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A New Method For Calculating the Distance Between Two Points On A Plane

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Abstract

Using generalized cosine and sine , a new method for calculating the distance between two points in a plane coordinate system and a polar coordinate system is given. The new method shows that using the Pythagorean theorem to calculate the distance between two points in a Cartesian coordinate system is the most troublesome! This almost overturns people's cognition !

Keywords affine coordinate system, Cartesian coordinate system, polar coordinate system,generalized cosine,generalized sine

1 Distance between two points in a planar affine coordinate system

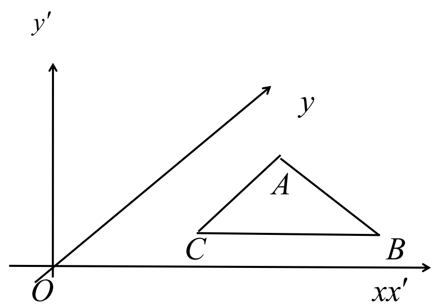

Figure 1

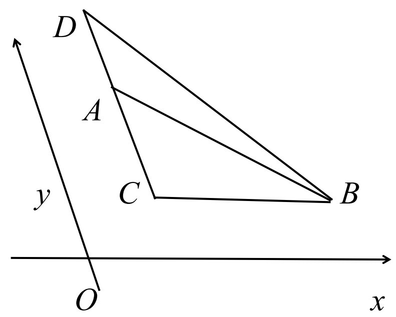

Figure 2

In the affine coordinate system xoy shown in Figure 1, the angle between the two coordinate axes is less than or equal to 60^0, the edge BC is parallel to the x axis, and the edge AC is parallel to the y axis. Given the edges AC, BC and angles $\angle C$, calculate AB. In other words, given the coordinates of two points A, B, calculate the distance between them. This problem can certainly be solved using the cosine theorem, but we have a simpler method, which is to use generalized cosine and generalized sine to solve a triangle: because $\angle C \le 60^0$ so it may not be the maximum angle of the triangle ABC. When the maximum angle of the triangle ABC is $\angle A$, $\angle C$ has a generalized cosine $\cos(r, \angle C) = \dfrac{AC}{BC}$,which can be used to find the generalized sine $\sin(r, \angle C) = \dfrac{AB}{BC}$,so, $AB = BC \sin(r, \angle C)$. When the maximum angle of the triangle ABC is $\angle B$, $\angle C$ has a generalized cosine $\cos(r, \angle C) = \dfrac{BC}{AC}$,which can be used to find the generalized sine $\sin(r, \angle C) = \dfrac{AB}{AC}$,so, $AB = AC \sin(r, \angle C)$,where r is the rank of the triangle ABC. This method can be found in [1]. Here is an example to illustrate:

Example 1: In the sketch shown in Figure 1, the angle between the positive directions of the x axis and the y axis is 40^0, and the coordinates of the two points A, B are $A(1,3), B(4,2)$ respectively . Find the distance between the two points A, B.

Solution $AC = 1, BC = 3, \angle C = 40^0 < 60^0, \cos(r, \angle C) = \dfrac{AC}{BC} = \dfrac{1}{3}$,.According to the table,

$\sin(r, 40^0) \approx 0.97$, thus $AB = BC \sin(r, \angle C) \approx 2.91$ .Here, we need to extend the electronic mathematical table of trigonometric functions to the electronic mathematical table of generalized trigonometric functions. As this electronic mathematical table has not yet been established, we will temporarily use formulas $\sin^2(r, \alpha) = \cos^2(r, \alpha) - 2\cos(r, \alpha)\cos\alpha + 1$ to calculate $\sin(r, \alpha)$,more precisely, it is to create a mathematical table through formulas $\sin^2(r, \alpha) = \cos^2(r, \alpha) - 2\cos(r, \alpha)\cos\alpha + 1$ for $\sin(r, \alpha)$, and the same will not be explained below.

The process of this method involves performing one division, one lookup (simpler than arithmetic), and one multiplication! To establish a Cartesian coordinate system and use the Pythagorean theorem to calculate the distance between two points, it is necessary to go through two squares, one addition, and one square root. So,using generalized cosine and sine to calculate the distance between two points in an affine coordinate system like Figure 1 is simpler than using Pythagorean theorem in a Cartesian coordinate system! Of course, is also simpler than using the cosine theorem! Because the Pythagorean theorem is simpler than the cosine theorem.

In the affine coordinate system xoy shown in Figure 2, the angle between the two coordinate axes is greater than 60^0, the edge BC is parallel to the x axis, and the edge AC is parallel to the y axis. Given the edges AC, BC and angles $\angle C$, calculate AB. In other words, given the coordinates of two points A, B, calculate the distance between them. At this point, $\angle C$ may be the maximum angle of the triangle ABC,there is no generalized cosine or sine at the $\angle C$, hence this article adopts the following method:

There is no harm in $CA \le CB$, extending the edge CA to the point D,such that $CD = CB$, we obtain an isosceles triangle BCD, and knowing the waist BC and the top angle of this isosceles triangle, so,it is easy to find its base edge BD and bottom angle $\angle D$. Because $\angle C > 60^0$, therefore $\angle D < 60^0$, $\angle D$ is not the maximum angle of triangle ABD, so AB can be obtained by applying the method in Example 1 in the triangle ABD. Here is an example to illustrate:

Example 2: In the sketch shown in Figure 2, the angle between the positive directions of the x axis and the y axis is 80^0, and the coordinates of the two points A, B are $A(2,4), B(8,2)$ respectively . Find the distance between the two points A, B.

