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Multi-beam bathymetry model

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Abstract

Multi-beam bathymetry is an important means to improve the efficiency of seabed topographic surveying, so it is of great significance to study the layout of multi-beam survey lines for ocean surveying and mapping. Based on basic mathematical principles, this paper mainly applies the relevant mathematical knowledge of solid geometry, trigonometric functions to establish a multi-beam bathymetry model. The model has a solid theoretical foundation. The article analyzes the geometric relationships in the model in detail.

Keywords: stereo geometry, trigonometric functions, multi-beam bathymetry.

Introduction

In order to improve the efficiency of seabed topographic surveying, the multi - beam bathymetry technology[1] is developed on the basis of single-beam bathymetry technology. This technology can measure a full-coverage water depth strip with a certain width, taking the survey line of the survey ship as the axis. It overcomes the shortcomings of single-beam bathymetry to a certain extent. The multi-beam bathymetry system realizes comprehensive, efficient and accurate detection of underwater terrain by emitting multiple beams at the same time, which not only greatly improves the measurement efficiency, but also significantly enhances the integrity and reliability of the data.

In the sea area where the seabed is flat, multi-beam bathymetry can emit dozens or even hundreds of beams at a time in a plane perpendicular to the survey track, and then receive the sound waves returned from the seabed by the receiving transducer. Its working principle is shown in Figure 1, where α is the angle between the submarine slope surface and the horizontal plane, θ is opening angle of the transducer.

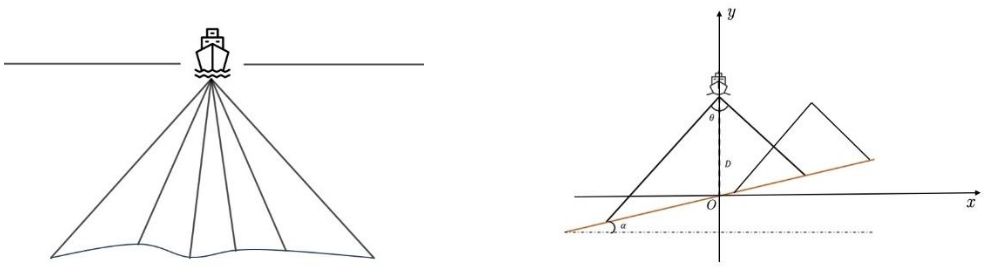

Figure1 Principle of multi-beam bathymetry and the diagram of establishing a coordinate system

Modeling process steps

I. Two-dimensional in-plane

i. Establish a planar Cartesian coordinate system

Taking the projection point of the center point of the sea area on the submarine slope as the coordinate origin, the opening direction of the angle between the submarine slope and the horizontal plane is the positive direction of the *x*-axis, the vertical direction is the *y*-axis, and the direction pointing to the sea level is the positive direction, and the coordinate diagram is shown in Figure 1.

We assume the abscissa and the depth of the current position of the measuring vessel are x_i and D_i, respectively. Figure 2 shows the position of the beam relative to the center of the surveyed area in the ocean.

Table 1 The abscissa represented by x_i

x_i	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
	-800	-600	-400	-200	0	200	400	600	800

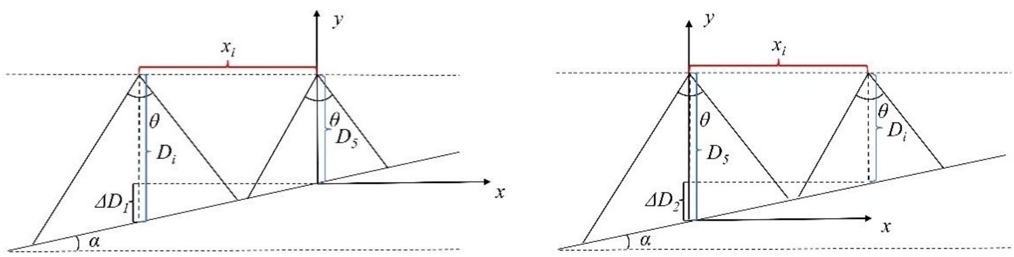

Figure 2 The position of the beam relative to the center of the ocean

ii. The depth of the sea

Let ΔD_i be the difference between the seawater depth at the *i*-th survey line and the seawater depth at the central point of the sea area, with the expression being

$$\Delta D_i = \begin{cases} D_i - D_5, i = 1,2,3,4, \\ D_5 - D_i, i = 6,7,8,9. \end{cases} \tag{1}$$

Then

$$\tan\alpha = \frac{\Delta D_i}{x_i}, i = 1,2,...,9. \tag{2}$$

So the expression of ΔD_j is:

$$\Delta D_i = \tan\alpha \cdot x_i, i = 1,2,...,9. \tag{3}$$

Finally, it is obtained that D_i is:

$$D_i = \begin{cases} D_5 + \Delta D_i, i = 1,2,3,4, \\ D_5 - \Delta D_i, i = 6,7,8,9. \end{cases} \tag{4}$$

iii. Coverage width

The coverage width [2] is divided into w_{i1} , w_{i2} $(i=1,2,\dots,9)$ two parts, as shown in Figure 3.

