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On the right alternative color algebras

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Abstract

In this paper, we establish the color version of identities in (right) alternative algebras.

Keywords: *Color algebras, ε-Commutator, ε-Jordan product, Associator, Right alternative color algebras.*

1. Introduction

The study of nonassociative algebras was originally motivated by certain problems in physics and other branches of mathematics. However, most types of nonassociative algebras are now studied more for their own sake.

The class of alternative algebras is defined by the two identities

$$x(xy) = x^2y \text{ and } yx^2 = (yx)x.$$

Algebras for which $yx^2 = (yx)x$ have been called right alternative algebras. They were first studied by A. A. Albert, who showed that a semisimple, right alternative algebra over a field of characteristic 0 is alternative [1]. In [8] some properties of such algebras were used to solve a problem in projective planes. In [5] and [8] fundamental identities characterizing right alternative algebras were found. In this paper we give some of these identities in case of right alternative color algebras.

The paper is organized as follows. In Section 2 we give some definitions. In Section 3 we establish some fundamental identities in right alternative color algebras.

Throughout this paper, all vector spaces and algebras considered are assumed to be finite dimensional over a fixed ground field $\mathbb{K}$ of characteristic not 2 and G is an abelian group.

2. Definitions

Definition 2.1. 1. A $\mathbb{K}$-vector space V is said to be G-graded whenever we are given a family $\left(V_g\right)_{g\in G}$ of subspaces of V such that $V = \bigoplus_{g\in G} V_g$ (direct sum).

2. An element v of $V = \bigoplus_{g\in G} V_g$ is said to be homogeneous of degree $g \in G$ if $v \in V_g$.

Definition 2.2. An algebra $(A, +, ., \times)$ is called a G-graded algebra if:

1. A is a G-graded vector space $A = \bigoplus_{g\in G} A_g$,
2. $A_g A_h \subseteq A_{g+h}$ for all g and h in G.

Definition 2.3. A mapping $\varepsilon: G \times G \to \mathbb{K}^*$ is called a bicharacter on G if the following identities hold for all i, j, k in G :

1. $\varepsilon(i, j + k) = \varepsilon(i, j)\varepsilon(i, k)$
2. $\varepsilon(i + j, k) = \varepsilon(i, k)\varepsilon(j, k)$
3. $\varepsilon(i, j)\varepsilon(j, i) = 1$.

We will define as color the pair (G, ε) as above [4]. We assume throughout this paper that ε is a fixed bicharacter on G. All elements in a graded algebra A are assumed to be homogeneous and we write $\bar{x}$ for the degree of x.

In a color algebra A, the ε-commutator and the ε-Jordan product of any two elements x, y of A are defined respectively as

$$[x, y] := xy - \varepsilon(\bar{y}, \bar{x})yx$$

and

$$x \circ y := xy + \varepsilon(\bar{y}, \bar{x})yx.$$

For any elements x, y, z of A, the associator (x, y, z) is defined as

$$(x, y, z) := (xy)z - x(yz).$$

Definition 2.4. A G-graded algebra A is a right alternative color algebra if for all elements x, y, z of A

$$(x, y, z) = -\varepsilon(\bar{z}, \bar{y})(x, z, y).$$

If, moreover, left alternative color algebra

$$(x, y, z) = -\varepsilon(\bar{y}, \bar{x})(y, x, z)$$

holds in A, then A is said to be color alternative.

The trilinear function $(x, y, z) + (y, z, x) + (z, x, y)$ is shown to be very useful in the study of nonassociative algebras. Here we define its color version as

$$S(x, y, z) := (x, y, z) + \varepsilon(\bar{y} + \bar{z}, \bar{x})(y, z, x) + \varepsilon(\bar{z}, \bar{x} + \bar{y})(z, x, y).$$

In order to facilitate the calculations, we define the two following functions:

$$f(w, x, y, z) \coloneqq (wx, y, z) - (w, xy, z) + (w, x, yz) - w(x, y, z) - (w, x, y)z$$

$$g(x, w, y, z) \coloneqq \varepsilon(\bar{w}, \bar{y} + \bar{z})(x, w, yz) + \varepsilon(\bar{w}, \bar{z})(x, y, wz) - \varepsilon(\bar{w}, \bar{z})\varepsilon(\bar{w}, \bar{y})\varepsilon(\bar{z}, \bar{y})(x, w, z)y - (x, y, z)w.$$

We shall show presently that all two functions are identically zero.

3. Fundamental Identities

Proposition 3.1. Let A be a color algebra. Then for any elements w, x, y, z in A,

$$f(w,x,y,z) = 0.$$

Proof. Let w, x, y, z be any elements of the color algebra A. Then by the direct expresion of associators in $f(w,x,y,z)$ we have

$f(w,x,y,z) = (wx,y,z) - (w,xy,z) + (w,x,yz) - w(x,y,z) - (w,x,y)z$
$= ((wx)y)z - (wx)(yz) - (w(xy))z + w((xy)z) + (wx)(yz) - w(x(yz)) - w((xy)z + w(x(yz))$

$-((wx)y)z + (w(xy))z$
$= 0.$

Proposition 3.2. Let A be a color algebra. Then for any elements x, y, z in A,

$$[xy,z] - x[y,z] - \varepsilon(\bar{z},\bar{y})[x,z]y = (x,y,z) - \varepsilon(\bar{z},\bar{y})(x,z,y) + \varepsilon(\bar{z},\bar{x}+\bar{y})(z,x,y).$$

Proof. Let x, y, z be any elements of the color algebra A. Then we have

$[xy,z] - x[y,z] - \varepsilon(\bar{z},\bar{y})[x,z]y$
$= (xy)z - \varepsilon(\bar{z},\overline{xy})z(xy) - x(yz - \varepsilon(\bar{z},\bar{y})zy) - \varepsilon(\bar{z},\bar{y})(xz - \varepsilon(\bar{z},\bar{x})zx)y$
$= (xy)z - \varepsilon(\bar{z},\bar{x}+\bar{y})z(xy) - x(yz - \varepsilon(\bar{z},\bar{y})zy) - \varepsilon(\bar{z},\bar{y})(xz - \varepsilon(\bar{z},\bar{x})zx)y$ (because $\overline{xy} = \bar{x}+\bar{y}$)

$= (xy)z - \varepsilon(\bar{z},\bar{x}+\bar{y})z(xy) - x(yz) + \varepsilon(\bar{z},\bar{y})x(zy) - \varepsilon(\bar{z},\bar{y})(xz)y + \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{z},\bar{x})(zx)y$
$= (xy)z - \varepsilon(\bar{z},\bar{x}+\bar{y})z(xy) - x(yz) + \varepsilon(\bar{z},\bar{y})x(zy) - \varepsilon(\bar{z},\bar{y})(xz)y + \varepsilon(\bar{z},\bar{x}+\bar{y})(zx)y$
$= (xy)z - x(yz) - \varepsilon(\bar{z},\bar{y})[(xz)y - x(zy)] + \varepsilon(\bar{z},\bar{x}+\bar{y})[(zx)y - z(xy)]$
$= (x,y,z) - \varepsilon(\bar{z},\bar{y})(x,z,y) + \varepsilon(\bar{z},\bar{x}+\bar{y})(z,x,y).$

Proposition 3.3. Let A be a color algebra. Then for any x, y, z in A,

$$[xy,z] - [x,y]z + \varepsilon(\bar{z},\bar{y})[xz,y] - \varepsilon(\bar{z},\bar{y})[x,z]y = \varepsilon(\bar{y},\bar{x})(y,x,z) + \varepsilon(\bar{z},\bar{x}+\bar{y})(z,x,y).$$

Proof. Let x, y, z be any elements of the color algebra A. Then we have

$[xy,z] - [x,y]z + \varepsilon(\bar{z},\bar{y})[xz,y] - \varepsilon(\bar{z},\bar{y})[x,z]y$
$= (xy)z - \varepsilon(\bar{z},\overline{xy})z(xy) - (xy - \varepsilon(\bar{y},\bar{x})yx)z + \varepsilon(\bar{z},\bar{y})[(xz)y - \varepsilon(\bar{y},\overline{xz})y(xz)]$
$\quad - \varepsilon(\bar{z},\bar{y})(xz - \varepsilon(\bar{z},\bar{x})zx)y$
$= (xy)z - \varepsilon(\bar{z},\bar{x}+\bar{y})z(xy) - (xy)z + \varepsilon(\bar{y},\bar{x})(yx)z + \varepsilon(\bar{z},\bar{y})(xz)y - \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y},\bar{x}+\bar{z})y(xz)$
$\quad - \varepsilon(\bar{z},\bar{y})(xz)y + \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{z},\bar{x})(zx)y$
$=- \varepsilon(\bar{z},\bar{x}+\bar{y})z(xy) + \varepsilon(\bar{y},\bar{x})(yx)z + \varepsilon(\bar{z},\bar{y})(xz)y - \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y},\bar{x})\varepsilon(\bar{y},\bar{z})y(xz) - \varepsilon(\bar{z},\bar{y})(xz)y$
$\quad + \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{z},\bar{x})(zx)y$
$=- \varepsilon(\bar{z},\bar{x}+\bar{y})z(xy) + \varepsilon(\bar{y},\bar{x})(yx)z - \varepsilon(\bar{y},\bar{x})y(xz) + \varepsilon(\bar{z},\bar{x}+\bar{y})(zx)y$
$= \varepsilon(\bar{y},\bar{x})[(yx)z - y(xz)] + \varepsilon(\bar{z},\bar{x}+\bar{y})[(zx)y - z(xy)]$
$= \varepsilon(\bar{y},\bar{x})(y,x,z) + \varepsilon(\bar{z},\bar{x}+\bar{y})(z,x,y).$

