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Markov Birth–Death Models for Reliability Analysis of Repairable Systems

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Abstract

This study investigates the application of Markov birth–death models to the reliability assessment of repairable systems with redundancy operating under constant failure and repair rates. The systems considered are composed of identical units, with the assumption that no further failures occur while the system is in the down state. Mathematical models are developed for various system configurations, and analytical expressions are obtained for key reliability measures, including the stationary availability coefficient, mean time to failure (MTTF), mean time between failures (MTBF), and steady-state probabilities. Based on the derived formulas for MTTF and MTBF, as well as for the stationary availability coefficient, graphical dependencies on the input parameters are constructed and analyzed.

Keywords: *Markov birth–death models, reliability analysis, repairable systems with redundancy, series systems, parallel systems*

1. Introduction

Reliability analysis of repairable systems plays a crucial role in the design, maintenance, and operation of modern engineering systems. Such systems, which can be restored to working condition after a failure, are common in a wide range of applications, including telecommunications, transportation, energy, and manufacturing. Accurate modeling of their failure and repair behavior is essential for predicting system availability, optimizing maintenance strategies, and reducing operational costs.

Reliability and availability analysis of repairable systems is generally performed using stochastic processes, including Markov, semi-Markov, and semi-regenerative processes [1]. One of the most widely used mathematical tools for modeling the stochastic behavior of repairable systems is the class of Markov processes, particularly birth–death processes [2]. These processes offer a tractable and intuitive framework for representing systems that transition between discrete states—such as operational, degraded, or failed – under random failure and repair events. The birth–death paradigm, in particular, allows for elegant analytical treatment of such transitions, where "births" represent the completion of repair actions, resulting in transitions to states with higher operational capacity, and "deaths" correspond to failure events, leading to transitions to lower-capacity states.

Birth-death stochastic process is one of the most important special cases of the continuous-time homogenous Markov process where the states represent the current size of a population. This process has many applications in queuing theory, reliability engineering, demography, biology and other areas [3-6].

In paper [7], Markov birth–death processes with constant transition intensities between neighboring states and possessing the ergodic property are considered. By utilizing the properties of exponential distributions, formulas are derived for the mean time of transition from state i to state j and for the reverse transitions from state j to state i, the mean time spent outside a given state i, the mean time spent in the group of states $(0, ..., i-1)$ to the left of state i, and the mean time spent in the group of states $(i+1, i+2, ...)$ to the right. The results obtained in [7] serve as the foundation for the present analysis.

This paper presents a study of Markov birth–death models applied to the reliability analysis of repairable systems with constant failure and repair rates. It is assumed that the system consists of identical units and that no additional failures occur while the system is in the down state.

Several system configurations are modeled, and key reliability metrics are derived, including the stationary availability coefficient, mean time to failure (MTTF), mean time between failures (MTBF), and steady-state probabilities.

2. Basic Definitions and Assumptions

2.1. The Birth-Death Process with a Finite Number of States

Let us denote states by natural numbers 0, 1, 2, ... and assume that the intensities λ_k, μ_i are constant. We consider a birth-death process with a finite number of states, illustrated by the state-transition graph in Figure 1.

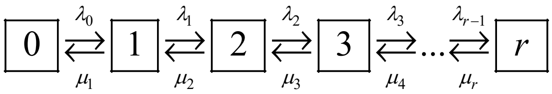

Figure 1. Transition graph for the birth-death process with a finite number of states

It is well known that the distribution of the time intervals between any two successive jumps in any Markov process with continuous time and discrete space of states is exponential. More precisely, let W_i be the instant of the ith jump of the birth-death process $Z(t)$ and $\tau_i = W_{i+1} - W_i$ be the sojourn time; suppose that $Z(W_i) = k,$ then the process spends exponentially distributed time τ_i in the state $Z(t) = k$ with the mean $E(\tau_i) = 1/\left(\lambda_k + \mu_k\right).$ When a jump occurs, the process moves to state $Z(t) = k+1$ with probability $\lambda_k / \left(\lambda_k + \mu_k\right),$ or to state $Z(t) = k-1$ with probability $\mu_k / \left(\lambda_k + \mu_k\right).$

Let us introduce notation T_{ij} to denote the mean time from the instant the system comes to state i to the transition to state j. The following equations hold [7]:

$$\lambda_k p_k = \mu_{k+1} p_{k+1}, \quad p_{k+1} = p_0 \prod_{i=0}^{k} \frac{\lambda_i}{\mu_{i+1}}, \quad 0 \le k \le r-1; \quad \sum_{k=0}^{r} p_k = 1.$$

Here p_k is the steady-state probability of the system being in the state $k.$ Thus, we have

$$p_0 = \left(1 + \sum_{k=0}^{r-1} P(0,k)\right)^{-1}, \quad p_k = p_0 P(0,k-1), \quad 1 \le k \le r; \; P(u,v) = \prod_{i=u}^{v} \frac{\lambda_i}{\mu_{i+1}}. \tag{1}$$

Let us denote by $T_{out(k-)}$ and $T_{out(k+)}$ the mean time the process spends in the group of states $(0,1,...,k-1)$ to the left of state $k,$ and in the group of states $(k+1,k+2,...,r)$ to the right of state $k,$ respectively. Then the following expression holds [7]:

$$T_{out(k-)} = \frac{1+\sum_{s=0}^{k-2} P(0,s)}{\mu_k P(0,k-1)}, \quad 1 \le k \le r; \quad T_{out(k+)} = \frac{1}{\lambda_k}\sum_{s=k}^{r-1} P(k,s), \quad 0 \le k \le r-1;$$

$$T_{ij} = \sum_{k=i+1}^{j} T_{out(k-)} = \sum_{k=i+1}^{j} \frac{1+\sum_{s=0}^{k-2} P(0,s)}{\mu_k P(0,k-1)}, \quad 0 \le i < j; \tag{2}$$

$$T_{ji} = \sum_{k=i}^{j-1} T_{out(k+)} = \sum_{k=i}^{j-1} \frac{1}{\lambda_k}\sum_{s=k}^{r-1} P(k,s), \quad 0 \le i < j \le r.$$

2.2. Repairable Systems

Consider a repairable system that consists of $r = m + c$ identical units, namely, m main operating units and c unloaded redundant units. At any given moment, a unit can be in one of two states: operational or failed. Suppose that the number n of repair facilities is limited, so failed units necessarily form a queue for repair. Assuming that the time to failure X and the repair time Y for each unit follow exponential distributions with parameters λ and μ, respectively, our task is to determine the reliability indices of the system.

The system states are numbered by non-negative integers 0, 1, 2, ..., such that each state number corresponds to the number of failed units.

In this paper, we consider parallel repairable systems (see the transition graphs in Figures 2 and 3) and series repairable systems (Figure 4). A parallel repairable system continues to operate as long as at least one unit remains functional. In a series repairable system, the entire system fails if any single unit fails, as all components must function simultaneously.

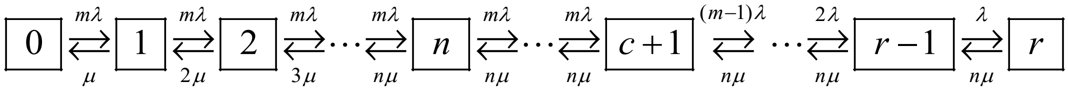

Figure 2. Transition graph for the parallel repairable system (case of *n*<*c*)

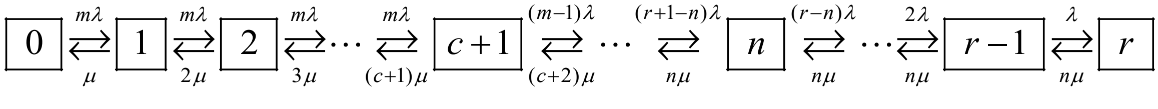

Figure 3. Transition graph for the parallel repairable system (case of *n*>*c*)

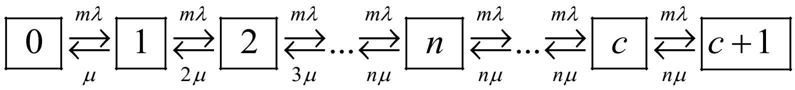

Figure 4. Transition graph for the series repairable system

For these systems, the time to failure (X_S) and the time between failures (X_{SB}) do not coincide, because the interval X_S begins at the moment of transition from state 1 to state 0, whereas X_{SB} begins at the transition from state r to state $r-1$ for parallel systems, and from state $c+1$ to state c for series systems. Both intervals end simultaneously: at the transition from state $r-1$ to state r for parallel systems, and from state c to state $c+1$ for series systems. The system downtime (X_{SD}) corresponds to the time spent in state r for parallel systems and in state $c+1$ for series systems; therefore, $E(X_{SD}) = 1/(n\mu)$.

The system's stationary availability coefficient is given by the formula

$$K = 1 - p_N = \frac{E(X_{SB})}{E(X_{SB}) + E(X_{SD})},$$

where $N = r$ for parallel repairable systems and $N = c+1$ for series repairable systems.