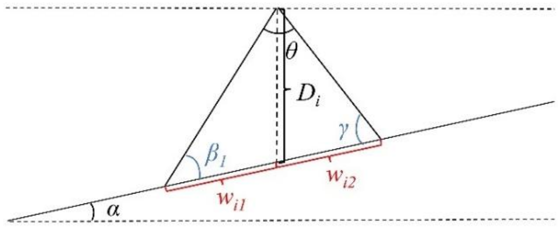

Figure 3 Schematic diagram of the coverage width

According to the geometric relation and the sinusoidal theorem of the triangle, the coverage width of the two parts is:

$$\frac{w_{i1}}{\sin\frac{\theta}{2}}=\frac{D_i}{\sin\beta_1}\text{ and }\frac{w_{i2}}{\sin\frac{\theta}{2}}=\frac{D_i}{\sin\gamma}, \tag{5}$$

β_1 and γ are as shown in Figure 3, and their expressions are:

$$\beta_1=\frac{\pi}{2}-\frac{\theta}{2}-\alpha\text{ and }\gamma=\frac{\pi}{2}-\frac{\theta}{2}+\alpha\ . \tag{6}$$

Therefore, the coverage width is obtained as:

$$W=\left(w_{i1}+w_{i2}\right)\cdot\cos\alpha\ . \tag{7}$$

iv. Overlap rate

Let the covering widths of the *i*-th and (*i*−1)-th survey lines be denoted by W_i and W_{i-1} $(i=2,3,\dots,9)$ respectively. The projection of the coverage width on the submarine slope is W_i' , where the length of the $W_i'=W_i\cdot\sec\alpha$. The overlapping part between the *i*-th survey line and the (*i*-1)-th survey line is denoted as f_i . The non - overlapping part in W_{i-1}' is denoted as F_{i-1} .Draw a line d_i parallel to *d*, and then a triangle *ABC* is obtained. As shown in Figure 4.

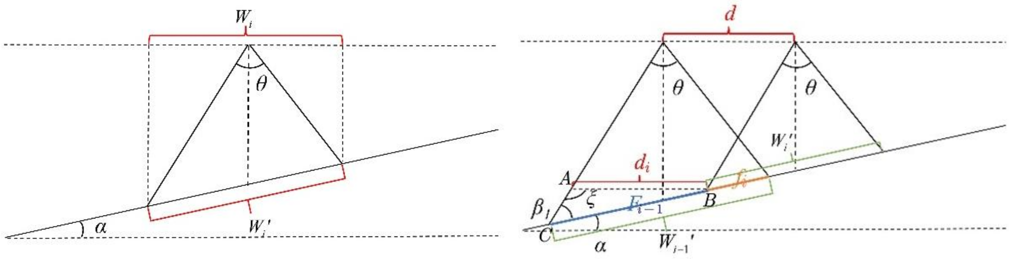

Figure 4 Schematic diagram of the overlap rate

From the geometric relation and the sine theorem:

$$\frac{d}{\sin\beta_1}=\frac{F_{i-1}}{\sin\xi}, \tag{8}$$

where $\xi=\pi-(\frac{\pi}{2}-\frac{\theta}{2})$. The length of the overlapping part is:

$$f_i=W_{i-1}'-F_{i-1}. \tag{9}$$

The overlap ratio is the length of the overlap divided by the width of the coverage multiplied by 100%, i.e.:

$$\eta=\frac{f_i}{W_i'}\times 100\%. \tag{10}$$

II. In three-dimensional space

i. Geometric foundations

Let the angle between the line where the coverage width is located and the horizontal plane be α_1.

The situation in three-dimensional space is simplified to Figure 5, where AB is perpendicular to horizontal plane BCD, $BD\perp DC$, plane ACD is the submarine slope, $BF\perp AD$, $DE\perp BC$, $\angle ADB=\alpha$, $\angle ACB=\alpha_1$.