Proposition 3.4. Let A be a color algebra. Then for any x, y, z in A,

$$[xy, z] + \varepsilon(\bar{y} + \bar{z}, \bar{x})[yz, x] + \varepsilon(\bar{z}, \bar{x} + \bar{y})[zx, y] = S(x, y, z).$$

Proof. Let x, y, z be any elements of the color algebra A. Then we have

$S(x,y,z) = (x,y,z) + \varepsilon(\bar{y}+\bar{z},\bar{x})(y,z,x) + \varepsilon(\bar{z},\bar{x}+\bar{y})(z,x,y)$
$= (xy)z - x(yz) + \varepsilon(\bar{y}+\bar{z},\bar{x})(yz)x - \varepsilon(\bar{y}+\bar{z},\bar{x})y(zx) + \varepsilon(\bar{z},\bar{x}+\bar{y})(zx)y - \varepsilon(\bar{z},\bar{x}+\bar{y})z(xy)$
$= (xy)z - \varepsilon(\bar{z},\bar{x}+\bar{y})z(xy) + \varepsilon(\bar{y}+\bar{z},\bar{x})(yz)x - x(yz) + \varepsilon(\bar{z},\bar{x}+\bar{y})(zx)y - \varepsilon(\bar{y}+\bar{z},\bar{x})y(zx)$
$= (xy)z - \varepsilon(\bar{z},\overline{xy})z(xy) + \varepsilon(\bar{y}+\bar{z},\bar{x})(yz)x - \varepsilon(\bar{y}+\bar{z},\bar{x})\varepsilon(\bar{x},\bar{y}+\bar{z})x(yz) + \varepsilon(\bar{z},\bar{x}+\bar{y})(zx)y$
$\quad - \varepsilon(\bar{z},\bar{x}+\bar{y})\varepsilon(\bar{x}+\bar{y},\bar{z})\varepsilon(\bar{y}+\bar{z},\bar{x})y(zx)$
$= [xy,z] + \varepsilon(\bar{y}+\bar{z},\bar{x})[(yz)x - \varepsilon(\bar{x},\bar{y}+\bar{z})x(yz)]$

$\quad +\varepsilon(\bar{z},\bar{x}+\bar{y})[(zx)y - \varepsilon(\bar{x}+\bar{y},\bar{z})\varepsilon(\bar{y}+\bar{z},\bar{x})y(zx)]$

$= [xy,z] + \varepsilon(\bar{y}+\bar{z},\bar{x})[(yz)x - \varepsilon(\bar{x},\overline{yz})x(yz)]$
$\quad + \varepsilon(\bar{z},\bar{x}+\bar{y})[(zx)y - \varepsilon(\bar{x},\bar{z})\varepsilon(\bar{y},\bar{z})\varepsilon(\bar{y},\bar{x})\varepsilon(\bar{z},\bar{x})y(zx)]$
$= [xy,z] + \varepsilon(\bar{y}+\bar{z},\bar{x})[yz,x] + \varepsilon(\bar{z},\bar{x}+\bar{y})[(zx)y - \varepsilon(\bar{y},\bar{z})\varepsilon(\bar{y},\bar{x})y(zx)]$
$= [xy,z] + \varepsilon(\bar{y}+\bar{z},\bar{x})[yz,x] + \varepsilon(\bar{z},\bar{x}+\bar{y})[(zx)y - \varepsilon(\bar{y},\bar{z}+\bar{x})y(zx)]$
$= [xy,z] + \varepsilon(\bar{y}+\bar{z},\bar{x})[yz,x] + \varepsilon(\bar{z},\bar{x}+\bar{y})[(zx)y - \varepsilon(\bar{y},\overline{zx})y(zx)]$
$= [xy,z] + \varepsilon(\bar{y}+\bar{z},\bar{x})[yz,x] + \varepsilon(\bar{z},\bar{x}+\bar{y})[zx,y].$

Lemma 3.5. Let A be a color algebra. Then for any x, y, z in A,

$$x(yz) - \varepsilon(\bar{y}+\bar{z},\bar{x})[y,z]x - \varepsilon(\bar{z},\bar{y})(xz)y = [x,[y,z]] - \varepsilon(\bar{z},\bar{y})(x,z,y).$$

Proof. Let x, y, z be any elements of the color algebra A. Then we have

$[x,[y,z]] - \varepsilon(\bar{z},\bar{y})(x,z,y)$

$= x(yz - \varepsilon(\bar{z},\bar{y})zy) - \varepsilon(\bar{y}+\bar{z},\bar{x})[y,z]x - \varepsilon(\bar{z},\bar{y})(xz)y - \varepsilon(\bar{z},\bar{y})x(zy)$
$= x(yz) - \varepsilon(\bar{z},\bar{y})x(zy) - \varepsilon(\bar{y}+\bar{z},\bar{x})[y,z]x - \varepsilon(\bar{z},\bar{y})(xz)y + \varepsilon(\bar{z},\bar{y})x(zy)$
$= x(yz) - \varepsilon(\bar{y}+\bar{z},\bar{x})[y,z]x - \varepsilon(\bar{z},\bar{y})(xz)y.$

Theorem 3.6. Let A be a color algebra. Then for any x, y, z in A,

$$(x \circ y) \circ z - \varepsilon(\bar{z},\bar{y})(x \circ z) \circ y = S(x,y,z) - 2\varepsilon(\bar{z},\bar{x}+\bar{y})(z,x,y) + 2\varepsilon(\bar{y},\bar{x})(y,x,z) - \varepsilon(\bar{y},\bar{x})S(y,x,z) + [x,[y,z]]$$

Proof. Let x, y, z be any elements of the color algebra A. Then we have

$(x \circ y) \circ z - \varepsilon(\bar{z},\bar{y})(x \circ z) \circ y$

$= (xy + \varepsilon(\bar{y},\bar{x})yx) \circ z - \varepsilon(\bar{z},\bar{y})(xz + \varepsilon(\bar{z},\bar{x})zx) \circ y$
$= (xy + \varepsilon(\bar{y},\bar{x})yx)z + \varepsilon(\bar{z},\bar{x}+\bar{y})z(xy + \varepsilon(\bar{y},\bar{x})yx) - \varepsilon(\bar{z},\bar{y})(xz + \varepsilon(\bar{z},\bar{x})zx)y$
$\quad -\varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y},\bar{x}+\bar{z})y(xz + \varepsilon(\bar{z},\bar{x})zx)$

$= (xy)z + \varepsilon(\bar{y},\bar{x})(yx)z + \varepsilon(\bar{z},\bar{x}+\bar{y})z(xy) + \varepsilon(\bar{z},\bar{x})\varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y},\bar{x})z(yx) - \varepsilon(\bar{z},\bar{y})(xz)y$
$\quad - \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{z},\bar{x})(zx)y - \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y},\bar{x})\varepsilon(\bar{y},\bar{z})y(xz) - \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y},\bar{x})\varepsilon(\bar{y},\bar{z})\varepsilon(\bar{z},\bar{x})y(zx)$
$= (xy)z + \varepsilon(\bar{y},\bar{x})(yx)z + \varepsilon(\bar{z},\bar{x}+\bar{y})z(xy) + \varepsilon(\bar{z},\bar{x})\varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y},\bar{x})z(yx) - \varepsilon(\bar{z},\bar{y})(xz)y$

$$- \varepsilon(\bar{z}, \bar{x} + \bar{y})(zx)y - \varepsilon(\bar{y}, \bar{x})y(xz) - \varepsilon(\bar{y}, \bar{x})\varepsilon(\bar{z}, \bar{x})y(zx)$$
$$= (xy)z + \varepsilon(\bar{z}, \bar{x})\varepsilon(\bar{z}, \bar{y})\varepsilon(\bar{y}, \bar{x})z(yx) - \varepsilon(\bar{z}, \bar{y})(xz)y - \varepsilon(\bar{y} + \bar{z}, \bar{x})y(zx)$$

$$- \varepsilon(\bar{z}, \bar{x} + \bar{y})[(zx)y - z(xy)] + \varepsilon(\bar{y}, \bar{x})[(yx)z - y(xz)]$$
$$= (xy)z + \varepsilon(\bar{z}, \bar{x})\varepsilon(\bar{z}, \bar{y})\varepsilon(\bar{y}, \bar{x})z(yx) - \varepsilon(\bar{z}, \bar{y})(xz)y - \varepsilon(\bar{y} + \bar{z}, \bar{x})y(zx) - \varepsilon(\bar{z}, \bar{x} + \bar{y})(z, x, y)$$

$$+ \varepsilon(\bar{y}, \bar{x})(y, x, z)$$
$$= (xy)z - x(yz) + x(yz) + \varepsilon(\bar{z}, \bar{x})\varepsilon(\bar{z}, \bar{y})\varepsilon(\bar{y}, \bar{x})z(yx) - \varepsilon(\bar{z}, \bar{x})\varepsilon(\bar{z}, \bar{y})\varepsilon(\bar{y}, \bar{x})(zy)x$$

$$+ \varepsilon(\bar{z}, \bar{x})\varepsilon(\bar{z}, \bar{y})\varepsilon(\bar{y}, \bar{x})(zy)x - \varepsilon(\bar{z}, \bar{y})(xz)y - \varepsilon(\bar{y} + \bar{z}, \bar{x})y(zx) + \varepsilon(\bar{y} + \bar{z}, \bar{x})(yz)x$$

