,n)|^2\right|\le\left|f^2(1+y,n)-g^2(y,n)\right|\le C_2n^2\log^{-80+\epsilon}n\,.$$

By the change of variable $y=(x-1)$ we have

$$T(1,1)=\int_{-x_0}^{x_0}|f(1+y,n)|^2\epsilon(-2y)\,dy;$$

so that

$$\left|\int_{-x_0}^{x_0} |f(1+y,n)|^2 \epsilon(2y)dy - \int_{-x_0}^{x_0} |g(y,n)|^2 \epsilon(-2y)dy\right|$$
$$\leq \int_{-x_0}^{x_0} \left|f^2(1+y,n) - g^2(y,n)\right| dy < C_6 n \log^{-78} n\,.$$

Let

$$T_1(n) - \int_{-x_o}^{x_0} |g(y,n)|^2 \epsilon(-2y)dy.$$

Then

$$|T(1,1) - T_1(n)| \leq C_6 n \log^{-78} n.$$

Let

$$T(n) = \sum_{\substack{m_1 m_2 \\ m_1 \geq 2, m_2 \geq 2 \text{ and } m_2 - m_1 = 2 \\ m_1 \leq n, m_2 \leq n}} \log^{-1} m_1 \log^{-1} m_2.$$

Clearly,

$$T(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} |g(y,n)|^2 \epsilon(-2y)dy.$$

Also, the number of terms on the right hand side is $(n-3)$ and each term is greater than $\log^{-1} n$ and less than 1 so that

$$(n-3)\log^{-2} n < T(n) < n$$
$$\left|\sum_{m=2}^{m_1} e(my)\right| \leq \frac{1}{|\sin \pi y|} \leq \frac{1}{2y} \quad \left(m_1 \geq 2, 0 < |y| \leq \frac{1}{2}\right)$$

Hence by definition of $g(y,n)$ and Abel's lemma we have

$$|g(y,n)| < |y|^{-1} \quad \left(0 < |y| < \frac{1}{2}\right);$$

so that

$$|T(n) - T_1(n)| \leq 2\int_{x_0}^{\frac{1}{2}} y^{-2} dy \leq C_4 n \log^{-2+\epsilon} n\,.$$

Hence

(1)

$$|T(1,1) - T(n)| \leq C_6 n \log^{-96+2\epsilon} n + C_4 n \log^{-(2+\epsilon)} n\,.$$

Clearly,

$$r(n,2) = \int_{x_0}^{1+x_0} |f(x,n)|^2 \cos(4\pi x) dx = \int_{x_0}^{1-x_0} |f(x,n)|^2 \cos(4\pi x) dx + T(n) + o\left(\frac{n}{\log^2 n}\right).$$

and

$$|f(x,n)|^2 = \pi(n) + 2 \sum_{\substack{2\leq m\leq n \\ p_i\leq n, p_j\leq n \\ m=p_i-p_j}} r(n,m) \cos(2\pi m x).$$

Hence,

$$\int_{x_0}^{1-x_0} |f(x,n)|^2 \cos(4\pi x) dx = \int_{x_0}^{1-x_0} \pi(n) \cos(4\pi x) dx + \int_{x_0}^{1-x_0} r(n,2) \cos^2(4\pi x) dx + 2 \sum_{2 < m < \frac{n}{\log^{2+\epsilon} n}} r(n,m) \int_{x_0}^{1-x_0} \cos(2\pi m x) \cos(4\pi x) dx + 2 \sum_{\frac{n}{\log^{2+\epsilon} n} \leq m \leq n} r(n,m) \int_{x_0}^{1-x_0} \cos(2\pi m x) \cos(4\pi x) dx = l_1(n) + l_2(n) + l_3(n) + l_4(n).$$

Lemma 6

$$\frac{\sin\theta}{\theta} < \frac{2+\cos\theta}{3} \quad (0 < \theta \leq 2\pi)\ [3],\ \text{p. } 107.$$

Lemma 7

If $m > 2$

$$2\int_{x_0}^{1-x_0} \cos(2\pi m x) \cos(4\pi x)\, dx = \frac{-\sin(2\pi(m-2)x_0)}{\pi(m-2)} - \frac{\sin(2\pi(m+2)x_0)}{\pi(m+2)}$$

Proof.