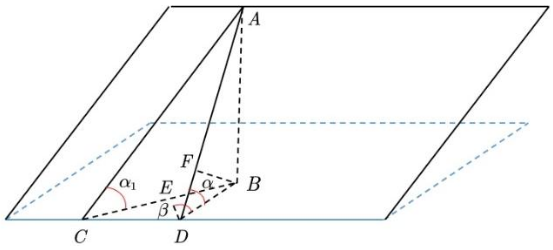

Figure 5 Geometric diagram of the submarine slope when it is an inclined plane

According to the geometric relationship, it is known that $DE \perp$ plane ABC, so $\overrightarrow{DE}$ is the direction vector of the survey line. Since $BF \perp$ plane ACD, $\overrightarrow{BF}$ is the normal vector of the seabed slope. BD is the projection of $\overrightarrow{BF}$ on the horizontal plane, $\angle BDE = \beta$. If $AB = 1$, then in the right triangle ABC, $BC = \cot\alpha_1$. In the right triangle ABD, $BD = \cot\alpha$. And in the right triangle BDE, $\angle EBD = 90° - \beta$.

In the right triangle BDC:

$$\cos\angle EBD = \sin\beta = \frac{BD}{BC} = \frac{\cot\alpha}{\cot\alpha_1}, \tag{11}$$

that is $\tan\alpha_1 = \sin\beta \cdot \tan\alpha$.

ii. Actual sea area model

In a rectangular sea area, let the angle between the direction of the survey line and the normal direction of the seabed slope projection on the horizontal plane be β, the opening angle of the multi-beam transducer is θ, the angle between the seabed slope and the horizontal plane is α, and the seawater depth at the center point of the sea area is h_0 meters.

Let's take $\beta = 90^\circ$ as a demarcation, and discuss it in three cases below.

1) When $90° < \beta < 270°$

Let the distance from the measurement location to the center of the sea area be denoted as *r*. When $90° < \beta < 270°$, the water depth decreases and the coverage width decreases along the direction of the upward slope of the seabed. The expression for $\tan\alpha_1$ has been obtained previously. The situation at this moment is the same as that in the two-dimensional plane, and after arrangement, the expression for the coverage width is as follows:

$$W = \left\{ \frac{(h_0 - r \cdot \tan\alpha_1)\sin\frac{\theta}{2}}{\sin\beta_1} + \frac{(h_0 - r \cdot \tan\alpha_1)\sin\frac{\theta}{2}}{\sin\gamma} \right\} \cdot \cos\alpha . \tag{12}$$

Here, β_1 and γ are the same as those in the two-dimensional plane.

When $\beta = 180^\circ$, it is a special case. At this time, $\alpha_1 = \alpha$, that is, $\tan\alpha_1 = \tan\alpha$.

2) When $0° \le \beta < 90°$ or $270° < \beta < 360°$

In this case, the water depth increases and the coverage width increases along the direction of the downward slope of the seabed. Transform it into the scenario in the two-dimensional plane, and after arrangement, the expression for the coverage width at this time is:

$$W = \left\{ \frac{(h_0 + r \cdot \tan\alpha_1)\sin\frac{\theta}{2}}{\sin\beta_1} + \frac{(h_0 + r \cdot \tan\alpha_1)\sin\frac{\theta}{2}}{\sin\gamma} \right\} \cdot \cos\alpha . \tag{13}$$

When $\beta = 0^\circ$, it is a special case, which is the same as the case when $\beta = 180^\circ$. At this time, $\alpha_1 = \alpha$, that is, $\tan\alpha_1 = \tan\alpha$.

3) When $\beta = 90^\circ$ or $\beta = 270^\circ$

In this case, the covering widths are all the same. Here, we take $\beta = 90^\circ$ and $r = 0$ as an example. At this point, the plane where the beam is located is the same as that in Figure 2. After arrangement, the coverage width is obtained as:

$$W = \left\{ \frac{h_0 \cdot \sin\frac{\theta}{2}}{\sin\beta_1} + \frac{h_0 \cdot \sin\frac{\theta}{2}}{\sin\gamma} \right\} \cdot \cos\alpha. \quad (14)$$

Summary

This paper establish a model to serve the line design of multi-beam bathymetry. Through theoretical analysis and from a geometric perspective, a relationship model among the seawater depth, coverage width, and overlap rate in both the two-dimensional plane and three-dimensional space has been established. It is analyzed by using basic mathematical knowledge, which has a solid theoretical foundation and is relatively easy to understand.

Acknowledgement

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