$$- \varepsilon(\bar{y} + \bar{z}, \bar{x})(yz)x - \varepsilon(\bar{z}, \bar{x} + \bar{y})(z, x, y) + \varepsilon(\bar{y}, \bar{x})(y, x, z)$$
$$= (x, y, z) + x(yz) - \varepsilon(\bar{z}, \bar{x})\varepsilon(\bar{z}, \bar{y})\varepsilon(\bar{y}, \bar{x})[(zy)x - z(yx)] + \varepsilon(\bar{y} + \bar{z}, \bar{x})[(yz)x - y(zx)]$$

$$- \varepsilon(\bar{z}, \bar{x} + \bar{y})(z, x, y) + \varepsilon(\bar{y}, \bar{x})(y, x, z) - \varepsilon(\bar{y} + \bar{z}, \bar{x})(yz)x + \varepsilon(\bar{z}, \bar{x})\varepsilon(\bar{z}, \bar{y})\varepsilon(\bar{y}, \bar{x})(zy)x$$

$$- \varepsilon(\bar{z}, \bar{y})(xz)y$$
$$= (x, y, z) + x(yz) - \varepsilon(\bar{z}, \bar{x})\varepsilon(\bar{z}, \bar{y})\varepsilon(\bar{y}, \bar{x})(z, y, x) + \varepsilon(\bar{y} + \bar{z}, \bar{x})(y, z, x) - \varepsilon(\bar{z}, \bar{x} + \bar{y})(z, x, y)$$
$$+ \varepsilon(\bar{y}, \bar{x})(y, x, z) - \varepsilon(\bar{y} + \bar{z}, \bar{x})(yz)x + \varepsilon(\bar{y} + \bar{z}, \bar{x})\varepsilon(\bar{z}, \bar{y})(zy)x - \varepsilon(\bar{z}, \bar{y})(xz)y$$
$$= (x, y, z) + x(yz) - \varepsilon(\bar{z}, \bar{x})\varepsilon(\bar{z}, \bar{y})\varepsilon(\bar{y}, \bar{x})(z, y, x) + \varepsilon(\bar{y} + \bar{z}, \bar{x})(y, z, x) - \varepsilon(\bar{z}, \bar{x} + \bar{y})(z, x, y)$$
$$+ \varepsilon(\bar{y}, \bar{x})(y, x, z) - \varepsilon(\bar{y} + \bar{z}, \bar{x})(yz - \varepsilon(\bar{z}, \bar{y})zy)x - \varepsilon(\bar{z}, \bar{y})(xz)y$$
$$= (x, y, z) + x(yz) - \varepsilon(\bar{z}, \bar{x})\varepsilon(\bar{z}, \bar{y})\varepsilon(\bar{y}, \bar{x})(z, y, x) + \varepsilon(\bar{y} + \bar{z}, \bar{x})(y, z, x) - \varepsilon(\bar{z}, \bar{x} + \bar{y})(z, x, y)$$
$$+ \varepsilon(\bar{y}, \bar{x})(y, x, z) - \varepsilon(\bar{y} + \bar{z}, \bar{x})(yz - \varepsilon(\bar{z}, \bar{y})zy)x - \varepsilon(\bar{z}, \bar{y})(xz)y$$
$$= (x, y, z) - \varepsilon(\bar{z}, \bar{x})\varepsilon(\bar{z}, \bar{y})\varepsilon(\bar{y}, \bar{x})(z, y, x) + \varepsilon(\bar{y} + \bar{z}, \bar{x})(y, z, x) - \varepsilon(\bar{z}, \bar{x} + \bar{y})(z, x, y)$$
$$+ \varepsilon(\bar{y}, \bar{x})(y, x, z) - \varepsilon(\bar{y} + \bar{z}, \bar{x})[y, z]x - \varepsilon(\bar{z}, \bar{y})(xz)y + x(yz)$$

According to the Lemma 3.5 we have:

$$x(yz) - \varepsilon(\bar{y} + \bar{z}, \bar{x})[y, z]x - \varepsilon(\bar{z}, \bar{y})(xz)y = \big[x, [y, z]\big] - \varepsilon(\bar{z}, \bar{y})(x, z, y).$$

Therefore

$$(x \circ y) \circ z - \varepsilon(\bar{z}, \bar{y})(x \circ z) \circ y$$
$$= (x, y, z) - \varepsilon(\bar{z}, \bar{x})\varepsilon(\bar{z}, \bar{y})\varepsilon(\bar{y}, \bar{x})(z, y, x) + \varepsilon(\bar{y} + \bar{z}, \bar{x})(y, z, x) - \varepsilon(\bar{z}, \bar{x} + \bar{y})(z, x, y)$$
$$+\varepsilon(\bar{y}, \bar{x})(y, x, z) + \big[x, [y, z]\big] - \varepsilon(\bar{z}, \bar{y})(x, z, y)$$
$$= (x, y, z) + \varepsilon(\bar{y} + \bar{z}, \bar{x})(y, z, x) + \varepsilon(\bar{z}, \bar{x} + \bar{y})(z, x, y) - 2\varepsilon(\bar{z}, \bar{x} + \bar{y})(z, x, y) + 2\varepsilon(\bar{y}, \bar{x})(y, x, z)$$
$$-\varepsilon(\bar{y}, \bar{x})(y, x, z) - \varepsilon(\bar{z}, \bar{y})(x, z, y) - \varepsilon(\bar{z}, \bar{x})\varepsilon(\bar{z}, \bar{y})\varepsilon(\bar{y}, \bar{x})(z, y, x) + \big[x, [y, z]\big]$$
$$= S(x, y, z) - 2\varepsilon(\bar{z}, \bar{x} + \bar{y})(z, x, y) + 2\varepsilon(\bar{y}, \bar{x})(y, x, z) - \varepsilon(\bar{y}, \bar{x})(y, x, z)$$

$$-\varepsilon(\bar{x}, \bar{y})\varepsilon(\bar{y}, \bar{x})\varepsilon(\bar{z}, \bar{y})(x, z, y) - \varepsilon(\bar{z}, \bar{x})\varepsilon(\bar{z}, \bar{y})\varepsilon(\bar{y}, \bar{x})(z, y, x) + [x, [y, z]]$$
$$= S(x, y, z) - 2\varepsilon(\bar{z}, \bar{x} + \bar{y})(z, x, y) + 2\varepsilon(\bar{y}, \bar{x})(y, x, z) - \varepsilon(\bar{y}, \bar{x})(y, x, z) - \varepsilon(\bar{y}, \bar{x})\varepsilon(\bar{x} + \bar{z}, \bar{y})(x, z, y)$$
$$-\varepsilon(\bar{z}, \bar{x} + \bar{y})\varepsilon(\bar{y}, \bar{x})(z, y, x) + [x, [y, z]]$$
$$= S(x, y, z) - 2\varepsilon(\bar{z}, \bar{x} + \bar{y})(z, x, y) + 2\varepsilon(\bar{y}, \bar{x})(y, x, z)$$
$$-\varepsilon(\bar{y}, \bar{x})[(y, x, z) + \varepsilon(\bar{x} + \bar{z}, \bar{y})(x, z, y) + \varepsilon(\bar{z}, \bar{x} + \bar{y})(z, y, x)] + [x, [y, z]]$$
$$= S(x, y, z) - 2\varepsilon(\bar{z}, \bar{x} + \bar{y})(z, x, y) + 2\varepsilon(\bar{y}, \bar{x})(y, x, z) - \varepsilon(\bar{y}, \bar{x})S(y, x, z) + [x, [y, z]].$$

Thus

$$(x \circ y) \circ z - \varepsilon(\bar{z}, \bar{y})(x \circ z) \circ y = S(x, y, z) - 2\varepsilon(\bar{z}, \bar{x} + \bar{y})(z, x, y) + 2\varepsilon(\bar{y}, \bar{x})(y, x, z)$$

$-\varepsilon(\bar{y}, \bar{x})S(y, x, z) + [x, [y, z]]$ for all x, y, z in A.

Corollary 3.7. Let A be a right alternative color algebra. Then for any x, y, z in A,

$$(x \circ y) \circ z - \varepsilon(\bar{z}, \bar{y})(x \circ z) \circ y = 2(x, y, z) + [x, [y, z]].$$

Proof. Let x, y, z be any elements of the right alternative color algebra A. Then we have

