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Generalized Pythagorean Theorem

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Abstract

As we all know, Pythagorean theorem is aimed at right triangle. In this paper, Pythagorean theorem is extended to any triangle except equilateral triangle, generalized Pythagorean theorem is introduced, so that the so-called Pythagorean number becomes history and no longer exists, and it is very simple to judge whether a triangle is obtuse triangle, right triangle or acute triangle. According to the generalized Pythagorean theorem, the rank of a triangle,generalized sine and generalized cosine are introduced, so that when the rank of a triangle is known, the triangle can be solved as a right triangle; when the rank of a triangle is not known, a simple and direct method to solve the triangle is also obtained,and right now there is no necessary to use the sine theorem and cosine theorem as usual. Pythagorean theorem mainly studies straight and studies distance,but the generalized Pythagorean theorem can study skew. Finally, the geometric significance and theoretical significance of the rank of a triangle are given.

Keywords: generalized Pythagorean theorem，rank of triangle，generalized sine，generalized cosine

1. Generalized Pythagorean theorem

Theorem 1 For positive real numbers a,b,c , if satisfied $a \le b < c$, there must exist a unique positive real number n , such that $a^n + b^n = c^n$,when $n<1$, a,b,c do not form a triangle, when $n=1$, a,b,c form a straight triangle, when $1<n<2$, a,b,c form an obtuse triangle, when $n=2$, a,b,c form a right triangle, and when $n>2$, a,b,c form an acute triangle.

Proof Taking logarithms on both sides of the equation $a^x + b^x = c^x$,obtains $x\ln c = x\ln b + \ln(1+(\frac{a}{b})^x)$ .Let the function $f(x) = x\ln c - x\ln b - \ln(1+(\frac{a}{b})^x)$. On one hand, $f'(x) = \ln c - \ln b + \frac{\ln b - \ln a}{1+(\frac{b}{a})^x} > 0$,on the other hand, $f(0) = -\ln 2 < 0$ and $\lim_{x\to+\infty} f(x) = +\infty$ .thus the existence and uniqueness of the theorem 1 hold true [1-3]. Let the function $f(x) = 1 + x^n - (1+x)^n$, then $f'(x) = nx^{n-1} - n(1+x)^{n-1}$, $f(0) = 0$ .When $n < 1$, $f'(x) > 0$, $f(x) > 0$, $1+(\frac{a}{b})^n > (1+\frac{a}{b})^n$, $(a+b)^n < a^n + b^n = c^n$, $a+b < c$,so, a,b and c do not form a triangle. . When $n > 1$, $f'(x) < 0$, $f(x) < 0$, then, $1+(\frac{a}{b})^n < (1+\frac{a}{b})^n$, $(a+b)^n > a^n + b^n = c^n$, $a+b > c$,so, a,b and c form a triangle . Let the function $f(x) = (1+x^n)^{\frac{2}{n}} - 1 - x^2$, then $f'(x) = 2x(x^{n-2}(1+x^n)^{\frac{2-n}{n}} - 1)$, $f(0) = 0$ .When $n < 2$, $f'(x) > 0$, $f(x) > 0$, then, $\left(1+(\frac{a}{b})^n\right)^{\frac{2}{n}} > 1 + \left(\frac{a}{b}\right)^2$, $a^2 + b^2 < c^2$ then, a,b,c form an obtuse triangle,when $n > 2$, $f'(x) < 0$, $f(x) < 0$, then, $\left(1+(\frac{a}{b})^n\right)^{\frac{2}{n}} < 1 + \left(\frac{a}{b}\right)^2$, $a^2 + b^2 > c^2$,then, a,b,c forms an acute triangle.