3. Formulas for Reliability Metrics of Repairable Systems

3.1. Parallel Repairable Systems (Case of *n*<*c*)

The number of failed units forms a birth-death process with a finite number of states and the following transition intensities (see Figure 2):

$$\lambda_k = m\lambda,\ \ 0 \le k \le c; \qquad \lambda_k = (m + c - k)\lambda,\ \ c+1 \le k \le r-1;$$
$$\mu_k = k\mu,\ \ 1 \le k \le n < c; \qquad \mu_k = n\mu,\ \ n+1 \le k \le r.$$

Let $\rho = \lambda / \mu$. With the help of (1) and (2), we obtain:

$$p_0 = \frac{1}{\sum_{s=0}^{n} \frac{(m\rho)^s}{s!} + \sum_{s=n+1}^{c+1} \frac{(m\rho)^s}{n!n^{s-n}} + \sum_{s=c+2}^{r} \frac{(m-1)!m^{c+1}\rho^s}{n!(r-s)!n^{s-n}}}; \quad p_k = p_0 \frac{(m\rho)^k}{k!},\ \ 1 \le k \le n;$$

$$p_k = p_0 \frac{(m\rho)^k}{n!n^{k-n}},\ \ n+1 \le k \le c+1; \qquad p_k = p_0 \frac{(m-1)!m^{c+1}\rho^k}{n!(r-k)!n^{k-n}},\ \ c+2 \le k \le r;$$

$$E(X_S)=T_{0,r}=\sum_{k=1}^{r}T_{out(k-)}=\frac{1}{\mu}\sum_{k=1}^{n}\frac{(k-1)!}{(m\rho)^k}\sum_{s=0}^{k-1}\frac{(m\rho)^s}{s!}+$$
$$+\frac{1}{\mu}\sum_{k=n+1}^{c+1}\frac{(n-1)!n^{k-n}}{(m\rho)^k}\left(\sum_{s=0}^{n}\frac{(m\rho)^s}{s!}+\sum_{s=n+1}^{k-1}\frac{(m\rho)^s}{n!n^{s-n}}\right)+$$
$$+\frac{1}{\mu}\sum_{k=c+2}^{r}\frac{(n-1)!(r-k)!n^{k-n}}{(m-1)!m^{c+1}\rho^k}\left(\sum_{s=0}^{n}\frac{(m\rho)^s}{s!}+\sum_{s=n+1}^{c+1}\frac{(m\rho)^s}{n!n^{s-n}}+\sum_{s=c+2}^{k-1}\frac{(m-1)!m^{c+1}\rho^s}{n!(r-s)!n^{s-n}}\right);$$
$$E(X_{SB})=T_{r-1,r}=T_{out(r-)}=$$
$$=\frac{(n-1)!n^{r-n}}{(m-1)!m^{c+1}\rho^r\mu}\left(\sum_{s=0}^{n}\frac{(m\rho)^s}{s!}+\sum_{s=n+1}^{c+1}\frac{(m\rho)^s}{n!n^{s-n}}+\sum_{s=c+2}^{r-1}\frac{(m-1)!m^{c+1}\rho^s}{n!(r-s)!n^{s-n}}\right).$$

3.2. Parallel Repairable Systems (Case of *n*>*c*)

The number of failed units forms a birth-death process with a finite number of states and the following transition intensities (see Figure 3):

$$\lambda_k=m\lambda,\ \ 0\le k\le c<n;\qquad \lambda_k=(m+c-k)\lambda,\ \ c+1\le k\le r-1;$$
$$\mu_k=k\mu,\ \ 1\le k\le n;\qquad \mu_k=n\mu,\ \ n+1\le k\le r.$$

With the help of (1) and (2), we obtain:

$$p_0=\frac{1}{\sum_{s=0}^{c+1}\frac{(m\rho)^s}{s!}+\sum_{s=c+2}^{n}\frac{(m-1)!m^{c+1}\rho^s}{s!(r-s)!}+\sum_{s=n+1}^{r}\frac{(m-1)!m^{c+1}\rho^s}{n!(r-s)!n^{s-n}}};\quad p_k=p_0\frac{(m\rho)^k}{k!},\ \ 1\le k\le c+1;$$
$$p_k=p_0\frac{(m-1)!m^{c+1}\rho^k}{k!(r-k)!},\ \ c+2\le k\le n;\qquad p_k=p_0\frac{(m-1)!m^{c+1}\rho^k}{n!(r-k)!n^{k-n}},\ \ n+1\le k\le r;$$

$$1\le n\le r\Rightarrow$$
$$E(X_S)=T_{0,r}=\sum_{k=1}^{r}T_{out(k-)}=\frac{1}{\mu}\sum_{k=1}^{c+1}\frac{(k-1)!}{(m\rho)^k}\sum_{s=0}^{k-1}\frac{(m\rho)^s}{s!}+$$
$$+\frac{1}{\mu}\sum_{k=c+2}^{n}\frac{(k-1)!(r-k)!}{(m-1)!m^{c+1}\rho^k}\left(\sum_{s=0}^{c+1}\frac{(m\rho)^s}{s!}+\sum_{s=c+2}^{k-1}\frac{(m-1)!m^{c+1}\rho^s}{s!(r-s)!}\right)+$$
$$+\frac{1}{\mu}\sum_{k=n+1}^{r}\frac{(n-1)!(r-k)!n^{k-n}}{(m-1)!m^{c+1}\rho^k}\left(\sum_{s=0}^{c+1}\frac{(m\rho)^s}{s!}+\sum_{s=c+2}^{n}\frac{(m-1)!m^{c+1}\rho^s}{s!(r-s)!}+\sum_{s=n+1}^{k-1}\frac{(m-1)!m^{c+1}\rho^s}{n!(r-s)!n^{s-n}}\right);$$
$$E(X_{SB})=T_{r-1,r}=T_{out(r-)}=$$
$$=\frac{(n-1)!n^{r-n}}{(m-1)!m^{c+1}\rho^r\mu}\left(\sum_{s=0}^{c+1}\frac{(m\rho)^s}{s!}+\sum_{s=c+2}^{n-1}\frac{(m-1)!m^{c+1}\rho^s}{s!(r-s)!}+\sum_{s=n}^{r-1}\frac{(m-1)!m^{c+1}\rho^s}{n!(r-s)!n^{s-n}}\right).$$

The values of $E(X_S)$ for $n=r-1$ and $n=r$ are the same, as are the values of $E(X_{SB})$.

In the case of $c=0,$ the above formulas yield the expressions for the reliability metrics of a non-redundant repairable system:

$$p_0=\frac{1}{\sum_{s=0}^{n}C_m^s\rho^s+\sum_{s=n+1}^{m}\frac{m!\rho^s}{n!(m-s)!n^{s-n}}};\qquad C_m^k=\frac{m!}{k!(m-k)!};$$

$$p_k=p_0C_m^k\rho^k,\ \ 1\le k\le n;\qquad p_k=p_0\frac{m!\rho^k}{n!(m-k)!n^{k-n}},\quad n+1\le k\le m;$$

$$E(X_S)=\frac{1}{m!\mu}\sum_{k=1}^{n}\frac{(k-1)!(m-k)!}{\rho^k}\sum_{s=0}^{k-1}C_m^s\rho^s+$$
$$+\frac{(n-1)!}{m!\mu}\sum_{k=n+1}^{m}\frac{(m-k)!n^{k-n}}{\rho^k}\left(\sum_{s=0}^{n}C_m^s\rho^s+\sum_{s=n+1}^{k-1}\frac{m!\,\rho^s}{n!(m-s)!n^{s-n}}\right);$$
$$E(X_{SB})=\frac{(n-1)!n^{m-n}}{m!\mu\rho^m}\left(\sum_{s=0}^{n}C_m^s\rho^s+\sum_{s=n+1}^{m-1}\frac{m!\,\rho^s}{n!(m-s)!n^{s-n}}\right).$$

3.3. Parallel Repairable Systems (Case of *n*=*c*)

In the case of $n=c,$ we have:

$$p_0=\frac{1}{\sum_{s=0}^{c}\frac{(m\rho)^s}{s!}+\sum_{s=c+1}^{r}\frac{(m-1)!m^{c+1}\rho^s}{c!(r-s)!c^{s-c}}};$$

$$p_k=p_0\frac{(m\rho)^k}{k!},\ \ 1\le k\le c;\qquad p_k=p_0\frac{(m-1)!m^{c+1}\rho^k}{c!(r-k)!c^{k-c}},\ \ c+1\le k\le r;$$

$$E(X_S)=\frac{1}{\mu}\sum_{k=1}^{c+1}\frac{(k-1)!}{(m\rho)^k}\sum_{s=0}^{k-1}\frac{(m\rho)^s}{s!}+$$
$$+\frac{1}{\mu}\sum_{k=c+2}^{r}\frac{(c-1)!(r-k)!c^{k-c}}{(m-1)!m^{c+1}\rho^k}\left(\sum_{s=0}^{c+1}\frac{(m\rho)^s}{s!}+\sum_{s=c+2}^{k-1}\frac{(m-1)!m^{c+1}\rho^s}{c!(r-s)!c^{s-c}}\right);$$
$$E(X_{SB})=\frac{(c-1)!c^{r-c}}{(m-1)!m^{c+1}\rho^r\mu}\left(\sum_{s=0}^{c+1}\frac{(m\rho)^s}{s!}+\sum_{s=c+2}^{r-1}\frac{(m-1)!m^{c+1}\rho^s}{c!(r-s)!c^{s-c}}\right).$$