$$\begin{aligned}
&(x \circ y) \circ z - \varepsilon(\bar{z}, \bar{y})(x \circ z) \circ y \\
&= S(x, y, z) - \varepsilon(\bar{y}, \bar{x})S(y, x, z) - 2\varepsilon(\bar{z}, \bar{x} + \bar{y})(z, x, y) + 2\varepsilon(\bar{y}, \bar{x})(y, x, z) + [x, [y, z]] \\
&= (x, y, z) + \varepsilon(\bar{y} + \bar{z}, \bar{x})(y, z, x) + \varepsilon(\bar{z}, \bar{x} + \bar{y})(z, x, y) - \varepsilon(\bar{y}, \bar{x})(y, x, z) - \varepsilon(\bar{y}, \bar{x})\varepsilon(\bar{x} + \bar{z}, \bar{y})(x, z, y) \\
&- \varepsilon(\bar{y}, \bar{x})\varepsilon(\bar{z}, \bar{y} + \bar{x})(z, y, x) - 2\varepsilon(\bar{z}, \bar{x} + \bar{y})(z, x, y) + 2\varepsilon(\bar{y}, \bar{x})(y, x, z) + [x, [y, z]] \\
&= (x, y, z) + \varepsilon(\bar{y} + \bar{z}, \bar{x})(y, z, x) + \varepsilon(\bar{z}, \bar{x} + \bar{y})(z, x, y) + \varepsilon(\bar{y}, \bar{x})\varepsilon(\bar{z}, \bar{x})(y, z, x) \\
&- \varepsilon(\bar{y}, \bar{x})\varepsilon(\bar{x}, \bar{y})\varepsilon(\bar{z}, \bar{y})(x, z, y) + \varepsilon(\bar{y}, \bar{x})\varepsilon(\bar{z}, \bar{y} + \bar{x})\varepsilon(\bar{x}, \bar{y})(z, x, y) - 2\varepsilon(\bar{z}, \bar{x} + \bar{y})(z, x, y) \\
&+ 2\varepsilon(\bar{y}, \bar{x})(y, x, z) + [x, [y, z]] \\
&= (x, y, z) + \varepsilon(\bar{y} + \bar{z}, \bar{x})(y, z, x) + \varepsilon(\bar{z}, \bar{x} + \bar{y})(z, x, y) + \varepsilon(\bar{y} + \bar{z}, \bar{x})(y, z, x) - \varepsilon(\bar{z}, \bar{y})(x, z, y) \\
&+ \varepsilon(\bar{z}, \bar{y} + \bar{x})(z, x, y) - 2\varepsilon(\bar{z}, \bar{x} + \bar{y})(z, x, y) - 2\varepsilon(\bar{y}, \bar{x})\varepsilon(\bar{z}, \bar{x})(y, z, x) + [x, [y, z]] \\
&= (x, y, z) + 2\varepsilon(\bar{y} + \bar{z}, \bar{x})(y, z, x) + \varepsilon(\bar{z}, \bar{y})\varepsilon(\bar{y}, \bar{z})(x, y, z) - 2\varepsilon(\bar{y} + \bar{z}, \bar{x})(y, z, x) + [x, [y, z]] \\
&= (x, y, z) + (x, y, z) + [x, [y, z]] \\
&= 2(x, y, z) + [x, [y, z]].
\end{aligned}$$

Proposition 3.8. Let A be a right alternative color algebra. Then for any elements x, y, z in A,

$$[x, y]z - x[y, z] - \varepsilon(\bar{z}, \bar{y})[xz, y] = 2(x, y, z) + \varepsilon(\bar{y} + \bar{z}, \bar{x})(y, z, x).$$

Proof. Let x, y, z be any elements of the right alternative color algebra A. We have:

(1) $[xy, z] - x[y, z] - \varepsilon(\bar{z}, \bar{y})[x, z]y = (x, y, z) - \varepsilon(\bar{z}, \bar{y})(x, z, y) + \varepsilon(\bar{z}, \bar{x} + \bar{y})(z, x, y)$,

(2) $[xy, z] - [x, y]z + \varepsilon(\bar{z}, \bar{y})[xz, y] - \varepsilon(\bar{z}, \bar{y})[x, z]y = \varepsilon(\bar{y}, \bar{x})(y, x, z) + \varepsilon(\bar{z}, \bar{x} + \bar{y})(z, x, y)$.

Subtracting member wise (1) from (2) and next using the right color alternativity, we have

$$\begin{aligned}
[x, y]z - x[y, z] - \varepsilon(\bar{z}, \bar{y})[xz, y] &= (x, y, z) - \varepsilon(\bar{z}, \bar{y})(x, z, y) + \varepsilon(\bar{y}, \bar{x})(y, x, z) \\
&= (x, y, z) + \varepsilon(\bar{z}, \bar{y})\varepsilon(\bar{y}, \bar{z})(x, y, z) - \varepsilon(\bar{y}, \bar{x})\varepsilon(\bar{z}, \bar{x})(y, z, x) \\
&= (x, y, z) + (x, y, z) - \varepsilon(\bar{y} + \bar{z}, \bar{x})(y, z, x) \\
&= 2(x, y, z) - \varepsilon(\bar{y} + \bar{z}, \bar{x})(y, z, x).
\end{aligned}$$

Thus $[x, y]z - x[y, z] - \varepsilon(\bar{z}, \bar{y})[xz, y] = 2(x, y, z) + \varepsilon(\bar{y} + \bar{z}, \bar{x})(y, z, x)$ for all x, y, z in A.

Remark 3.9. If A is color alternative, then $[x, y]z - x[y, z] - \varepsilon(\bar{z}, \bar{y})[xz, y] = 3(x, y, z)$ for all x, y, z in A. Indeed,

$$
\begin{aligned}
[x,y]z - x[y,z] - \varepsilon(\bar{z},\bar{y})[xz,y] &= 2(x,y,z) + \varepsilon(\bar{y}+\bar{z},\bar{x})(y,z,x)\\
&= 2(x,y,z) - \varepsilon(\bar{y}+\bar{z},\bar{x})\varepsilon(\bar{x},\bar{z})(y,x,z)\\
&= 2(x,y,z) + \varepsilon(\bar{y}+\bar{z},\bar{x})\varepsilon(\bar{x},\bar{z})\varepsilon(\bar{x},\bar{y})(x,y,z)\\
&= 2(x,y,z) + \varepsilon(\bar{y}+\bar{z},\bar{x})\varepsilon(\bar{x},\bar{y}+\bar{z})(x,y,z)\\
&= 2(x,y,z) + (x,y,z)\\
&= 3(x,y,z)
\end{aligned}
$$

Proposition 3.10. Let A be a right alternative color algebra. Then for any elements x, y, z in A,

$$S(x,y,z) + \varepsilon(\bar{z},\bar{y})S(x,z,y) = 0.$$

Proof. Let x, y, z be any elements of the right alternative color algebra A. Then we have by definition

$$S(x,y,z) := (x,y,z) + \varepsilon(\bar{y}+\bar{z},\bar{x})(y,z,x) + \varepsilon(\bar{z},\bar{x}+\bar{y})(z,x,y).$$

And

$$
\begin{aligned}
&\varepsilon(\bar{z},\bar{y})S(x,z,y)\\
&= \varepsilon(\bar{z},\bar{y})(x,z,y) + \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y}+\bar{z},\bar{x})(z,y,x) + \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y},\bar{x}+\bar{z})(y,x,z)\\
&=- \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y},\bar{z})(x,y,z) - \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y}+\bar{z},\bar{x})\varepsilon(\bar{x},\bar{y})(z,x,y) - \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y},\bar{x}+\bar{z})\varepsilon(\bar{z},\bar{x})(y,z,x)\\
&=- (x,y,z) - \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y},\bar{x})\varepsilon(\bar{z},\bar{x})\varepsilon(\bar{x},\bar{y})(z,x,y) - \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y},\bar{x})\varepsilon(\bar{y},\bar{z})\varepsilon(\bar{z},\bar{x})(y,z,x)\\
&=- (x,y,z) - \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{z},\bar{x})(z,x,y) - \varepsilon(\bar{y},\bar{x})\varepsilon(\bar{z},\bar{x})(y,z,x)\\
&=- (x,y,z) - \varepsilon(\bar{z},\bar{x}+\bar{y})(z,x,y) - \varepsilon(\bar{y}+\bar{z},\bar{x})(y,z,x).
\end{aligned}
$$

Therefore

$$S(x,y,z) + \varepsilon(\bar{z},\bar{y})S(x,z,y) = 0.$$

Proposition 3.11. Let A be a right alternative color algebra. Then for any elements x, y, z in A,

$$[x \circ y, z] + \varepsilon(\bar{y}+\bar{z},\bar{x})[y \circ z, x] + \varepsilon(\bar{z},\bar{x}+\bar{y})[z \circ x, y] = 0.$$

Proof. Let x, y, z be any elements of the right alternative color algebra A. We have

$$S(x,y,z) = [xy,z] + \varepsilon(\bar{y}+\bar{z},\bar{x})[yz,x] + \varepsilon(\bar{z},\bar{x}+\bar{y})[zx,y]$$

and

$$S(x,z,y) = [xz,y] + \varepsilon(\bar{z}+\bar{y},\bar{x})[zy,x] + \varepsilon(\bar{y},\bar{x}+\bar{z})[yx,z].$$

As

$$S(x,y,z) + \varepsilon(\bar{z},\bar{y})S(x,z,y) = 0$$

therefore

$$
\begin{aligned}
0 = {}& [xy,z] + \varepsilon(\bar{y}+\bar{z},\bar{x})[yz,x] + \varepsilon(\bar{z},\bar{x}+\bar{y})[zx,y] + \varepsilon(\bar{z},\bar{y})[xz,y] + \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{z}+\bar{y},\bar{x})[zy,x]\\
&+ \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y},\bar{x}+\bar{z})[yx,z]
\end{aligned}
$$