Solution $AC = 2, BC = 6, \angle C = 80^0, AD = 4, \angle D = 50^0, BD = 2BC \sin 40^0$

$\approx 7.7134513162384, \cos(r, \angle D) = \dfrac{AD}{BD} \approx 0.5185746089534$ .According to the table:

$\sin(r, \angle D) \approx 0.7760495850038$,thus $AB = BD \sin(r, \angle D) \approx 5.9860206929138.$

The process of this example involves one subtraction, three multiplications (including one multiplication by 2), two divisions (including one division by 2), and two lookup tables. In fact, its main operations are only two multiplications and one division, without square root. To use the Pythagorean theorem to calculate the distance between two points in a Cartesian coordinate system, four operations are required. It should be emphasized that these four operations include one square root, which is to find the arithmetic square root of a certain number!According to the method of finding the arithmetic square root of a number introduced in junior high school textbooks, it can be seen that besides being able to memorize what the arithmetic square root of some numbers is, finding the arithmetic square root of a number is more troublesome than the main operation done in this example adding two multiplications and one division together! For example 20 this is how I beg! So, using generalized cosine and sine to calculate the distance between two points in an affine coordinate system like Figure 2 is simpler than using Pythagorean theorem in a Cartesian coordinate system! Of course, its also simpler than using the cosine theorem! Because the Pythagorean theorem is simpler than the cosine theorem.

If the angle between two axes in an affine coordinate system satisfies $\sin\frac{\angle C}{2}$ is a true fraction, then $BD = 2BC\sin\frac{\angle C}{2}$ is easy to find, and the distance AB is easy to find. So, building an affine coordinate system cannot be done casually, but should make the $\sin\frac{\angle C}{2}$ of the angle $\angle C$ between the two coordinate axes as simple as possible! For example, if the angle between the positive directions of the x axis and the y axis is $\angle C = 2\sin^{-1} 0.9$, $128^0 < \angle C < 130^0$,then another angle between the two axes is $180^0 - \angle C < 60^0$ .There are two situations to calculate the distance between two points in the affine coordinate system, as shown in Figure 1 and Figure 2. In the case of Figure 1, as shown in Example 1, AB is easy to find; In the case of Figure 2, it is easy to find $BD = 2BC\sin\frac{\angle C}{2}$ and the subsequent steps are the same as Example 1, AB is also easy to find! So, this affine coordinate system is great, it is a good affine coordinate system!

In summary, in the affine coordinate system, there are at most two situations for calculating the distance between two points, as shown in Figure 1 and Figure 2. The situation in Figure 1 is very simple, while the situation in Figure 2 mainly involves BD ,an additional step of calculating the distance compared to Figure 1, and it can make this step very easy, so it also very simple. So when calculating distance, we need to use an affine coordinate system, especially the good affine coordinate system mentioned in this article! It is worth mentioning here:Affine coordinate systems with familiar angles $30^0, 45^0, 60^0, 90^0, 120^0, 135^0, 150^0$ between two coordinate axes are not good! Especially, the commonly used Cartesian coordinate system is not good! Because the values $\sin\frac{\angle C}{2}$ of the angles $\angle C$ between the two axes of these coordinate systems are not simple!

2 Distance between two points in a polar coordinate system

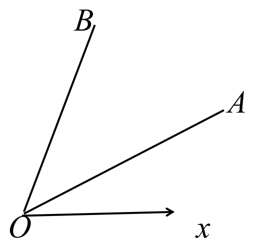

Figure 3

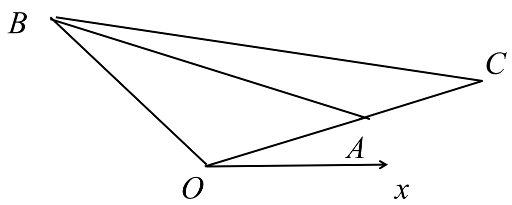

Figure 4

In the polar coordinate system shown in Figure 3, the polar coordinates of two points A, B are $A(\rho_1, \theta_1), B(\rho_2, \theta_2)$ respectively, find the distance between the two points A, B.

In the triangle ABC, $OA = \rho_1, OB = \rho_2, \angle AOB = \theta_2 - \theta_1$.

It is a common way to immediately calculate the distance between two points using the cosine theorem. But obviously, this problem can also be solved using the methods of Example 1 and Example 2. As pointed out above, the methods of Example 1 and Example 2 are definitely simpler than using the cosine theorem! So applying the methods of Example 1 and Example 2, which use generalized cosine and generalized sine to calculate distance, to polar coordinate systems can better demonstrate its superiority!