3.4. Series Repairable Systems

The number of failed units forms a birth-death process with a finite number of states and the following transition intensities (see Figure 4):

$$\lambda_k=m\lambda,\ \ 0\le k\le c;$$
$$\mu_k=k\mu,\ \ 1\le k\le n;\qquad \mu_k=n\mu,\ \ n+1\le k\le c+1.$$

With the help of (1) and (2), we obtain:

$$p_0 = \frac{1}{\sum_{s=0}^{n}\frac{(m\rho)^s}{s!} + \sum_{s=n+1}^{c+1}\frac{(m\rho)^s}{n!n^{s-n}}};$$

$$p_k = p_0\frac{(m\rho)^k}{k!},\quad 1 \le k \le n; \qquad p_k = p_0\frac{(m\rho)^k}{n!n^{k-n}},\quad n+1 \le k \le c+1; \tag{3}$$

$1 \le n \le c+1 \Rightarrow$

$$E(X_S) = T_{0,c+1} = \sum_{k=1}^{c+1} T_{out(k-)} = \frac{1}{\mu}\sum_{k=1}^{n}\frac{(k-1)!}{(m\rho)^k}\sum_{s=0}^{k-1}\frac{(m\rho)^s}{s!} + \\ + \frac{(n-1)!}{\mu}\sum_{k=n+1}^{c+1}\frac{n^{k-n}}{(m\rho)^k}\left(\sum_{s=0}^{n}\frac{(m\rho)^s}{s!} + \sum_{s=n+1}^{k-1}\frac{(m\rho)^s}{n!n^{s-n}}\right); \tag{4}$$

$$E(X_{SB}) = T_{c,c+1} = T_{out(c+1-)} = \frac{(n-1)!n^{c+1-n}}{\mu(m\rho)^{c+1}}\left(\sum_{s=0}^{n-1}\frac{(m\rho)^s}{s!} + \sum_{s=n}^{c}\frac{(m\rho)^s}{n!n^{s-n}}\right).$$

The values of $E(X_S)$ for $n = c$ and $n = c+1$ are the same, as are the values of $E(X_{SB})$.

To obtain the corresponding formulas for a single-unit repairable system with redundancy from (3) and (4), it is sufficient to set $m = 1$.

4. Graphical Analysis of Series Repairable System Metrics

4.1. Analysis of a Repairable System with one main operating unit

Since the mean time to failure ($E(X_S)$) and the mean time between failures ($E(X_{SB})$) depend not only on ρ but also on μ, it is convenient to consider the dependencies of $\mu E(X_S)$ and $\mu E(X_{SB})$ on ρ.

For a repairable system with one main operating unit and one unloaded redundant unit, the values of $\mu E(X_S)$ and $\mu E(X_{SB})$ increase significantly as ρ approaches zero, but decrease rapidly as ρ approaches one (see Figures 5–8).

An increase in the number of unloaded redundant units results in improved system metrics (see Figures 9–12).

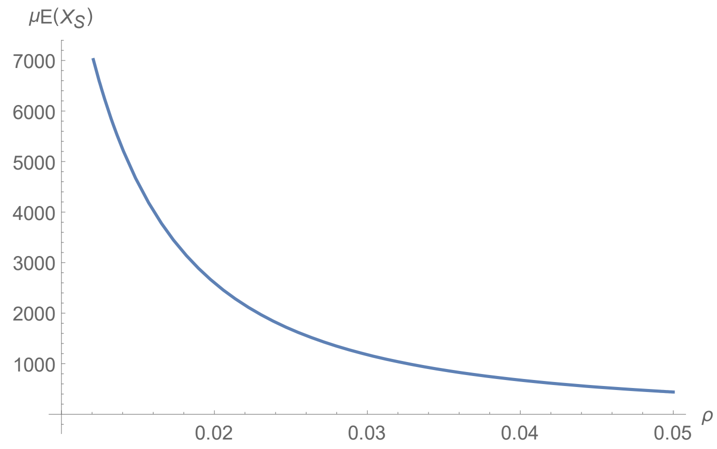

Figure 5. Dependence of $\mu E(X_S)$ on ρ for a repairable system with one main operating unit and one unloaded redundant unit (case of m=n=c=1)

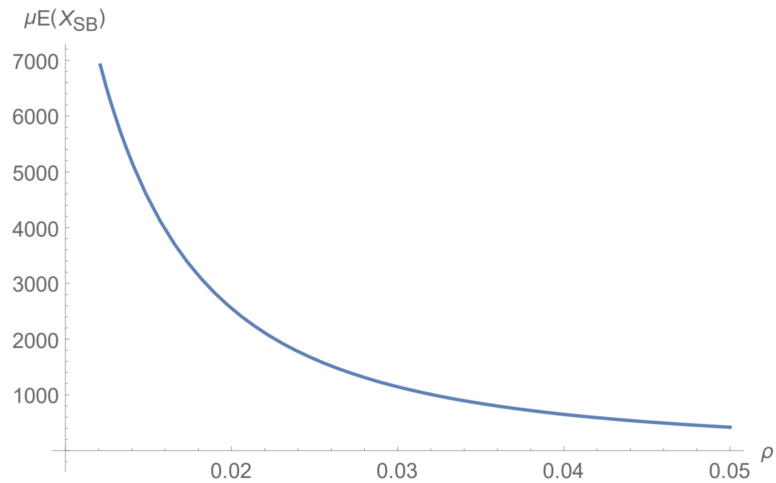

Figure 6. Dependence of $\mu E(X_{SB})$ on ρ for a repairable system with one main operating unit and one unloaded redundant unit (case of m=n=c=1)

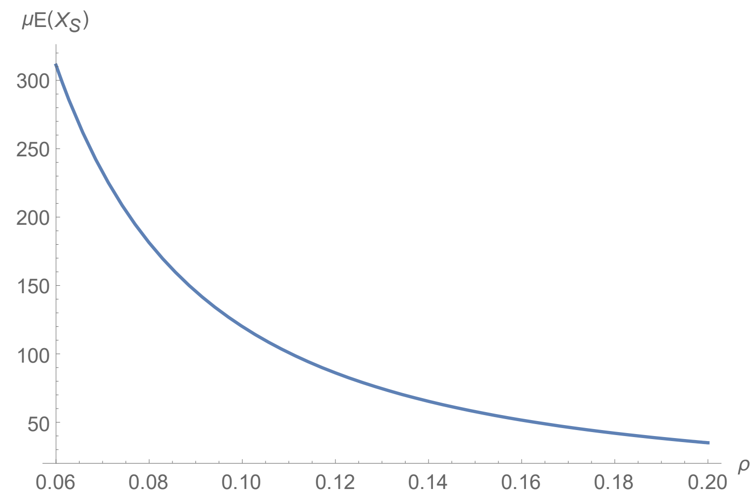

Figure 7. Dependence of $\mu E(X_S)$ on ρ for a repairable system with one main operating unit and one unloaded redundant unit (case of m=n=c=1)

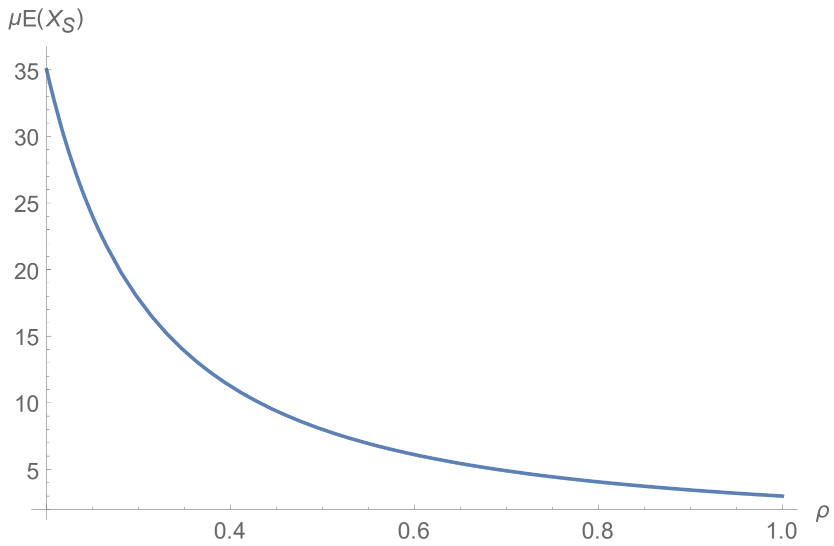

Figure 8. Dependence of $\mu E(X_S)$ on ρ for a repairable system with one main operating unit and one unloaded redundant unit (case of m=n=c=1)