$$= (xy)z - \varepsilon(\bar{z},\overline{xy})z(xy) + \varepsilon(\bar{y}+\bar{z},\bar{x})(yz)x - \varepsilon(\bar{y}+\bar{z},\bar{x})\varepsilon(\bar{x},\overline{yz})x(yz) + \varepsilon(\bar{z},\bar{x}+\bar{y})(zx)y$$
$$- \varepsilon(\bar{z},\bar{x}+\bar{y})\varepsilon(\bar{y},\overline{zx})y(zx) + \varepsilon(\bar{z},\bar{y})(xz)y - \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y},\overline{xz})y(xz) + \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{z}+\bar{y},\bar{x})(zy)x$$
$$- \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{z}+\bar{y},\bar{x})\varepsilon(\bar{x},\overline{zy})x(zy) + \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y},\bar{x}+\bar{z})(yx)z - \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y},\bar{x}+\bar{z})\varepsilon(\bar{z},\overline{yx})z(yx)$$
$$= (xy)z - \varepsilon(\bar{z},\bar{x}+\bar{y})z(xy) + \varepsilon(\bar{y}+\bar{z},\bar{x})(yz)x - \varepsilon(\bar{y}+\bar{z},\bar{x})\varepsilon(\bar{x},\bar{y}+\bar{z})x(yz) + \varepsilon(\bar{z},\bar{x}+\bar{y})(zx)y$$
$$- \varepsilon(\bar{z},\bar{x}+\bar{y})\varepsilon(\bar{y},\bar{z}+\bar{x})y(zx) + \varepsilon(\bar{z},\bar{y})(xz)y - \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y},\bar{x}+\bar{z})y(xz) + \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{z}+\bar{y},\bar{x})(zy)x$$
$$-\varepsilon(\bar{z},\bar{y})\varepsilon(\bar{z}+\bar{y},\bar{x})\varepsilon(\bar{x},\bar{z}+\bar{y})x(zy) + \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y},\bar{x}+\bar{z})(yx)z - \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y},\bar{x}+\bar{z})\varepsilon(\bar{z},\bar{y}+\bar{x})z(yx)$$
$$= (xy)z - \varepsilon(\bar{z},\bar{x}+\bar{y})z(xy) + \varepsilon(\bar{y}+\bar{z},\bar{x})(yz)x - x(yz) + \varepsilon(\bar{z},\bar{x}+\bar{y})(zx)y$$
$$- \varepsilon(\bar{z},\bar{x})\varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y},\bar{z})\varepsilon(\bar{y},\bar{x})y(zx) + \varepsilon(\bar{z},\bar{y})(xz)y - \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y},\bar{x})\varepsilon(\bar{y},\bar{z})y(xz)$$
$$+ \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{z}+\bar{y},\bar{x})(zy)x - \varepsilon(\bar{z},\bar{y})x(zy) + \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y},\bar{x})\varepsilon(\bar{y},\bar{z})(yx)z$$
$$- \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y},\bar{x})\varepsilon(\bar{y},\bar{z})\varepsilon(\bar{z},\bar{y}+\bar{x})z(yx)$$
$$= (xy)z - \varepsilon(\bar{z},\bar{x}+\bar{y})z(xy) + \varepsilon(\bar{y}+\bar{z},\bar{x})(yz)x - x(yz) + \varepsilon(\bar{z},\bar{x}+\bar{y})(zx)y - \varepsilon(\bar{z}+\bar{y},\bar{x})y(zx)$$
$$+ \varepsilon(\bar{z},\bar{y})(xz)y - \varepsilon(\bar{y},\bar{x})y(xz) + \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{z}+\bar{y},\bar{x})(zy)x - \varepsilon(\bar{z},\bar{y})x(zy) + \varepsilon(\bar{y},\bar{x})(yx)z$$
$$- \varepsilon(\bar{y},\bar{x})\varepsilon(\bar{z},\bar{y}+\bar{x})z(yx)$$
$$= (xy)z + \varepsilon(\bar{y},\bar{x})(yx)z - \varepsilon(\bar{z},\bar{x}+\bar{y})z(xy) - \varepsilon(\bar{y},\bar{x})\varepsilon(\bar{z},\bar{x}+\bar{y})z(yx) + \varepsilon(\bar{y}+\bar{z},\bar{x})(yz)x$$
$$+ \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{z}+\bar{y},\bar{x})(zy)x - x(yz) - \varepsilon(\bar{z},\bar{y})x(zy) + \varepsilon(\bar{z},\bar{x}+\bar{y})(zx)y + \varepsilon(\bar{z},\bar{y})(xz)y$$
$$- \varepsilon(\bar{z}+\bar{y},\bar{x})y(zx) - \varepsilon(\bar{y},\bar{x})y(xz)$$
$$= (xy + \varepsilon(\bar{y},\bar{x})yx)z - \varepsilon(\bar{z},\bar{x}+\bar{y})z(xy + \varepsilon(\bar{y},\bar{x})yx) + \varepsilon(\bar{y}+\bar{z},\bar{x})(yz + \varepsilon(\bar{z},\bar{y})zy)x$$
$$- \varepsilon(\bar{y}+\bar{z},\bar{x})\varepsilon(\bar{x},\bar{y}+\bar{z})x(yz + \varepsilon(\bar{z},\bar{y})zy) + \varepsilon(\bar{z},\bar{x}+\bar{y})[(zx)y + \varepsilon(\bar{x}+\bar{y},\bar{z})\varepsilon(\bar{z},\bar{y})(xz)y]$$
$$- \varepsilon(\bar{z},\bar{x}+\bar{y})\varepsilon(\bar{y},\bar{z}+\bar{x})[y(zx) + \varepsilon(\bar{x},\bar{z})y(xz)]$$
$$= [xy + \varepsilon(\bar{y},\bar{x})yx, z] + \varepsilon(\bar{y}+\bar{z},\bar{x})[yz + \varepsilon(\bar{z},\bar{y})zy, x] + \varepsilon(\bar{z},\bar{x}+\bar{y})(zx + \varepsilon(\bar{x},\bar{z})xz)y$$
$$- \varepsilon(\bar{z},\bar{x}+\bar{y})\varepsilon(\bar{y},\bar{z}+\bar{x})y(zx + \varepsilon(\bar{x},\bar{z})xz)$$
$$= [x \circ y, z] + \varepsilon(\bar{y}+\bar{z},\bar{x})[y \circ z, x] + \varepsilon(\bar{z},\bar{x}+\bar{y})(z \circ x)y - \varepsilon(\bar{z},\bar{x}+\bar{y})\varepsilon(\bar{y},\bar{z}+\bar{x})y(z \circ x)$$
$$= [x \circ y, z] + \varepsilon(\bar{y}+\bar{z},\bar{x})[y \circ z, x] + \varepsilon(\bar{z},\bar{x}+\bar{y})[z \circ x, y].$$

Thus

$$[x \circ y, z] + \varepsilon(\bar{y}+\bar{z},\bar{x})[y \circ z, x] + \varepsilon(\bar{z},\bar{x}+\bar{y})[z \circ x, y] = 0 \text{ for all } x, y, z \text{ in A.}$$

Lemma 3.12. Let A be a right alternative color algebra. Then for any x, y, z in A,

1. $[xy, z] - [\varepsilon(\bar{y},\bar{x})yx, z] = [[x, y], z]$.
2. $\varepsilon(\bar{z}+\bar{y},\bar{x})[y, z]x - x[y, z] = \varepsilon(\bar{y}+\bar{z},\bar{x})[[y, z], x]$.
3. $-\varepsilon(\bar{z},\bar{y})[x, z]y + \varepsilon(\bar{y},\bar{x})y[x, z] = \varepsilon(\bar{z},\bar{x}+\bar{y})[[z, x], y]$.

Proof. Let x, y, z be any elements of the right alternative color algebra A. By the definition of ε-commutator we have:

1. $[[xy,z]-[\varepsilon(\bar{y},\bar{x})yx,z]$

$$
\begin{aligned}
&= (xy)z-\varepsilon(\bar{z},\overline{xy})z(xy)-\varepsilon(\bar{y},\bar{x})(yx)z+\varepsilon(\bar{y},\bar{x})\varepsilon(\bar{z},\overline{yx})z(yx)\\
&= (xy-\varepsilon(\bar{y},\bar{x})yx)z-\varepsilon(\bar{z},\overline{xy})z(xy-\varepsilon(\bar{y},\bar{x})yx)\\
&= [x,y]z-\varepsilon(\bar{z},\bar{x}+\bar{y})z[x,y]\\
&= [[x,y],z],
\end{aligned}
$$

2. $\varepsilon(\bar{z}+\bar{y},\bar{x})[y,z]x-x[y,z]$

$$
\begin{aligned}
&= \varepsilon(\bar{y}+\bar{z},\bar{x})[y,z]x-\varepsilon(\bar{y}+\bar{z},\bar{x})\varepsilon(\bar{x},\bar{y}+\bar{z})x[y,z]\\
&= \varepsilon(\bar{y}+\bar{z},\bar{x})\{[y,z]x-\varepsilon(\bar{x},\bar{y}+\bar{z})x[y,z]\}\\
&= \varepsilon(\bar{y}+\bar{z},\bar{x})[[y,z],x],
\end{aligned}
$$

3. $\varepsilon(\bar{z},\bar{x}+\bar{y})[[z,x],y]$

$$
\begin{aligned}
&= \varepsilon(\bar{z},\bar{x}+\bar{y})[z,x]y-\varepsilon(\bar{z},\bar{x}+\bar{y})\varepsilon(\bar{y},\bar{z}+\bar{x})y[z,x]\\
&= \varepsilon(\bar{z},\bar{x}+\bar{y})(zx)y-\varepsilon(\bar{z},\bar{x}+\bar{y})\varepsilon(\bar{x},\bar{z})(xz)y-\varepsilon(\bar{z},\bar{x}+\bar{y})\varepsilon(\bar{y},\bar{z}+\bar{x})y[z,x]\\
&= \varepsilon(\bar{z},\bar{x}+\bar{y})(zx)y-\varepsilon(\bar{z},\bar{y})(xz)y-\varepsilon(\bar{z}+\bar{y},\bar{x})y[z,x]\\
&= \varepsilon(\bar{z},\bar{x}+\bar{y})(zx)y-\varepsilon(\bar{z},\bar{y})(xz)y-\varepsilon(\bar{z}+\bar{y},\bar{x})y(zx)+\varepsilon(\bar{z}+\bar{y},\bar{x})\varepsilon(\bar{x},\bar{z})y(xz)\\
&= \varepsilon(\bar{z},\bar{x})\varepsilon(\bar{z},\bar{y})(zx)y-\varepsilon(\bar{z},\bar{y})(xz)y-\varepsilon(\bar{z},\bar{x})\varepsilon(\bar{y},\bar{x})y(zx)+\varepsilon(\bar{y},\bar{x})y(xz)\\
&=-\varepsilon(\bar{z},\bar{y})(xz-\varepsilon(\bar{z},\bar{x})zx)y+\varepsilon(\bar{y},\bar{x})y(xz-\varepsilon(\bar{z},\bar{x})zx)\\
&=-\varepsilon(\bar{z},\bar{y})[x,z]y+\varepsilon(\bar{y},\bar{x})y[x,z].
\end{aligned}
$$