This theorem can be regarded as generalized Pythagorean theorem.Thus, not only right triangles have Pythagorean theorems, every triangle has a Pythagorean theorem,except for isosceles triangles with base length less than or equal to waist length. That is to say, except for isosceles triangles with base length less than or equal to waist length, each triangle corresponds to a unique real number n greater than or equal to 1. We call this real number the rank of the triangle. For any given real number n greater than 1, there is a family of triangles with rank n , such as (right angle) triangles with rank 2, which can be considered a family. Similarly, an isosceles triangle with a base less than waist has a rank of positive infinity, such a triangle is clearly an acute triangle (which can be considered to have a rank greater than 2), and it also constitutes a family of triangles. An isosceles triangle with a base equal to waist, i.e. an equilateral triangle, is the only triangle without rank. Obviously, equilateral triangles can also be considered a family, which contains the least number of triangles The rank of any triangle can be calculated using MATLAB. According to this theorem, a triangle with sides $a, b, (a^3 + b^3)^{\frac{1}{3}}$ must be an acute triangle. Based on this, for any given real number $n \geq 1$, a triangle of rank n can be easily constructed. Conversely, for any given triangle of rank n , this method can be used to construct it.

2. Generalized sine and generalized cosine

Define For any triangle ABC of rank r, let its maximum angle be $\angle C$ and $AB = c, BC = a, CA = b$,define the generalized sine $\sin(r, A) = \frac{a}{c}$ and the generalized cosine $\cos(r, \angle A) = \frac{b}{c}$ with a smaller acute angle $\angle A$. Note that only smaller acute angles have generalized sine and cosine, obviously.

Theorem 2 For any rank $r>1$ and any acute angle α, their generalized cosine $\cos(r,\alpha)$ and generalized sine $\sin(r,\alpha)$ exist and are unique

Prove Suppose the rank of a triangle ABC is r, $AB=c, BC=a, CA=b$, the smaller acute angle $\angle A=\alpha$, and the longest side is c. According to the cosine theorem, $a=\left(b^2+c^2-2bc\cos\alpha\right)^{\frac{1}{2}}$, because the rank of the triangle ABC is r, therefore $b^r+(b^2+c^2-2bc\cos\alpha)^{\frac{r}{2}}-c^r=0$. Let the functon $f(x)=x^r+(x^2-2x\cos\alpha+1)^{\frac{r}{2}}-1$. If it is proven that the function f has a unique zero point in $(0,1)$ then the theorem holds. $f(0)=0, f(1)=(2\sin\frac{\alpha}{2})^r>0$,

After a simple calculation, obtain $f'(x)=rx^{r-1}+r(x-\cos\alpha)(x^2-2x\cos\alpha+1)^{\frac{r-2}{2}}$. When $x\geq\cos\alpha$, $f'(x)>0$.

Let $g(x)=x^{r-1}+(x-\cos\alpha)(x^2-2x\cos\alpha+1)^{\frac{r-2}{2}}$, $h(x)=(x-\cos\alpha)(x^2-2x\cos\alpha+1)^{\frac{r-2}{2}}$

then $g(0)=-\cos\alpha<0$, $g(\cos\alpha)=(\cos\alpha)^{r-1}>0$

$h'(x)=(x^2-2x\cos\alpha+1)^{\frac{r-2}{2}-1}(x^2-2x\cos\alpha+1+(r-2)(x-\cos\alpha)^2)$, it is easy to calculate the minimum value of $k(x)=x^2-2x\cos\alpha+1+(r-2)(x-\cos\alpha)^2$ over $[0,\cos\alpha]$ is $\sin^2\alpha$ .so, $k(x)>0$, $h'(x)>0$, $g'(x)>0$ over $(0,\cos\alpha)$ .So, $g(x)$ there must be a unique zero point x_0 over $(0,\cos\alpha)$ .So, $f'(x)<0$ over $(0,x_0)$; $f'(x)>0$ over $(x_0,1)$, thus $f(x_0)<0$ and the function $f(x)$ has a unique zero point over (0,1).The theorem is proven

According to this theorem, it can be known that the only solution of the equation $x^r+(x^2-2x\cos\alpha+1)^{\frac{r}{2}}-1=0$ over $(0,1)$ is $\cos(r,\alpha)$, so $\cos(r,\alpha)$ is solely determined by r,α and $\cos(r,\alpha)$ can be calculated using Matlab.Obviously $\cos(2,\alpha)=\cos\alpha, \sin(2,\alpha)=\sin\alpha$. Therefore, the generalized cosine and generalized sine are natural extensions of the cosine and sine. To the given rank $r>1, \cos(r,\alpha)$ is a univariate function of α, which is a new function with almost no expression. One question is whether it is possible to use this new univariate function to represent a function without an expression?