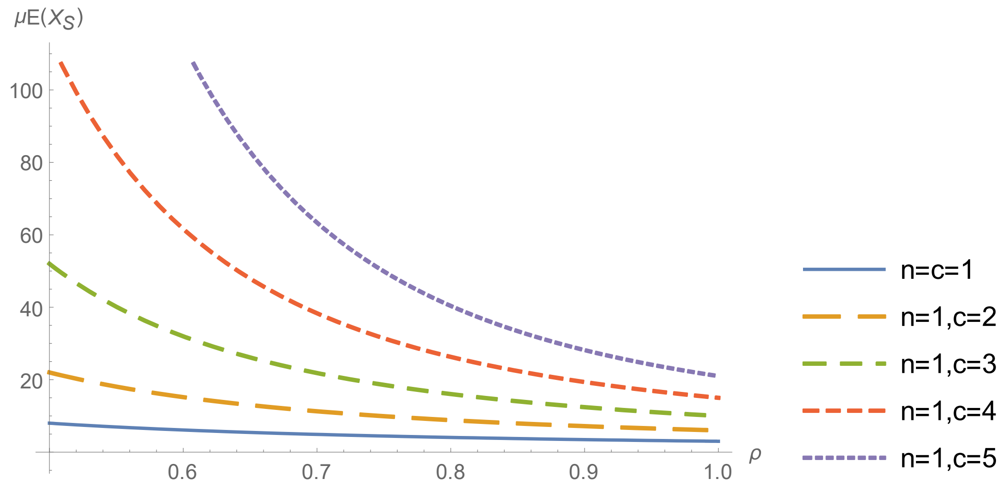

Figure 9. Dependencies of $\mu E(X_S)$ on ρ for a repairable system with one main operating unit and c unloaded redundant units (case of m=n=1 and various values of c)

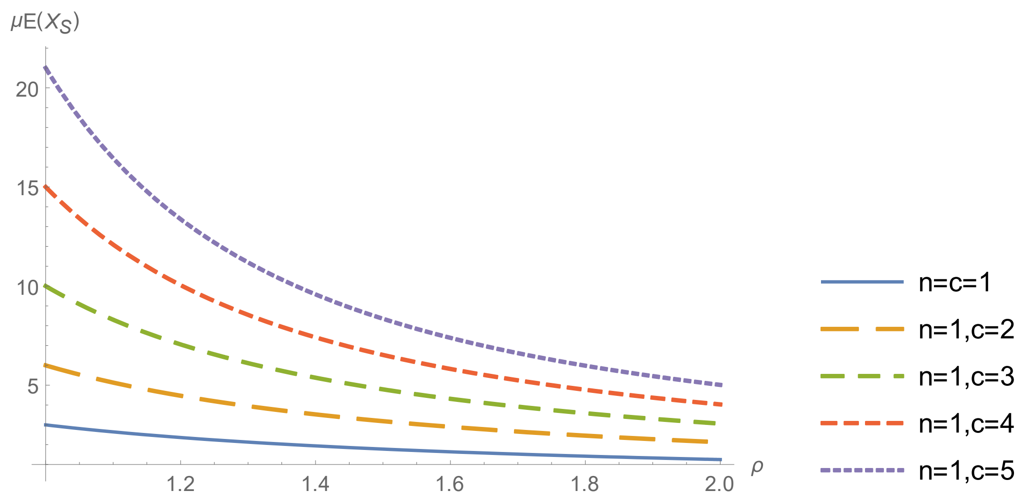

Figure 10. Dependencies of $\mu E(X_S)$ on ρ for a repairable system with one main operating unit and c unloaded redundant units (case of m=n=1 and various values of c)

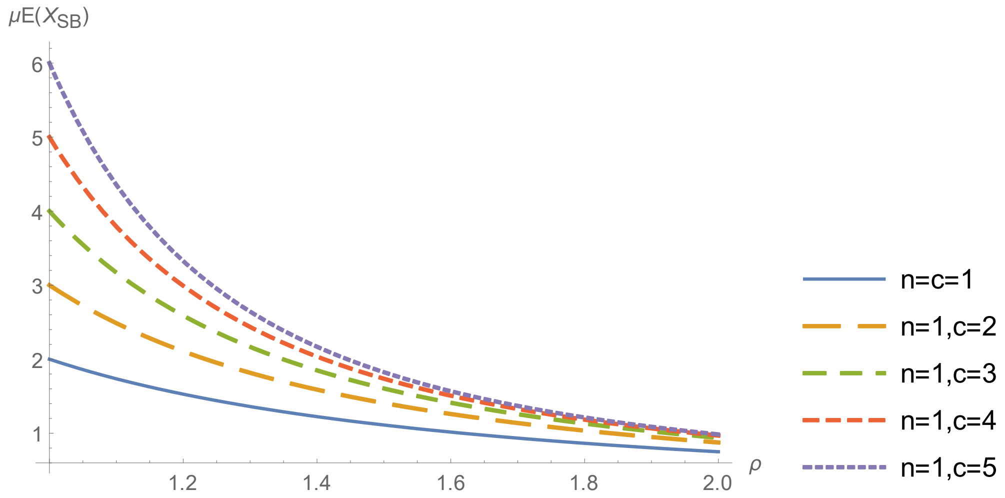

Figure 11. Dependencies of $\mu E(X_{SB})$ on ρ for a repairable system with one main operating unit and c unloaded redundant units (case of m=n=1 and various values of c)

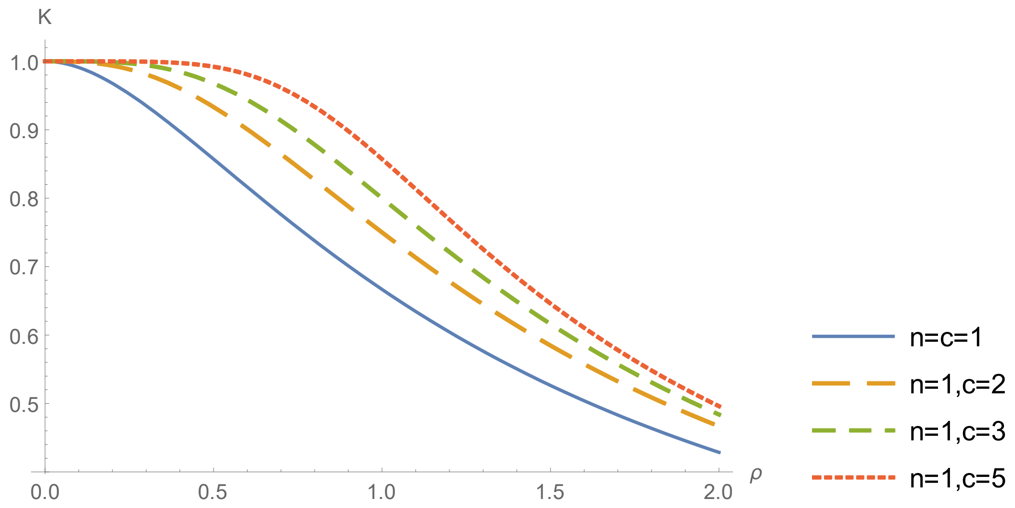

Figure 12. Dependencies of stationary availability coefficient K on ρ for a repairable system with one main operating unit and c unloaded redundant unit (case of m=n=1 and various values of c)

4.2. Analysis of the Impact of Increasing the Parameters *m*, *c*, and *n*

As the number of main operating units increases, system metrics deteriorate (see Figures 13-15). When the parameters m, c, and n are increased simultaneously, the values of K increase (see Figure 16), whereas the values of $\mu E(X_S)$ increase only for $\rho < 0.5$ and decrease for $\rho > 1$ (see Figures 17-19).

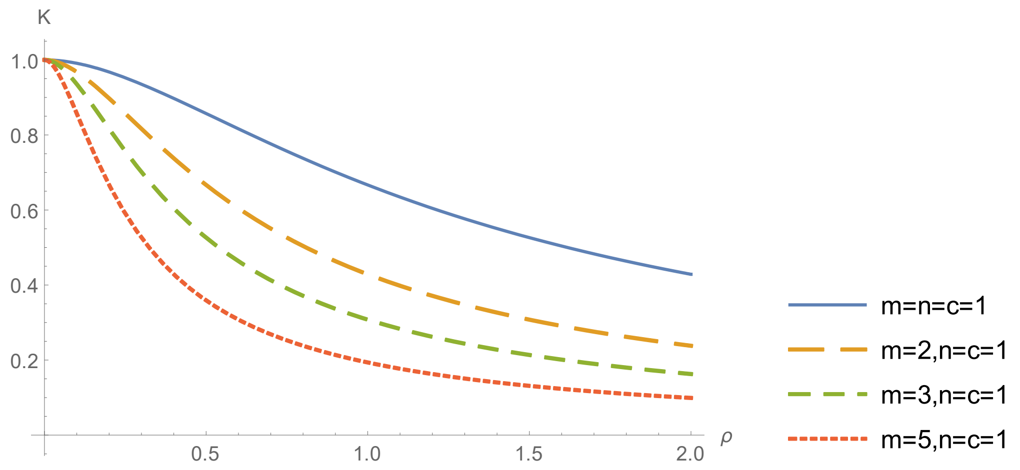

Figure 13. Dependencies of stationary availability coefficient K on ρ for a series repairable system with m main operating units and one unloaded redundant unit (case of n=1 and various values of m)