Theorem 3.13. Let A be a right alternative color algebra. Then for any x,y,z in A,

$$[[x,y],z]+\varepsilon(\bar{y}+\bar{z},\bar{x})[[y,z],x]+\varepsilon(\bar{z},\bar{x}+\bar{y})[[z,x],y]=2S(x,y,z).$$

Proof. Let x,y,z be any elements of the right alternative color algebra A. We have

$$[xy,z]-x[y,z]-\varepsilon(\bar{z},\bar{y})[x,z]y=(x,y,z)-\varepsilon(\bar{z},\bar{y})(x,z,y)+\varepsilon(\bar{z},\bar{x}+\bar{y})(z,x,y)$$

Switching x and y and next multiply by $\varepsilon(\bar{y},\bar{x})$ we obtain

$$
\begin{aligned}
[\varepsilon(\bar{y},\bar{x})yx,z]-\varepsilon(\bar{y},\bar{x})y[x,z]-\varepsilon(\bar{z},\bar{x})\varepsilon(\bar{y},\bar{x})[y,z]x &= \varepsilon(\bar{y},\bar{x})(y,x,z)-\varepsilon(\bar{y},\bar{x})\varepsilon(\bar{z},\bar{x})(y,z,x)\\
&\quad+\varepsilon(\bar{y},\bar{x})\varepsilon(\bar{z},\bar{y}+\bar{x})(z,y,x).
\end{aligned}
$$

Now, subtracting member wise the equality above from the first equality, we get

$$
\begin{aligned}
&[xy,z]-x[y,z]-\varepsilon(\bar{z},\bar{y})[x,z]y-[\varepsilon(\bar{y},\bar{x})yx,z]+\varepsilon(\bar{y},\bar{x})y[x,z]+\varepsilon(\bar{z}+\bar{y},\bar{x})[y,z]x\\
&=(x,y,z)-\varepsilon(\bar{z},\bar{y})(x,z,y)+\varepsilon(\bar{z},\bar{x}+\bar{y})(z,x,y)-\varepsilon(\bar{y},\bar{x})(y,x,z)+\varepsilon(\bar{y}+\bar{z},\bar{x})(y,z,x)\\
&\quad-\varepsilon(\bar{y},\bar{x})\varepsilon(\bar{z},\bar{x}+\bar{y})(z,y,x);
\end{aligned}
$$

that is

$$
\begin{aligned}
&\{[xy,z]-[\varepsilon(\bar{y},\bar{x})yx,z]\}+\{\varepsilon(\bar{z}+\bar{y},\bar{x})[y,z]x-x[y,z]\}+\{-\varepsilon(\bar{z},\bar{y})[x,z]y+\varepsilon(\bar{y},\bar{x})y[x,z]\}\\
&=\{(x,y,z)-\varepsilon(\bar{z},\bar{y})(x,z,y)\}+\{-\varepsilon(\bar{y},\bar{x})(y,x,z)+\varepsilon(\bar{y}+\bar{z},\bar{x})(y,z,x)\}\\
&\quad+\{\varepsilon(\bar{z},\bar{x}+\bar{y})(z,x,y)-\varepsilon(\bar{y},\bar{x})\varepsilon(\bar{z},\bar{x}+\bar{y})(z,y,x)\}.
\end{aligned}
$$

According to the Lemma 3.12 we have

$$
\begin{aligned}
&\{[xy,z]-[\varepsilon(\bar{y},\bar{x})yx,z]\}+\{\varepsilon(\bar{z}+\bar{y},\bar{x})[y,z]x-x[y,z]\}+\{-\varepsilon(\bar{z},\bar{y})[x,z]y+\varepsilon(\bar{y},\bar{x})y[x,z]\}\\
&=\big[[x,y],z\big]+\varepsilon(\bar{y}+\bar{z},\bar{x})\big[[y,z],x\big]+\varepsilon(\bar{z},\bar{x}+\bar{y})\big[[z,x],y\big].
\end{aligned}
$$

Therefore

$$\big[[x,y],z\big] + \varepsilon(\bar{y}+\bar{z},\bar{x})\big[[y,z],x\big] + \varepsilon(\bar{z},\bar{x}+\bar{y})\big[[z,x],y\big]$$
$$= \{(x,y,z) - \varepsilon(\bar{z},\bar{y})(x,z,y)\} + \{-\varepsilon(\bar{y},\bar{x})(y,x,z) + \varepsilon(\bar{y}+\bar{z},\bar{x})(y,z,x)\} + \{\varepsilon(\bar{z},\bar{x}+\bar{y})(z,x,y)$$
$$- \varepsilon(\bar{y},\bar{x})\varepsilon(\bar{z},\bar{x}+\bar{y})(z,y,x)\}$$
$$= \{(x,y,z) + \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y},\bar{z})(x,y,z)\} + \{\varepsilon(\bar{y},\bar{x})\varepsilon(\bar{z},\bar{x})(y,z,x) + \varepsilon(\bar{y}+\bar{z},\bar{x})(y,z,x)\}$$
$$+ \{\varepsilon(\bar{z},\bar{x}+\bar{y})(z,x,y) + \varepsilon(\bar{y},\bar{x})\varepsilon(\bar{x},\bar{y})\varepsilon(\bar{z},\bar{x}+\bar{y})(z,x,y)\}$$
$$= \{(x,y,z) + (x,y,z)\} + \{\varepsilon(\bar{y}+\bar{z},\bar{x})(y,z,x) + \varepsilon(\bar{y}+\bar{z},\bar{x})(y,z,x)\} + \{\varepsilon(\bar{z},\bar{x}+\bar{y})(z,x,y)$$
$$+ \varepsilon(\bar{z},\bar{x}+\bar{y})(z,x,y)\}$$
$$= 2\{(x,y,z) + \varepsilon(\bar{y}+\bar{z},\bar{x})(y,z,x) + \varepsilon(\bar{z},\bar{x}+\bar{y})(z,x,y)\}$$
$$= 2S(x,y,z).$$

Of all which precedes we have

$$[[x,y],z] + \varepsilon(\bar{y}+\bar{z},\bar{x})[[y,z],x] + \varepsilon(\bar{z},\bar{x}+\bar{y})[[z,x],y] = 2S(x,y,z) \text{ for all } x,y,z \text{ in A.}$$

Lemma 3.14. Let A be a right alternative color algebra. Then for any w,x,y,z in A,

$$g(x,w,y,z) = 0.$$

Proof. Let w,x,y,z be any elements of the right alternative color algebra A.