In classical theory, the cosine of an acute angle α, which appears to be the ratio of the adjacent side to the longest side (hypotenuse) of the acute angle α of a right triangle, is essentially determined solely by the rank 2 of the right triangle and this acute angle α. Similarly, on the surface, the generalized cosine of an acute angle α is the ratio of the adjacent side to the longest side of the smaller acute angle α of a triangle,in essence, this generalized cosine $\cos(r,\alpha)$ is uniquely determined by the rank r of the triangle and this smaller acute angle α. Therefore, after introducing the rank of triangles the generalized cosine becomes meaningful,and the generalized sine naturally becomes meaningful as well!

3. A simple method for solving triangles

According to Theorem 1 and Theorem 2, the rank of a triangle and its smaller angle uniquely determine the generalized cosine of the smaller angle, and conversely, the smaller angle of a triangle and its generalized cosine uniquely determine its rank ,at last the rank of a triangle and the generalized cosine uniquely determined that acute angle .So three tables of electronic mathematics are proposed based on equation $x^r+(x^2-2x\cos\alpha+1)^{\frac{r}{2}}-1=0$:

① $\cos(r,\alpha)=x$,② $\cos^{-1}(x,r)=\alpha$,③ $\cos^{-1}(x,\alpha)=r$ .Similarly the following three electronic mathematical tables can be obtained: ④ $\sin(r,\alpha)=y$,⑤ $\sin^{-1}(y,r)=\alpha$,⑥ $\sin^{-1}(y,\alpha)=r$ .Add three more electronic mathematics tables below:⑦ $r(\alpha,\beta)=r$. That is ,given two smaller acute angles α,β of a triangle ,its rank r can be determined ,this table is proposed based on equation $\sin^r\alpha+\sin^r\beta=\sin^r(\alpha+\beta)$.⑧ $r^{-1}(r,\alpha)=\beta$ .That is, given the rank r and a smaller acute α angle of a triangle, the other smaller acute angle β can be determined ,this table is proposed based on equation $\sin^r\alpha+\sin^r\beta=\sin^r(\alpha+\beta)$ also.⑨ $r(x,y)=r$ .That is, given the generalized sine and cosine x,y of a smaller acute angle α of a triangle ,its rank r can be determined ,this table is proposed based on equation $x^r+y^r=1$ .With the above nine electronic math tables,solving triangles will be very simple,basically,just look up the tables! Here are some examples to illustrate:

Example1 Let the rank of a triangle ABC be 3, $AC=3$, $BC=4$, try to find its longest side AB and its three angles.

Solution $AB^3=AC^3+BC^3=3^3+4^3=91$, therefore the longest side $AB\approx 4.498$. Obviously, $r=3>2$, hence this triangle is an acute triangle.Accoding to the table②: $\angle A=\cos^{-1}(\frac{AC}{AB},r)=\cos^{-1}(\frac{3}{4.98},3)\approx 53.435^0$,

$\angle B=\cos^{-1}(\frac{BC}{AB},r)=\cos^{-1}(\frac{4}{4.98},3)\approx 37.041^0$, $\angle C\approx 180^0-53.435^0-37.041^0=89.524^0$.

It is almost a right triangle!

Example 2 Assuming the ABC rank of a triangle is 4, $\angle A=\frac{\pi}{6}\approx 0.52$, with the longest side $AB=6$, try to find the other two sides AC , BC and the other two angles $\angle B,\angle C$,and the area of the triangle.

Solution According to the table①④② : $\cos(4,\frac{\pi}{6})=0.9822,\sin(4,\frac{\pi}{6})\approx 0.513$, so

$$AC=AB\cos(4,\angle A)=6\cos(4,\frac{\pi}{6})=5.8932,\ BC=AB\sin(4,\angle A)=6\times\sin(4,\frac{\pi}{6})\approx 3.078$$

$$\cos(4,B)=\frac{BC}{AB}\approx 0.513,\ \angle B=\cos^{-1}(0.513,4)\approx 1.28,\ \angle C=\pi-\angle A-\angle B\approx 3.14-0.52-1.28=1.34 .$$

Area $S=\frac{1}{2}AB^2\cos(4,A)\sin A=0.5\times 6^2\times 0.9822\times 0.5=8.8398$

Similarly,when the other edge AC or BC is known,it is also easy to solve this triangle. Note that angle $\angle A$ is clearly not the maximum angle of the triangle ABC here. When the angle $\angle A$ is the maximum angle of the triangle ABC, the discussion is as follows:Given the rank r and maximum angle $\angle A$ of a triangle, it is evident that the other two smaller angles of the triangle can be calculated using MATLAB based on the equation $\sin^r\alpha+\sin^r(\pi-\angle A-\alpha)=\sin^r\angle A$.