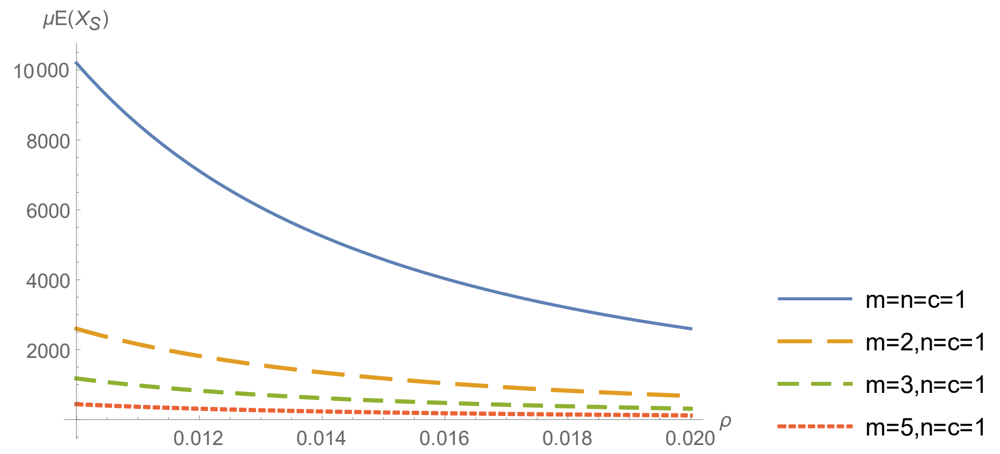

Figure 14. Dependencies of $\mu E(X_S)$ on ρ for a series repairable system with m operating units and one unloaded redundant unit (case of n=c=1 and various values of m)

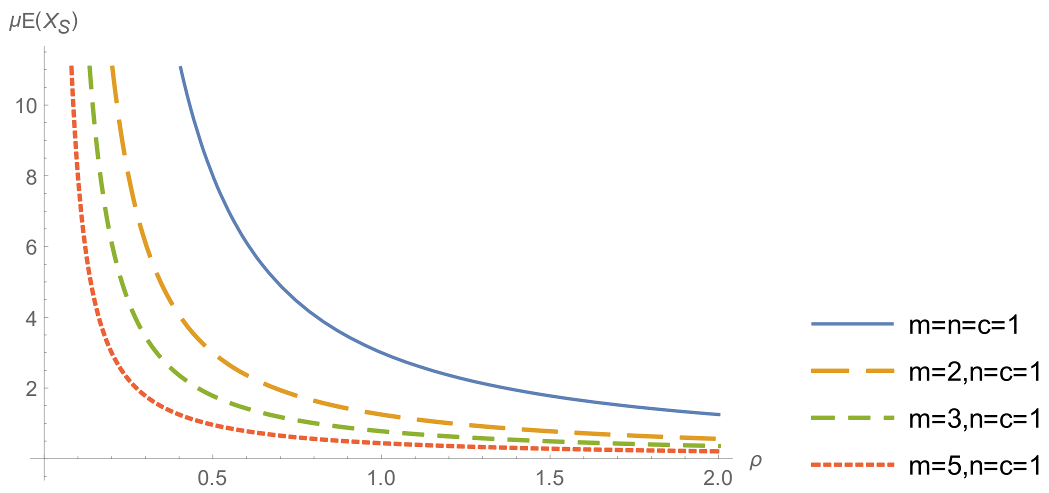

Figure 15. Dependencies of $\mu E(X_S)$ on ρ for a series repairable system with m operating units and one unloaded redundant unit (case of n=c=1 and various values of m)

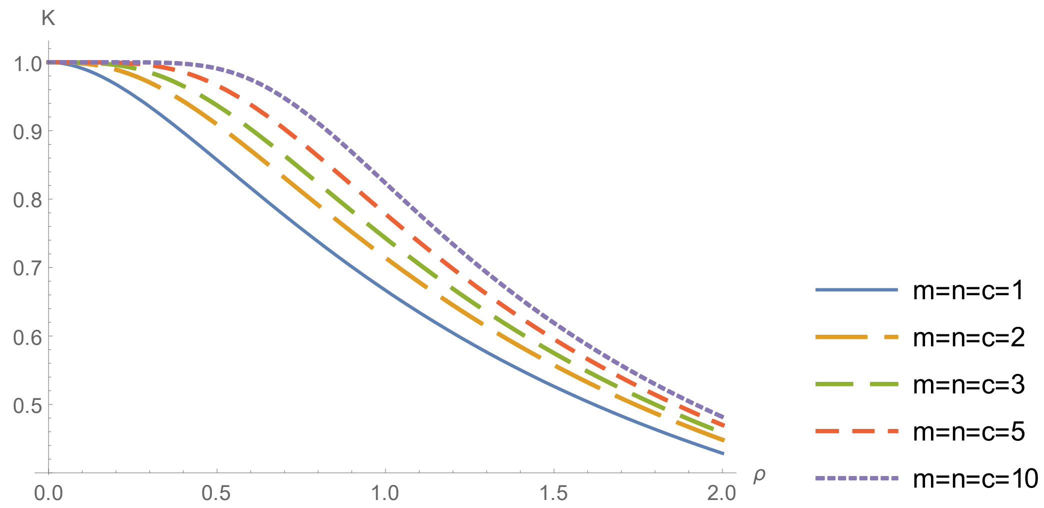

Figure 16. Dependencies of stationary availability coefficient K on ρ for a series repairable system (case of various values of m=c=n)

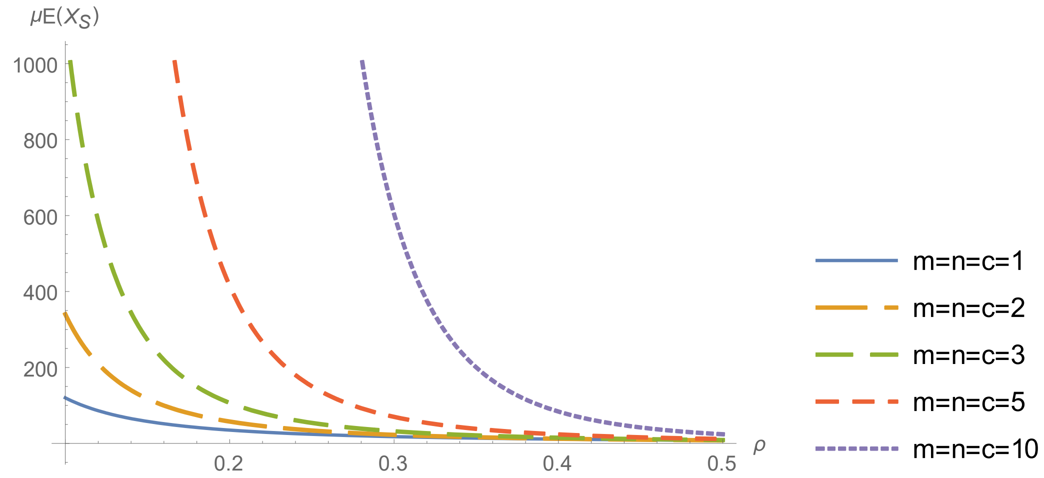

Figure 17. Dependencies of $\mu E(X_S)$ on ρ for a series repairable system (case of various values of m=c=n)

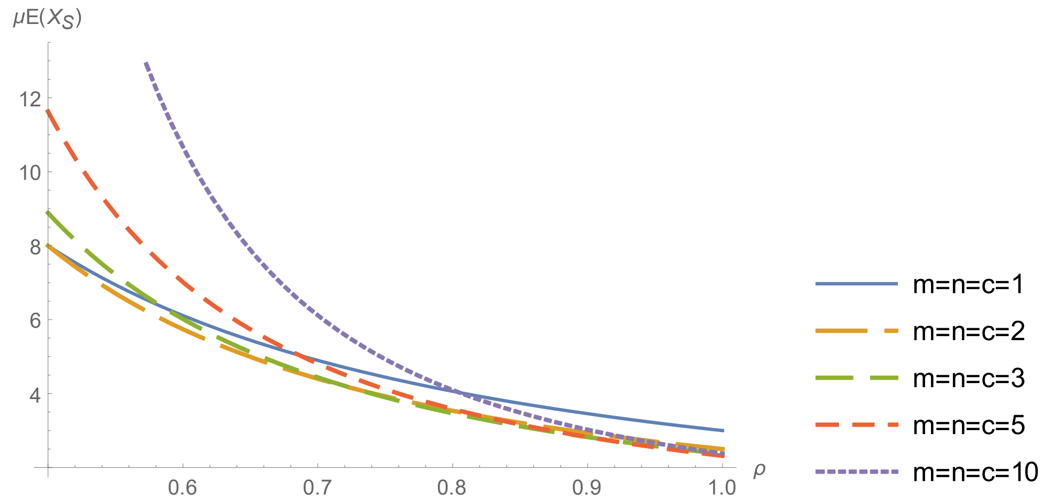

Figure 18. Dependencies of $\mu E(X_S)$ on ρ for a series repairable system (case of various values of m=c=n)