Example 3 In the polar coordinate system shown in Figure 4, the polar coordinates of two points A, B are $A(15, 20^0), B(25, 20^0 + 2\sin^{-1} 0.8)$ respectively . Find the distance between the two points A, B.

Solution Because $\angle AOB = 2\sin^{-1} 0.8 > 60^0$, therefore, extend the line segment OA to the point C , so that $OC = OB = 25$ .Hence,

$$\angle C = 90^0 - \sin^{-1} 0.8, CA = 10, CB = 2OC \sin \frac{\angle AOB}{2} = 50 \times 0.8 = 40, \cos(r, \angle C) = \frac{CA}{CB} = 0.25$$

According to the table: $\sin(r, \angle C) \approx 0.8139410298049$.

Thus, $AB = BC \sin(r, \angle C) \approx 32.55764119196$.

In this example, CB is easy to find, and after finding it, the steps are the same as in Example 1.

3 The superiority of affine coordinate system

This article provides a simpler method for calculating distance using generalized cosine and sine than using Pythagorean theorem!The simplicity lies in the fact that this method only requires two or three multiplication and division, while the Pythagorean theorem still requires square root!From this, it can be seen that the concept of verticality is no longer important, and even oblique intersection is better than verticality!

In the Cartesian coordinate system $x'oy'$ and affine coordinate system xoy shown in Figure 1, the coordinate origins of the two coordinate systems O coincide, the coordinate axes ox and ox' coincide with each other. Therefore, it is easy to obtain the coordinate transformation formula $\begin{cases} x' = x + y\cos\alpha \\ y' = y\sin\alpha \end{cases}$ (1),where the angle α is the angle between the positive directions of the x axis and the y axis of the affine coordinate system xoy. Obviously, this formula is reversible According to this reversible coordinate transformation formula, what do the equations of simple curves such as straight lines, circles, ellipses, hyperbolas, parabolas, etc. in Cartesian coordinate system look like in affine coordinate system,what curves do Simple binary equations in affine coordinate systems, such as binary linear equations, (especially simple)binary quadratic equations, etc., represent, these problems can be easily solved. That is to say, problems that can be studied in a Cartesian coordinate system can also be studied in an affine coordinate system, and vice versa. This indicates that the affine coordinate system is exactly the same as the Cartesian coordinate system, and according to this article, it can be seen that the affine coordinate system is better than the Cartesian coordinate system! So, when studying problems, we should use an affine coordinate system instead of a Cartesian coordinate system, but in reality, the opposite is true! This is because people are influenced by the narrow Pythagorean theorem, only accustomed to Cartesian coordinate systems and not to affine coordinate systems!

Let the angle $\angle C$ between the positive directions of the two axes of the affine coordinate system be $\angle C = 2\sin^{-1}\frac{3}{5}$,then $73.6^0 < \angle C < 73.8^0, \sin\frac{\angle C}{2} = \frac{3}{5}, 106.2^0 < 180^0 - \angle C <$ $106.4^0, \sin\frac{180^0 - \angle C}{2} = \frac{4}{5}$ .At this time, the two angles $\angle C, 180^0 - \angle C$ between the two coordinate axes are both greater than 60^0 ,and $\sin\frac{\angle C}{2} = 0.6, \sin\frac{180^0 - \angle C}{2} = 0.8$ are both very simple. According to the previous analysis, it is easy to find the distance between two points in this affine coordinate system. Furthermore, $\sin\angle C = 2\sin\frac{\angle C}{2}\cos\frac{\angle c}{2} = 2\times\frac{3}{5}\times\frac{4}{5} = \frac{24}{25}$,

$\cos\angle C = 1 - 2\sin^2\frac{\angle C}{2} = 1 - 2\times\frac{9}{25} = \frac{7}{25}$ .So, the coordinate transformation formula (1) $\begin{cases} x' = x + \frac{7}{25}y \\ y' = \frac{24}{25}y \end{cases}$ is also very simple. Generally, as long as $\sin\frac{\angle C}{2}$ is simple, $\sin\angle C$, $\cos\angle C$ are also simple. So, building a good affine coordinate system not only makes it easy to calculate the distance between any two points,but also simplifies the coordinate transformation formula (1) from the affine coordinate system to the Cartesian coordinate system , which fully demonstrates the advantages of a good affine coordinate system!

Note:① The methods of example 1, example 2, or example 3 are general! The operation in example 1 involves only one division and one multiplication, which is both simple and accurate. Example 2 and example 3 have an additional multiplication compared to example 1, and the multiplication can be greatly simplified,therefore, example 2 and example 3 are basically the same as example 1, and the results are also very accurate. For example, if we use the cosine theorem to solve example 2, the result is 5.9860206929139, if we use the cosine theorem to solve example 3,the result is 32.557641192199, the difference is only the last digit after the decimal point .

②Finding $\sin(r, \angle C)$ need to look up a table ,in fact, finding $\sin\angle C = \sin(2, \angle C)$ also need to look up a table, generally speaking !

③ The coordinate systems used in this article are all on a plane.

Reference

[1] Zhou Zhongwang. Generalized Pythagorean Theorem[J]. SCIREA Journal of Mathematics (accepted in 2025,awaiting publication).