However, further research is needed to determine how many pairs of these two smaller angles $\alpha, \pi - \angle A - \alpha$ there are. Therefore, in the seventh and eighth electronic mathematics table, it is necessary to limit the angle α, β to be the smaller angle of the triangle.

Example 3: Assuming the ABC rank of a triangle is 4, $\angle A = \frac{\pi}{6} \approx 0.52$, try to find the other two angles $\angle B$, $\angle C$ of this triangle.

Solution According to the table⑧ : $\angle B = r^{-1}(r, \angle A) = r^{-1}(4, \frac{\pi}{6}) \approx 1.28$

$$So, \angle C = \pi - \angle A - \angle B \approx 3.14 - 0.52 - 1.28 = 1.34.$$

Thus,under the condition of knowing the rank of a triangle,it is very easy to solve the triangle using the method of table lookup proposed in this article.

Example 4 In a triangle ABC, $\angle A = \frac{\pi}{3}, \angle B = \frac{\pi}{9}$, AB =4, try to find the rank of the triangle ABC and the length of AC,BC .

Solution According to the table⑨①④:

$$r = r(\frac{\pi}{9}, \frac{\pi}{3}) \approx 1.6, \cos(r, \angle A) \approx \cos(1.6, \frac{\pi}{3}) \approx 0.35, \sin(r, \angle A) = \sin(1.6, \frac{\pi}{3}) \approx 0.875$$

So, $AC = AB\cos(r, \angle A) \approx 4\cos(1.6, \frac{\pi}{3}) \approx 1.4$, $BC = AB\sin(r, \angle A) \approx 4\sin(1.6, \frac{\pi}{3}) = 3.5$

This solution illustrates how to use the method of looking up a table to solve a triangle when one side and the two angles of the triangle are known.

Example5 In a triangle ABC, $\angle A = 35^0, AB = 6, AC = 3$,, try to find the rank of the triangle ABC and the length of BC and $\angle B, \angle C$.

Solution $\cos(r, 35^0) = \frac{AC}{AB} = \frac{3}{6} = 0.5$ .According to the table③④②:

$$r = \cos^{-1}(0.5, 35^0) \approx 1.27, \sin(1.27, 35^0) \approx 0.66, \angle B = \cos^{-1}(0.66, 1.27) \approx 25.9^0$$

$$So, \angle C = 180^0 - \angle A - \angle B \approx 180^0 - 35^0 - 25.9^0 = 119.1^0, BC = AB\sin(r, \angle A) \approx 6 \times 0.66 = 3.96$$

This solution illustrates how to use the method of looking up a table to solve a triangle when two sides and the angle between the two sides are known and the angle is not maximum angle.

We will give a method of looking up a table to solve a triangle in the next paper, when two sides and the angle between the two sides are known and the angle is maximum angle .

when two sides and a angle are known and the angle is not between the two sides,we can find another angle by sine theorem,in this way,it becomes very simple to solve the triangle!

Example 6 In a triangle ABC , $AB = 7, AC = 6, BC = 5$,, try to find the rank of the triangle ABC and $\angle A, \angle B, \angle C$.

Solution According to the table⑨⑧:

$$r = r(\frac{6}{7}, \frac{5}{7}) \approx 2.97, \angle A = \cos^{-1}(2.97, \frac{6}{7}) \approx 44.42^0, \angle B = \cos^{-1}(2.97, \frac{5}{7}) \approx 57.12^0$$

So, $\angle C = 180^0 - \angle A - \angle B \approx 180^0 - 44.42^0 - 57.12^0 = 78.46^0$

Thus ,at any conditions,the method proposed in this article can be used to solve any triangles!