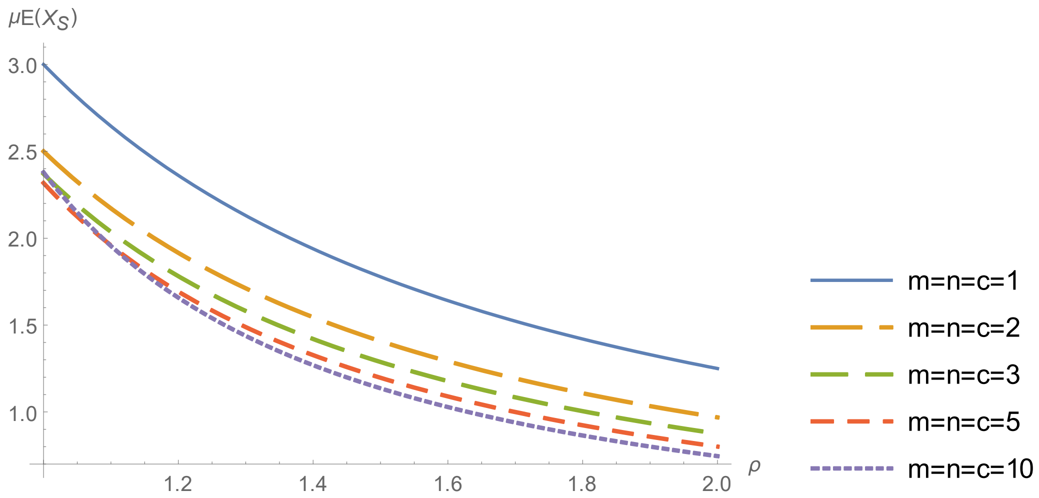

Figure 19. Dependencies of $\mu E(X_S)$ on ρ for a series repairable system (case of various values of m=c=n)

4.3. Comparison of Values $E(X_S)$ and $E(X_{SB})$

For all series repairable systems with $m \geq 1$ main operating units and unloaded redundant units, the ratio $E(X_{SB})/E(X_S)$ decreases as ρ increases. If only the number of main operating units increases while $c = n = 1$, the ratio $E(X_{SB})/E(X_S)$ also decreases (see Figure 20). A similar effect is observed when $m = n = 1$ and the number of unloaded redundant units c increases (Figure 21).

For a repairable system with one main operating unit, a simultaneous increase in c and n (i.e., $c = n$) leads to an increase in the ratio $E(X_{SB})/E(X_S)$ for $\rho < 0.5$, whereas for $\rho > 0.5$ the ratio increases only when c and n grow significantly (see Figure 22). As illustrated in Figure 23, in the case of a simultaneous increase in m, c, and n (i.e., $m = c = n$), the ratio $E(X_{SB})/E(X_S)$ decreases.

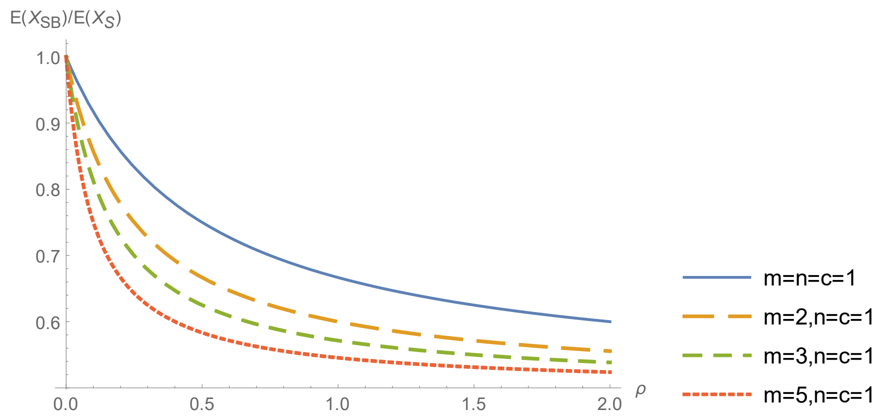

Figure 20. Dependencies of $E(X_{SB})/E(X_S)$ on ρ for a series repairable system with m operating units and one unloaded redundant unit (case of n=c=1 and various values of m)

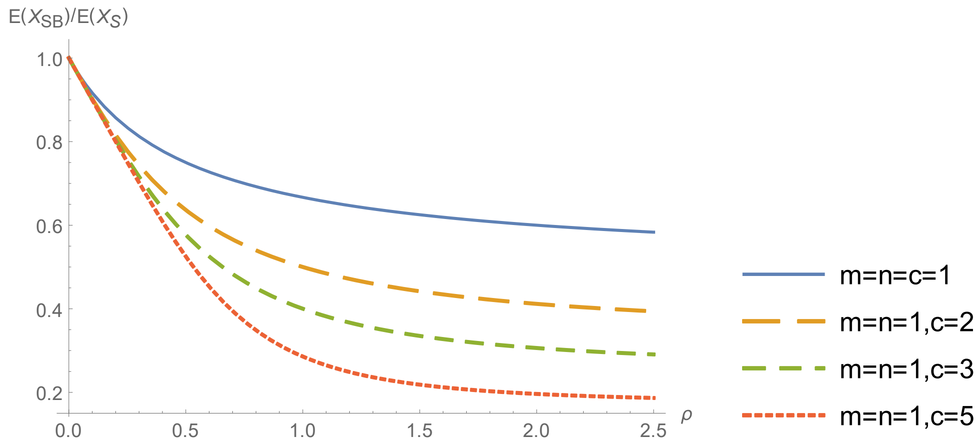

Figure 21. Dependencies of $E(X_{SB})/E(X_S)$ on ρ for a repairable system with one main operating unit and c unloaded redundant unit (case of m=n=1 and various values of c)

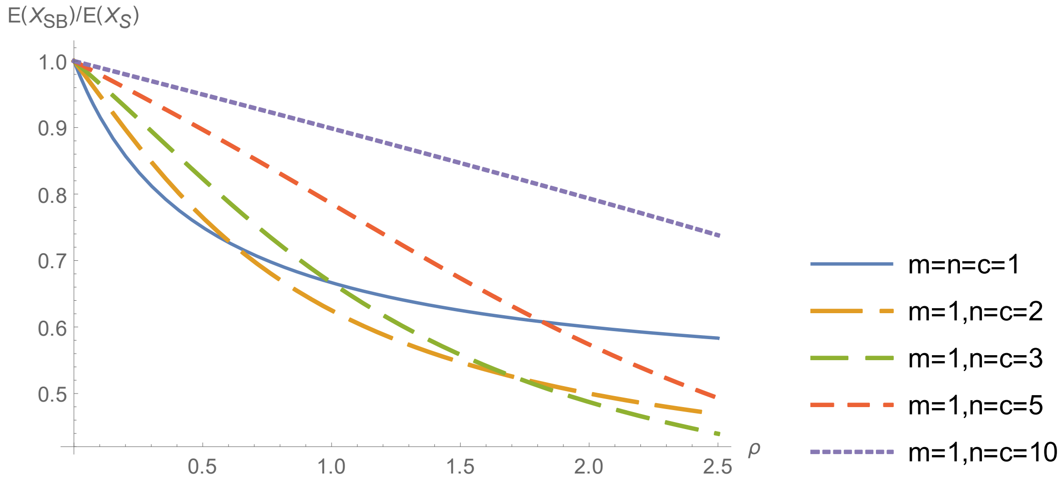

Figure 22. Dependencies of $E(X_{SB})/E(X_S)$ on ρ for a repairable system with one main operating unit and c unloaded redundant unit (case of m=1 and various values of n=c)

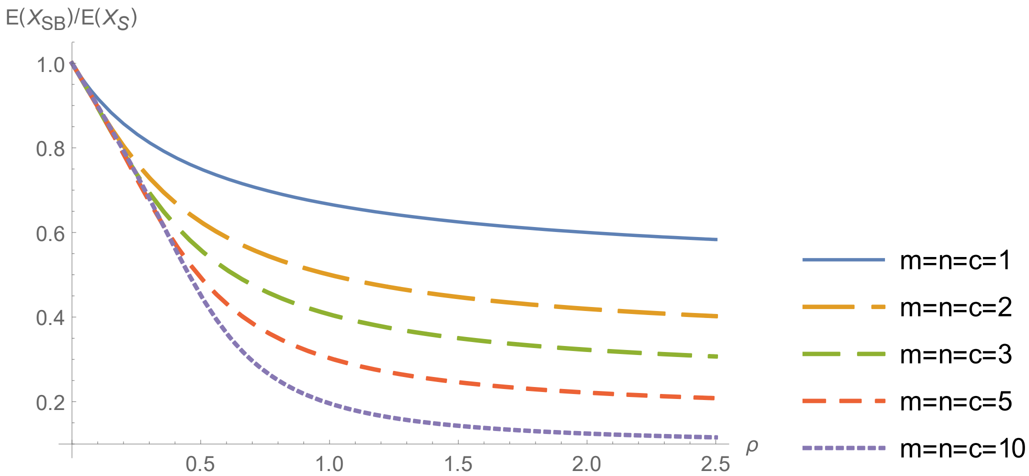

Figure 23. Dependencies of $E(X_{SB})/E(X_S)$ on ρ for a series repairable system (case of various values of m=c=n)

5. Graphical Analysis of Parallel Repairable System Metrics

5.1. Analysis of the Dependencies of $\mu E(X_S)$ and K on ρ, m, n, and c

As the number of main operating units increases, the values of $\mu E(X_S)$ and K increase for a non-redundant parallel system with one repair facility (see Figures 24 and 25). With further increases in m, the values of $\mu E(X_S)$ and K tend to stabilize (the curves showing the dependence of K on ρ for $m = 6, 10,$ and 14 nearly coincide). The stabilization of the values of $\mu E(X_S)$ and K is caused by an insufficient number of repair facilities.