If $m > 2$

$$2\int_{x_0}^{1-x_0} \cos(2\pi m x)\cos(4\pi x)dx = \frac{\sin(2\pi(m-2)x)}{2\pi(m-2)} + \frac{\sin(2\pi(m+2)x)}{2\pi(m+2)}\Bigg|_{x_0}^{1-x_0}$$
$$= \frac{\sin\left(2\pi(m-2)(1-x_0)\right)}{2\pi(m+2)}\Bigg|_{x_0}^{1-x_0}$$
$$-\left(\frac{\sin(2\pi(m-2)x_0)}{2\pi(m-2)} + \frac{\sin(2\pi(m+2)x_0)}{2\pi(m+2)}\right)$$
$$= \frac{\sin\left(2\pi(m-2)(-x_0)\right)}{2\pi(m-2)} + \frac{\sin\left(2\pi(m+2)(-x_0)\right)}{2\pi(m+2)}$$
$$-\left(\frac{\sin(2\pi(m-2)x_0)}{2\pi\left((m-2)\right)}\right) + \frac{\sin(2\pi(m+2)x_0)}{2\pi(m+2)}$$
$$=-\frac{\sin(2\pi(m-2)x_0)}{2\pi(m-2)} - \frac{\sin(2\pi(m+2)x_0)}{2\pi(m+2)}$$
$$-\frac{\sin(2\pi(m-2)x_0)}{2\pi(m-2)} - \frac{\sin(2\pi(m+2)x_0)}{2\pi(m+2)x_0}$$
$$=-\frac{\sin(2\pi(m-2)x_0)}{\pi(m-2)} - \frac{\sin(2\pi(m+2)x_0)}{\pi(m+2)}$$

To show $l_1(n) = o\left(\frac{n}{\log^2 n}\right)$.

$$\int_{x_0}^{1-x_0} \pi(n)\cos(4\pi x)dx = \frac{\pi(n)}{4\pi}\sin(4\pi x)\Bigg|_{x_0}^{1-x_0} =-\frac{\pi(n)}{2\pi}\sin(4\pi x_0).$$

Now apply Lemma 6.

To show $l_2(n) = o\left(\frac{n}{\log^2 n}\right)$.

This is immediate by Statement 1.

To show $l_3(n) = o\left(\frac{n}{\log^2 n}\right)$.

This follows immediately from Lemma 6, Lemma 7 and the facts that $2\pi(m-2)x_0 \leq 2\pi \leftrightarrow m \leq x_0^{-1} + 2$ and $2\pi(m+2)x_0 \leq 2\pi \leftrightarrow m \leq x_0^{-1} - 2$,

$$2 < m \leq \frac{n}{\log^{2+\epsilon} n}, r(n,m) \leq (1-\epsilon)\frac{n}{\log n} \quad \text{and } x_0 = \frac{\log^{2+\epsilon} n}{n}.$$

To show $I_4(n) = o\left(\frac{n}{\log^2 n}\right)$ for n in $a(n)$.

This follows immediately from the argument in the Introduction and the facts that

$$\frac{n}{\log^{2+\epsilon} n} \leq m \leq n, \quad \left|2\int_{x_0}^{1-x_0} \cos(2\pi m x)\cos(4\pi x)\,dx\right| \leq \frac{1}{\pi}\left(\frac{1}{m-2} + \frac{1}{m+2}\right) \quad \text{by Lemma 7,}$$

$$\sum_{\frac{n}{\log^{2+\epsilon} n}}^{n} \frac{1}{m} \leq \log n - \left\{\log\frac{n}{\log^{2+\epsilon} n}\right\} = (2+\epsilon)\log n \quad \text{and} \quad \frac{(2+\epsilon)\log n}{\log^{\epsilon} n} \to 0.$$

Counterexample to Statement 3

Assume $n \geq N_0$. By the PNT the number of primes in $\left(\frac{n}{2}, n\right)$ is asymptotic to $\frac{n}{2\log n}$ and the number of primes in $\left\{\frac{n}{8}, \frac{n}{4}\right\}$ is asymptotic to $\frac{n}{8\log n^*}$.

Let $S = \left\{m = p_i - p_j \,\middle|\, \frac{n}{2} \leq p_i \leq n, \frac{n}{8} \leq p_j \leq \frac{n}{4}\right\}$

$|S|$ is (asymptotically) equal to $\frac{n^2}{16\log^2 n}$ and $\frac{n}{4} \leq m \leq \frac{7n}{8}$.

But by the assumption we have $|S| \leq \frac{7n^2}{8\log^{2+\epsilon} n}$.

References

[1] T. Estermann, Introduction to Modern Prime Number Theory, London/New York: Cambridge University Press, 1961.

[2] G.H. Hardy and E.M. Wright, An Introduction to the Theory of Numbers, London/New York: Oxford University Press, 1965.