4. The geometric significance of the rank of a triangle

Two adjacent sides of the maximum angle $\angle C$ of a triangle with rank r are $a, b(a > b)$, The cosine of the $\angle C$ is $\cos\angle C = \frac{a^2 + b^2 - (a^r + b^r)^{\frac{2}{r}}}{2ab} = f(r), \quad f'(r) = \frac{(a^r + b^r)^{\frac{2}{r}-1}}{abr^2}(a^r \ln\frac{a^r + b^r}{a^r} + b^r \ln\frac{a^r + b^r}{b^r}) > 0$, so the larger r , the larger $\cos C$, and the smaller $\angle C$ .The geometric meaning of the rank of a triangle is that when the rank is equal to 1, the angle between the two sides a, b of the triangle is the largest, which is 180 degrees,as the rank increases, the maximum angle of the triangle sandwiched between the two sides gradually decreases,when the rank tends to infinity, the triangle tends to be an isosceles triangle with the waist a and the base b ,at this point, the maximum angle of the triangle sandwiched between the two sides a, b of the triangle reaches its minimum, if the base is equal to the waist, the maximum angle of the triangle reaches its minimum value of 60 degrees.

Based on the above analysis, we can use the rank of a triangle to determine its shape, that is, the smaller the rank of a triangle, the closer it is to a straight triangle, and the larger the rank, the closer it is to an isosceles triangle. For example, a triangle with a rank of 1.01 is very slender and basically a straight triangle, a right triangle with a rank of 2 and a relatively large rank is more like an isosceles triangle, and a triangle with a rank of 9 is basically an isosceles triangle.The rank of a triangle reflects this characteristic, which can also be seen from Example 1, Example 2, and Example 3, because the ranks of the triangles in these three examples are relatively large.

5. The theoretical significance of the rank of a triangle

Two adjacent sides of the maximum angle $\angle C$ of a triangle with rank r are $a, b(a > b)$, The cosine of the $\angle C$ is $\cos\angle C = \frac{a^2 + b^2 - (a^r + b^r)^{\frac{2}{r}}}{2ab} = f(r, a, b), \quad f'(r) = \frac{(a^r + b^r)^{\frac{2}{r}-1}}{abr^2}(a^r \ln\frac{a^r + b^r}{a^r} + b^r \ln\frac{a^r + b^r}{b^r}) > 0$

$$\lim_{a \to +\infty} \cos\angle C = 0, \lim_{a \to 0^+} \cos\angle C = 0$$

So, on a>0 and b>0, when 1<r<2, $f(r, a, b)$ must have a minimum value $f_1(r)$, a maximum value and an upper bound 0 of the maximum value; when r>2, $f(r,a,b)$ must have a maximum value $f_2(r)$,a minimum value and a lower bound 0 of the minimum value.If $r_1 < r_2$, then $f_1(r_1) < f_1(r_2), f_2(r_1) < f_2(r_2)$.

It follows that when a triangle is an obtuse triangle, knowing its rank means knowing the maximum value of its maximum angle,the lower bound of the minimum value of its maximum angle is $\frac{\pi}{2}$,as the rank gradually increases, the maximum value of its maximum angle gradually decreases. When a triangle is an acute triangle, knowing its rank means knowing the minimum value of its maximum angle,the upper bound of the maximum value of its maximum angle is $\frac{\pi}{2}$, as the rank gradually increases, the minimum value of its maximum angle gradually decreases.

In summary, the theoretical significance of the rank of a triangle is that, given its rank, one not only knows whether it is an obtuse triangle, a right triangle, or an acute triangle, but also further knows the interval of values for its maximum angle. Its maximum angle can take any number within this interval and cannot take any number outside of it. This interval is a semi open and semi closed interval with an open boundary of 90 degrees. Specifically, when the rank of a triangle is 2, this range degenerates into a number, which is 90 degrees.We conjecture that the closed boundary of this semi open and semi closed interval is $\cos^{-1}(1-2^{\frac{2}{r}-1})$.

6. Conclusion

After the extension of the Pythagorean theorem, any three non equal positive numbers that form a triangle are Pythagorean numbers, so the so-called Pythagorean numbers will no longer exist in history! The Pythagorean theorem is too narrow, and the generalized Pythagorean theorem has expanded it! The Pythagorean theorem mainly studies straight and distance, while the generalized Pythagorean theorem can study oblique.As for its application, please study and think about it!

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