A simultaneous increase in the parameters m and n, as well as in m, c, and n, leads to a substantial growth in K, which approaches 1 (Figures 26 and 27). As illustrated in Figure 28, in the case of a simultaneous increase in m, c, and n (i.e., $m = c = n$), the values of $\mu E(X_S)$ increase, but this growth slows down significantly for $\rho > 1$. As the number of main operating units increases while the sum $c + n$ remains constant, the values of $\mu E(X_S)$ and K increase (see Figures 29 and 30).

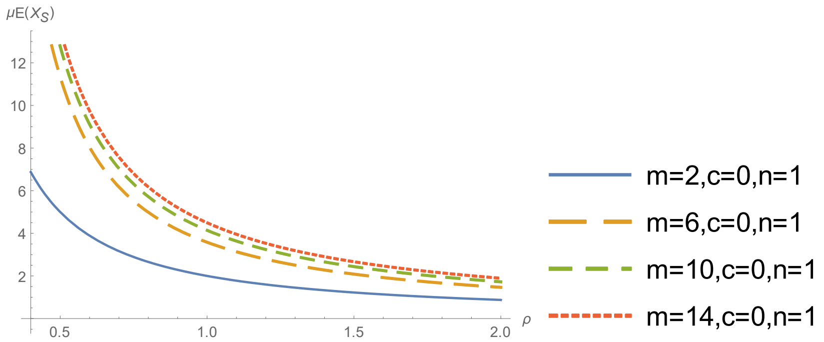

Figure 24. Dependencies of $\mu E(X_S)$ on ρ for a parallel non-redundant repairable system (case of c=0, n=1 and various values of m)

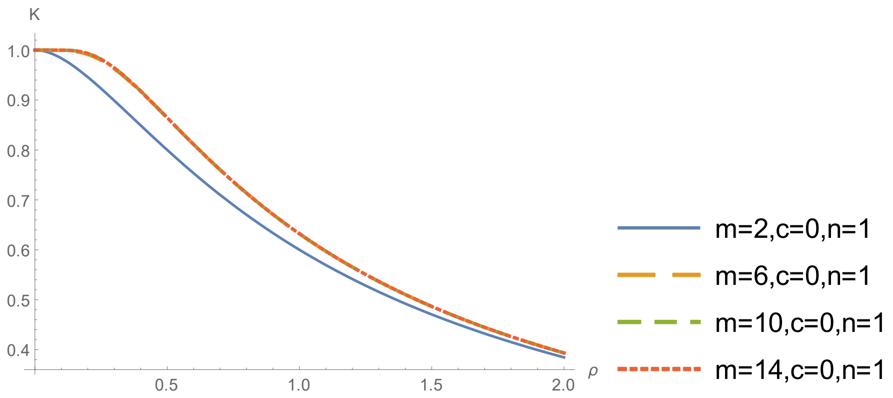

Figure 25. Dependencies of stationary availability coefficient K on ρ for a parallel non-redundant repairable system (case of c=0, n=1 and various values of m)

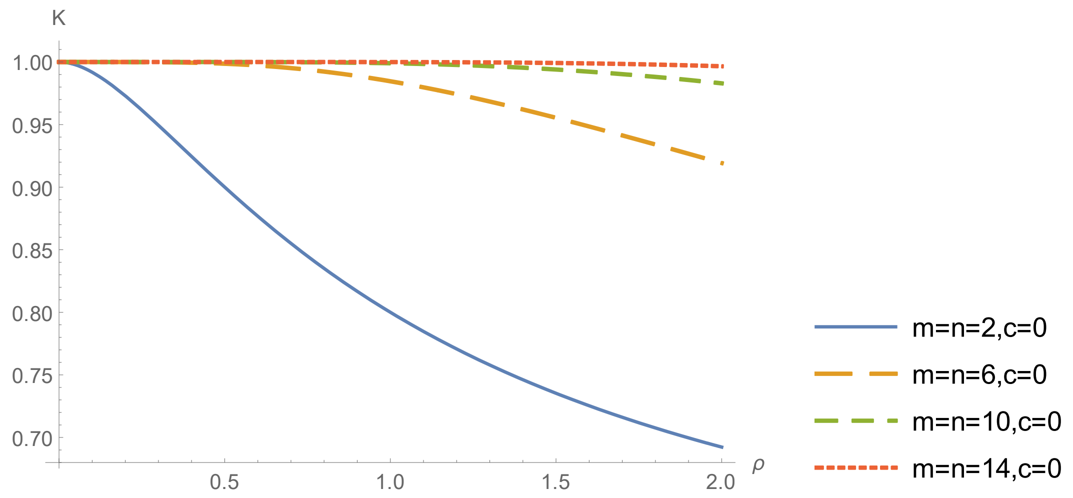

Figure 26. Dependencies of stationary availability coefficient *K* on *ρ* for a parallel non-redundant repairable system (case of *c*=0 and various values of *m*=*n*)

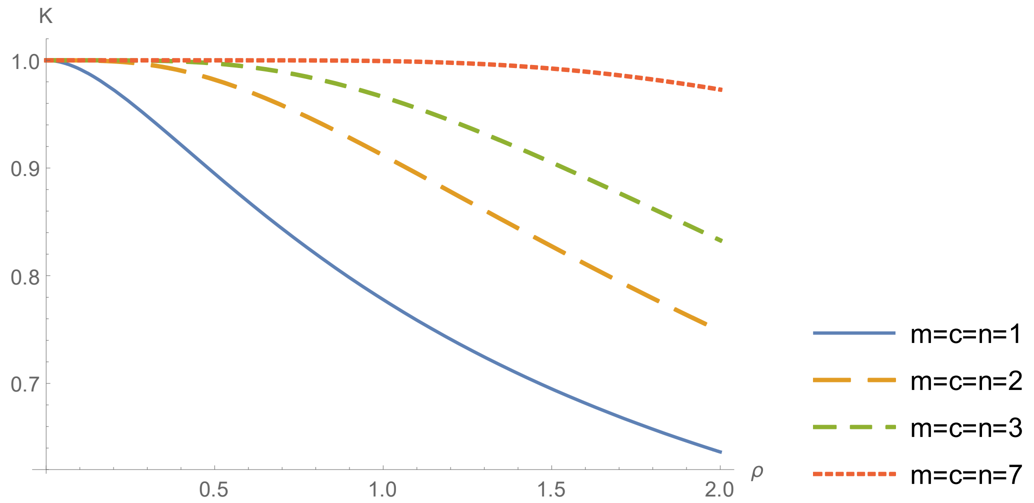

Figure 27. Dependencies of stationary availability coefficient *K* on *ρ* for a parallel repairable system (case of various values of *m*=c=*n*)

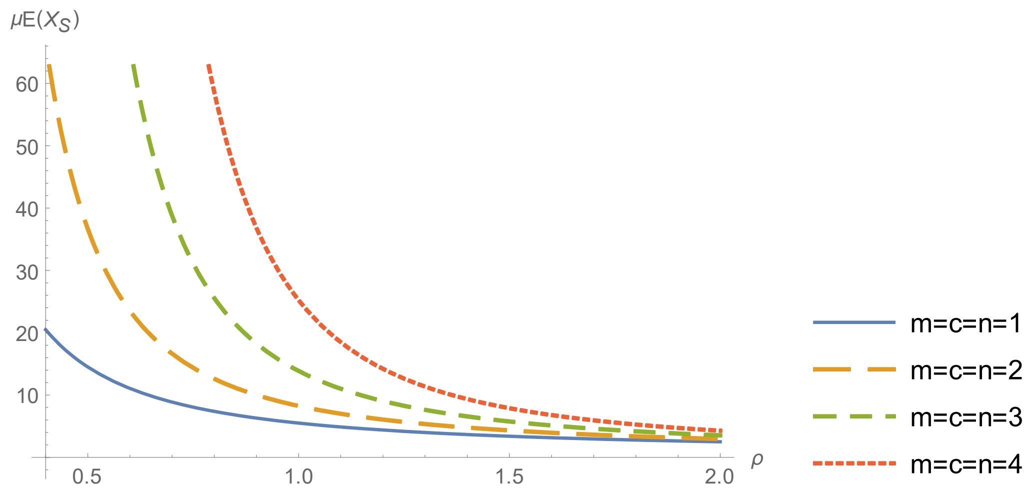

Figure 28. Dependencies of $\mu E(X_S)$ on *ρ* for a parallel repairable system (case of various values of *m*=c=*n*)