We have $f(w,x,y,z) = 0$ that is the function is identically zero. Then

$$
\begin{aligned}
0 &= \varepsilon(\bar{w},\bar{y}+\bar{z})f(x,w,y,z) - \varepsilon(\bar{z},\bar{y})f(x,z,y,w) + \varepsilon(\bar{w},\bar{z})\varepsilon(\bar{w},\bar{y})\varepsilon(\bar{z},\bar{y})f(x,w,z,y) \\
&\quad + \varepsilon(\bar{w},\bar{z})f(x,y,w,z) - \varepsilon(\bar{w},\bar{y})\varepsilon(\bar{z},\bar{y})f(x,z,w,y) + f(x,y,z,w) \\
&= \varepsilon(\bar{w},\bar{y})\varepsilon(\bar{w},\bar{z})\{(xw,y,z) - (x,wy,z) + (x,w,yz) - x(w,y,z) - (x,w,y)z\} \\
&\quad - \varepsilon(\bar{z},\bar{y})\{(xz,y,w) - (x,zy,w) + (x,z,yw) - x(z,y,w) - (x,z,y)w\} \\
&\quad + \varepsilon(\bar{w},\bar{z})\varepsilon(\bar{w},\bar{y})\varepsilon(\bar{z},\bar{y})\{(xw,z,y) - (x,wz,y) + (x,w,zy) - x(w,z,y) - (x,w,z)y\} \\
&\quad + \varepsilon(\bar{w},\bar{z})\{(xy,w,z) - (x,yw,z) + (x,y,wz) - x(y,w,z) - (x,y,w)z\} \\
&\quad - \varepsilon(\bar{w},\bar{y})\varepsilon(\bar{z},\bar{y})\{(xz,w,y) - (x,zw,y) + (x,z,wy) - x(z,w,y) - (x,z,w)y\} \\
&\quad + \{(xy,z,w) - (x,yz,w) + (x,y,zw) - x(y,z,w) - (x,y,z)w\} \\
&= \varepsilon(\bar{w},\bar{y})\varepsilon(\bar{w},\bar{z})\{(xw,y,z) - (x,wy,z) + (x,w,yz) - x(w,y,z) - (x,w,y)z\} \\
&\quad - \varepsilon(\bar{z},\bar{y})\{(xz,y,w) - (x,zy,w) + (x,z,yw) - x(z,y,w) - (x,z,y)w\} \\
&\quad + \varepsilon(\bar{w},\bar{z})\varepsilon(\bar{w},\bar{y})\varepsilon(\bar{z},\bar{y})\{ - \varepsilon(\bar{y},\bar{z})(xw,y,z) - (x,wz,y) - \varepsilon(\bar{z}+\bar{y},\bar{w})(x,zy,w) + \varepsilon(\bar{y},\bar{z})x(w,y,z) \\
&\quad - (x,w,z)y\} \\
&\quad + \varepsilon(\bar{w},\bar{z})\{(xy,w,z) + \varepsilon(\bar{z},\bar{y}+\bar{w})(x,z,yw) + (x,y,wz) - x(y,w,z) + \varepsilon(\bar{w},\bar{y})(x,w,y)z\} \\
&\quad - \varepsilon(\bar{w},\bar{y})\varepsilon(\bar{z},\bar{y})\{ - \varepsilon(\bar{y},\bar{w})(xz,y,w) - (x,zw,y) - \varepsilon(\bar{w}+\bar{y},\bar{z})(x,wy,z) + \varepsilon(\bar{y},\bar{w})x(z,y,w) - (x,z,w)y\} \\
&\quad + \{ - \varepsilon(\bar{w},\bar{z})(xy,w,z) - (x,yz,w) - \varepsilon(\bar{z}+\bar{w},\bar{y})(x,zw,y) + \varepsilon(\bar{w},\bar{z})x(y,w,z) - (x,y,z)w\} \\
&= \varepsilon(\bar{w},\bar{y})\varepsilon(\bar{w},\bar{z})(x,w,yz) + \varepsilon(\bar{z},\bar{y})(x,z,y)w - \varepsilon(\bar{w},\bar{z})\varepsilon(\bar{w},\bar{y})\varepsilon(\bar{z},\bar{y})(x,wz,y) \\
&\quad - \varepsilon(\bar{w},\bar{z})\varepsilon(\bar{w},\bar{y})\varepsilon(\bar{z},\bar{y})(x,w,z)y + \varepsilon(\bar{w},\bar{z})(x,y,wz) + \varepsilon(\bar{w},\bar{y})\varepsilon(\bar{z},\bar{y})(x,z,w)y - (x,yz,w) - (x,y,z)w \\
&= \varepsilon(\bar{w},\bar{y}+\bar{z})(x,w,yz) - \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{y},\bar{z})(x,y,z)w + \varepsilon(\bar{w},\bar{z})\varepsilon(\bar{w}+\bar{z},\bar{y})\varepsilon(\bar{y},\bar{w}+\bar{z})(x,y,wz) \\
&\quad - \varepsilon(\bar{w},\bar{z})\varepsilon(\bar{w},\bar{y})\varepsilon(\bar{z},\bar{y})(x,w,z)y + \varepsilon(\bar{w},\bar{z})(x,y,wz) - \varepsilon(\bar{w},\bar{z})\varepsilon(\bar{w},\bar{y})\varepsilon(\bar{z},\bar{y})(x,w,z)y \\
&\quad + \varepsilon(\bar{w},\bar{y}+\bar{z})(x,w,yz) - (x,y,z)w \\
&= 2\varepsilon(\bar{w},\bar{y}+\bar{z})(x,w,yz) - (x,y,z)w + \varepsilon(\bar{w},\bar{z})(x,y,wz) - 2\varepsilon(\bar{w},\bar{z})\varepsilon(\bar{w},\bar{y})\varepsilon(\bar{z},\bar{y})(x,w,z)y - (x,y,z)w \\
&\quad + \varepsilon(\bar{w},\bar{z})(x,y,wz) \\
&= 2\{\varepsilon(\bar{w},\bar{y}+\bar{z})(x,w,yz) + \varepsilon(\bar{w},\bar{z})(x,y,wz) - \varepsilon(\bar{w},\bar{z})\varepsilon(\bar{w},\bar{y})\varepsilon(\bar{z},\bar{y})(x,w,z)y - (x,y,z)w\} \\
&= 2g(x,w,y,z).
\end{aligned}
$$

And as the ground field $\mathbb{K}$ is of characteristic not 2 we obtain $g(x,w,y,z) = 0$ for all x,w,y,z in A.

We can now prove the following

Theorem 3.15. Let A be a right alternative color algebra. Then for any w,x,y,z in A,

$$(wx,y,z) + (w,x,[y,z]) = \varepsilon(\bar{y}+\bar{z},\bar{x})(w,y,z)x + w(x,y,z).$$

Proof. Let w,x,y,z be any elements of the right alternative color algebra A.

As $f(w,x,y,z) = 0$ and $g(w,z,x,y) = 0$, we have:

$$
\begin{aligned}
0 &= f(w,x,y,z) - g(w,z,x,y) \\
&= (wx,y,z) - (w,xy,z) + (w,x,yz) - w(x,y,z) - (w,x,y)z - \varepsilon(\bar{z},\bar{x}+\bar{y})(w,z,xy) \\
&\quad - \varepsilon(\bar{z},\bar{y})(w,x,zy) + \varepsilon(\bar{z},\bar{y})\varepsilon(\bar{z},\bar{x})\varepsilon(\bar{y},\bar{x})(w,z,y)x + (w,x,y)z \\
&= (wx,y,z) + \varepsilon(\bar{z},\bar{x}+\bar{y})(w,z,xy) + (w,x,yz) - w(x,y,z) - \varepsilon(\bar{z},\bar{x}+\bar{y})(w,z,xy) \\
&\quad - \varepsilon(\bar{z},\bar{y})(w,x,zy) - \varepsilon(\bar{y},\bar{z})\varepsilon(\bar{z},\bar{y})\varepsilon(\bar{z},\bar{x})\varepsilon(\bar{y},\bar{x})(w,y,z)x \\
&= (wx,y,z) + (w,x,yz) - \varepsilon(\bar{z},\bar{y})(w,x,zy) - \varepsilon(\bar{y}+\bar{z},\bar{x})(w,y,z)x - w(x,y,z) \\
&= (wx,y,z) + (w,x,yz) + (w,x,-\varepsilon(\bar{z},\bar{y})zy) - \varepsilon(\bar{y}+\bar{z},\bar{x})(w,y,z)x - w(x,y,z) \\
&= (wx,y,z) + (w,x,yz - \varepsilon(\bar{z},\bar{y})zy) - \varepsilon(\bar{y}+\bar{z},\bar{x})(w,y,z)x - w(x,y,z)
\end{aligned}
$$

$= (wx, y, z) + (w, x, [y, z]) - \varepsilon(\bar{y} + \bar{z}, \bar{x})(w, y, z)x - w(x, y, z)$
Therefore

$$(wx, y, z) + (w, x, [y, z]) = \varepsilon(\bar{y} + \bar{z}, \bar{x})(w, y, z)x + w(x, y, z) \text{ for all } w, x, y, z \text{ in A.}$$

Theorem 3.16. Let A be a right alternative color algebra. Then for any w, x, y, z in A,

$$(x, z, y \circ w) = (x, z \circ y, w) + \varepsilon(\bar{w}, \bar{y})(x, z \circ w, y).$$

Proof. Let w, x, y, z be any elements of the right alternative color algebra A. We have

$$\begin{aligned}
0 &= \varepsilon(\bar{w}, \bar{y})f(x, z, w, y) + f(x, z, y, w) \\
&= \{\varepsilon(\bar{w}, \bar{y})(xz, w, y) - \varepsilon(\bar{w}, \bar{y})(x, zw, y) + \varepsilon(\bar{w}, \bar{y})(x, z, wy) - \varepsilon(\bar{w}, \bar{y})x(z, w, y) - \varepsilon(\bar{w}, \bar{y})(x, z, w)y\} \\
&\quad + \{(xz, y, w) - (x, zy, w) + (x, z, yw) - x(z, y, w) - (x, z, y)w\} \\
&= - \varepsilon(\bar{w}, \bar{y})(x, zw, y) - (x, zy, w) + (x, z, yw) + \varepsilon(\bar{w}, \bar{y})(x, z, wy) + \varepsilon(\bar{w}, \bar{y} + \bar{z})(x, w, z)y \\
&\quad + \varepsilon(\bar{y}, \bar{z})(x, y, z)w + \varepsilon(\bar{w}, \bar{y})(xz, w, y) + (xz, y, w) - \varepsilon(\bar{w}, \bar{y})x(z, w, y) - x(z, y, w) \\
&= - \varepsilon(\bar{w}, \bar{y})(x, zw, y) - (x, zy, w) + (x, z, yw) + \varepsilon(\bar{w}, \bar{y})(x, z, wy) + \varepsilon(\bar{w}, \bar{y} + \bar{z})(x, w, z)y \\
&\quad + \varepsilon(\bar{y}, \bar{z})(x, y, z)w + \varepsilon(\bar{w}, \bar{y})(xz, w, y) - \varepsilon(\bar{w}, \bar{y})(xz, w, y) - \varepsilon(\bar{w}, \bar{y})x(z, w, y) + \varepsilon(\bar{w}, \bar{y})x(z, w, y) \\
&= - \varepsilon(\bar{w}, \bar{y})(x, zw, y) - (x, zy, w) + (x, z, yw) + \varepsilon(\bar{w}, \bar{y})(x, z, wy) + \varepsilon(\bar{w}, \bar{y} + \bar{z})(x, w, z)y \\
&\quad + \varepsilon(\bar{y}, \bar{z})(x, y, z)w.
\end{aligned}$$

According to the Lemma 3.14 we have $g(x, w, y, z) = 0$ that is

$$\varepsilon(\bar{w}, \bar{z})\varepsilon(\bar{w}, \bar{y})\varepsilon(\bar{z}, \bar{y})(x, w, z)y + (x, y, z)w = \varepsilon(\bar{w}, \bar{y} + \bar{z})(x, w, yz) + \varepsilon(\bar{w}, \bar{z})(x, y, wz).$$

Multiplying by $\varepsilon(\bar{y}, \bar{z})$ we obtain

$$\varepsilon(\bar{w}, \bar{y} + \bar{z})(x, w, z)y + \varepsilon(\bar{y}, \bar{z})(x, y, z)w = \varepsilon(\bar{w}, \bar{y} + \bar{z})\varepsilon(\bar{y}, \bar{z})(x, w, yz) + \varepsilon(\bar{w} + \bar{y}, \bar{z})(x, y, wz).$$

Therefore

$$\begin{aligned}
0 &= - \varepsilon(\bar{w}, \bar{y})(x, zw, y) - (x, zy, w) + (x, z, yw) + \varepsilon(\bar{w}, \bar{y})(x, z, wy) + \varepsilon(\bar{w}, \bar{y} + \bar{z})\varepsilon(\bar{y}, \bar{z})(x, w, yz) \\
&\quad + \varepsilon(\bar{w} + \bar{y}, \bar{z})(x, y, wz)
\end{aligned}$$

As $(x, z, yw) + \varepsilon(\bar{w}, \bar{y})(x, z, wy) = (x, z, y \circ w)$ we have

$$\begin{aligned}
0 &= - \varepsilon(\bar{w}, \bar{y})(x, zw, y) - (x, zy, w) + (x, z, y \circ w) - \varepsilon(\bar{w}, \bar{y} + \bar{z})\varepsilon(\bar{y}, \bar{z})\varepsilon(\bar{y} + \bar{z}, \bar{w})(x, yz, w) \\
&\quad - \varepsilon(\bar{w} + \bar{y}, \bar{z})\varepsilon(\bar{w} + \bar{z}, \bar{y})(x, wz, y) \\
&= - \varepsilon(\bar{w}, \bar{y})(x, zw, y) - (x, zy, w) + (x, z, y \circ w) - \varepsilon(\bar{y}, \bar{z})(x, yz, w) \\
&\quad - \varepsilon(\bar{w}, \bar{z})\varepsilon(\bar{y}, \bar{z})\varepsilon(\bar{w}, \bar{y})\varepsilon(\bar{z}, \bar{y})(x, wz, y) \\
&= - \varepsilon(\bar{w}, \bar{y})(x, zw, y) - (x, zy, w) + (x, z, y \circ w) - \varepsilon(\bar{y}, \bar{z})(x, yz, w) - \varepsilon(\bar{w}, \bar{z})\varepsilon(\bar{w}, \bar{y})(x, wz, y) \\
&= (x, z, y \circ w) - (x, zy, w) - \varepsilon(\bar{y}, \bar{z})(x, yz, w) - \varepsilon(\bar{w}, \bar{y})\{(x, zw, y) + \varepsilon(\bar{w}, \bar{z})(x, wz, y)\} \\
&= (x, z, y \circ w) - \{(x, zy, w) + \varepsilon(\bar{y}, \bar{z})(x, yz, w)\} - \varepsilon(\bar{w}, \bar{y})(x, z \circ w, y) \\
&= (x, z, y \circ w) - (x, z \circ y, w) - \varepsilon(\bar{w}, \bar{y})(x, z \circ w, y)
\end{aligned}$$

Thus

$$(x, z, y \circ w) = (x, z \circ y, w) + \varepsilon(\bar{w}, \bar{y})(x, z \circ w, y) \text{ for all } w, x, y, z \text{ in A.}$$

The following identity is proved to be valid in right alternative algebras:

$$(x,z,y\circ w)=2(x,z,w)y-2(x,y,z)w+(x,[z,y],w)+(x,[z,w],y)$$

(See identity (9) in [7]). Its color version is given by the following.

Theorem 3.17. Let A be a right alternative color algebra. Then for any w,x,y,z in A,

$$(x,z,y\circ w)=2\varepsilon(\bar{w},\bar{y})(x,z,w)y-2\varepsilon(\bar{y},\bar{z})(x,y,z)w+(x,[z,y],w)+\varepsilon(\bar{w},\bar{y})(x,[z,w],y).$$

Proof. Let w,x,y,z be any elements of the right alternative color algebra A. We have

$$\begin{aligned}
(x,z,y\circ w)&=(x,z\circ y,w)+\varepsilon(\bar{w},\bar{y})(x,z\circ w,y)\\
&=(x,zy+\varepsilon(\bar{y},\bar{z})yz,w)+\varepsilon(\bar{w},\bar{y})(x,zw+\varepsilon(\bar{w},\bar{z})wz,y)\\
&=(x,zy,w)+\varepsilon(\bar{y},\bar{z})(x,yz,w)+\varepsilon(\bar{w},\bar{y})(x,zw,y)+\varepsilon(\bar{w},\bar{y})\varepsilon(\bar{w},\bar{z})(x,wz,y)\\
&=(x,zy,w)-\varepsilon(\bar{y},\bar{z})(x,yz,w)+2\varepsilon(\bar{y},\bar{z})(x,yz,w)+\varepsilon(\bar{w},\bar{y})(x,zw,y)-\varepsilon(\bar{w},\bar{y})\varepsilon(\bar{w},\bar{z})(x,wz,y)\\
&\quad+2\varepsilon(\bar{w},\bar{y})\varepsilon(\bar{w},\bar{z})(x,wz,y)\\
&=(x,zy,w)+(x,-\varepsilon(\bar{y},\bar{z})yz,w)+2\varepsilon(\bar{y},\bar{z})(x,yz,w)+\varepsilon(\bar{w},\bar{y})\{(x,zw,y)+(x,-\varepsilon(\bar{w},\bar{z})wz,y)\}\\
&\quad+\varepsilon(\bar{w},\bar{y}+\bar{z})(x,wz,y)\\
&=(x,zy-\varepsilon(\bar{y},\bar{z})yz,w)+2\varepsilon(\bar{y},\bar{z})(x,yz,w)+\varepsilon(\bar{w},\bar{y})(x,zw-\varepsilon(\bar{w},\bar{z})wz,y)+2\varepsilon(\bar{w},\bar{y}+\bar{z})(x,wz,y)\\
&=(x,[z,y],w)+2\varepsilon(\bar{y},\bar{z})(x,yz,w)+\varepsilon(\bar{w},\bar{y})(x,[z,w],y)+2\varepsilon(\bar{w},\bar{y}+\bar{z})(x,wz,y)\\
&=(x,[z,y],w)+\varepsilon(\bar{w},\bar{y})(x,[z,w],y)+2\{\varepsilon(\bar{y},\bar{z})(x,yz,w)+\varepsilon(\bar{w},\bar{y}+\bar{z})(x,wz,y)\}\\
&=(x,[z,y],w)+\varepsilon(\bar{w},\bar{y})(x,[z,w],y)+2\{-\varepsilon(\bar{y},\bar{z})\varepsilon(\bar{w},\bar{y}+\bar{z})(x,w,yz)-\varepsilon(\bar{w},\bar{y}+\bar{z})\varepsilon(\bar{y},\bar{w}+\bar{z})(x,y,wz)\}\\
&=(x,[z,y],w)+\varepsilon(\bar{w},\bar{y})(x,[z,w],y)\\
&\quad-2\{\varepsilon(\bar{y},\bar{z})\varepsilon(\bar{w},\bar{y}+\bar{z})(x,w,yz)+\varepsilon(\bar{w},\bar{y})\varepsilon(\bar{w},\bar{z})\varepsilon(\bar{y},\bar{w})\varepsilon(\bar{y},\bar{z})(x,y,wz)\}\\
&=(x,[z,y],w)+\varepsilon(\bar{w},\bar{y})(x,[z,w],y)-2\varepsilon(\bar{y},\bar{z})\{\varepsilon(\bar{w},\bar{y}+\bar{z})(x,w,yz)+\varepsilon(\bar{w},\bar{z})(x,y,wz)\}
\end{aligned}$$

According to the Lemma 3.14 we have $g(x,w,y,z)=0$ that is

$$\varepsilon(\bar{w},\bar{y}+\bar{z})(x,w,yz)+\varepsilon(\bar{w},\bar{z})(x,y,wz)=\varepsilon(\bar{w},\bar{z})\varepsilon(\bar{w},\bar{y})\varepsilon(\bar{z},\bar{y})(x,w,z)y-(x,y,z)w$$

Therefore

$$\begin{aligned}
&(x,z,y\circ w)\\
&=(x,[z,y],w)+\varepsilon(\bar{w},\bar{y})(x,[z,w],y)-2\varepsilon(\bar{y},\bar{z})\{\varepsilon(\bar{w},\bar{z})\varepsilon(\bar{w},\bar{y})\varepsilon(\bar{z},\bar{y})(x,w,z)y+(x,y,z)w\}\\
&=(x,[z,y],w)+\varepsilon(\bar{w},\bar{y})(x,[z,w],y)-2\varepsilon(\bar{y},\bar{z})\{-\varepsilon(\bar{w},\bar{z})\varepsilon(\bar{w},\bar{y})\varepsilon(\bar{z},\bar{y})\varepsilon(\bar{z},\bar{w})(x,z,w)y+(x,y,z)w\}\\
&=(x,[z,y],w)+\varepsilon(\bar{w},\bar{y})(x,[z,w],y)+2\varepsilon(\bar{w},\bar{y})(x,z,w)y-2\varepsilon(\bar{y},\bar{z})(x,y,z)w.
\end{aligned}$$

Thus

$$(x,z,y\circ w)=(x,[z,y],w)+\varepsilon(\bar{w},\bar{y})(x,[z,w],y)+2\varepsilon(\bar{w},\bar{y})(x,z,w)y-2\varepsilon(\bar{y},\bar{z})(x,y,z)w \text{ for all } w,x,y,z \text{ in A}.$$

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