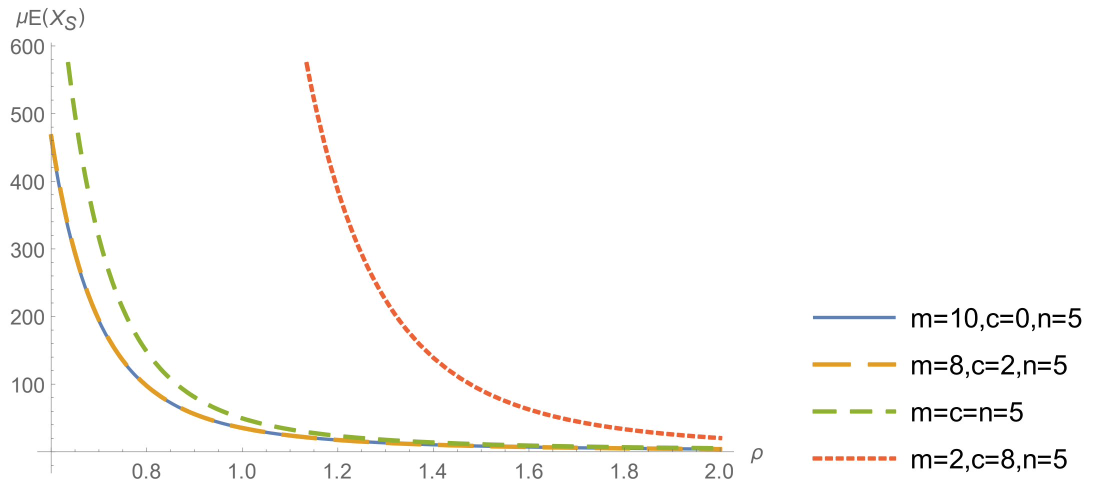

Figure 29. Dependencies of $\mu E(X_S)$ on ρ for a parallel repairable system (case of n=5 and various values of *m* and *c*)

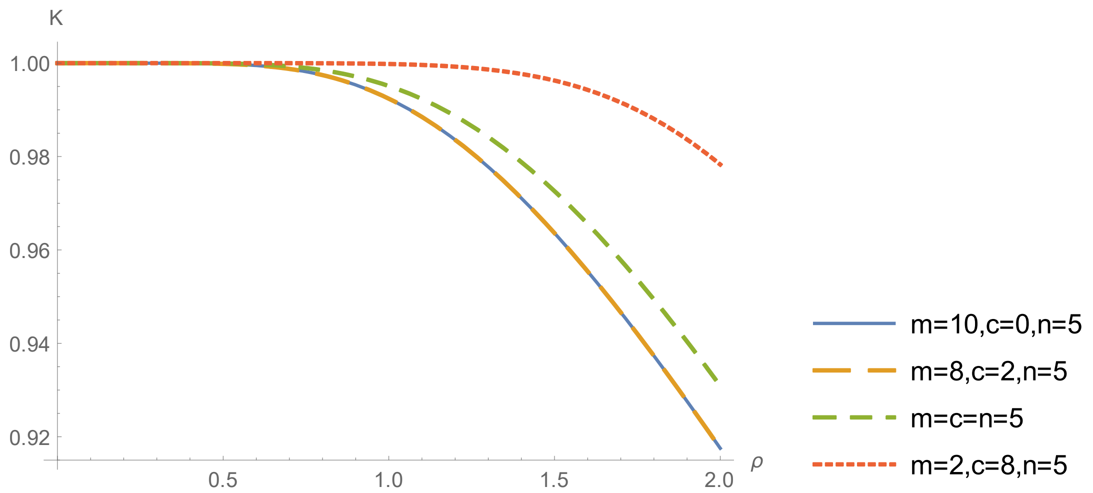

Figure 30. Dependencies of stationary availability coefficient *K* on ρ for a parallel repairable system (case of n=5 and various values of *m* and *c*)

5.2. Comparison of Values $E(X_S)$ and $E(X_{SB})$

As illustrated in Figures 31–34, the ratio $E(X_{SB})/E(X_S)$ decreases with increasing ρ for all parallel repairable systems, except in cases where the sum $m+c$ is small and the value of n is comparable to this sum. In some of these cases, the ratio $E(X_{SB})/E(X_S)$ not only increases as a function of ρ but can also exceed 1 (see Figure 32).

If the number n of repair facilities is insufficient and fixed, then for a fixed value of ρ, an increase in $m+c$ leads to a decrease in the ratio $E(X_{SB})/E(X_S)$. In the case of a simultaneous increase in m, c, and n, the ratio $E(X_{SB})/E(X_S)$ increases (Figure 34), in contrast to series repairable systems. Figure 34 shows that, in this case, the dependence of $E(X_{SB})/E(X_S)$ on ρ is non-monotonic in the range $0.5<\rho<1$.

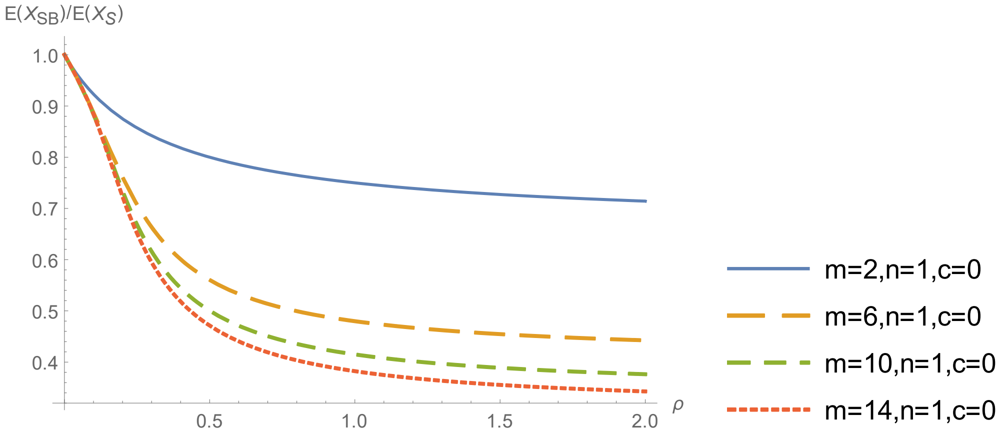

Figure 31. Dependencies of $E(X_{SB})/E(X_S)$ on ρ for a parallel non-redundant repairable system (case of *c*=0, *n*=1 and various values of *m*)

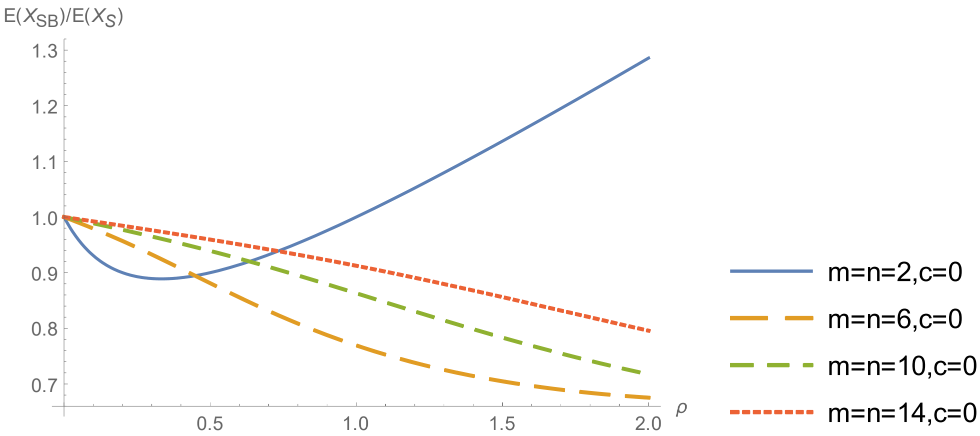

Figure 32. Dependencies of $E(X_{SB})/E(X_S)$ on ρ for a parallel non-redundant repairable system (case of *c*=0 and various values of *m*=*n*)

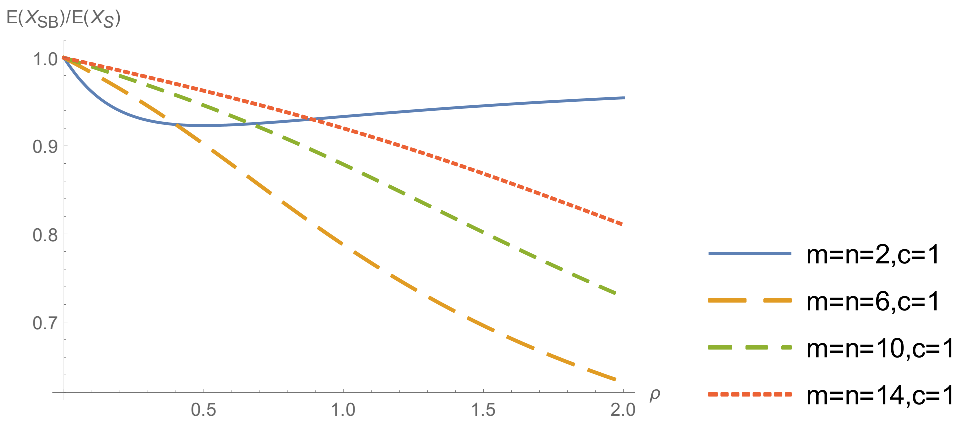

Figure 33. Dependencies of $E(X_{SB})/E(X_S)$ on ρ for a parallel repairable system (case of *c*=1 and various values of *m*=*n*)

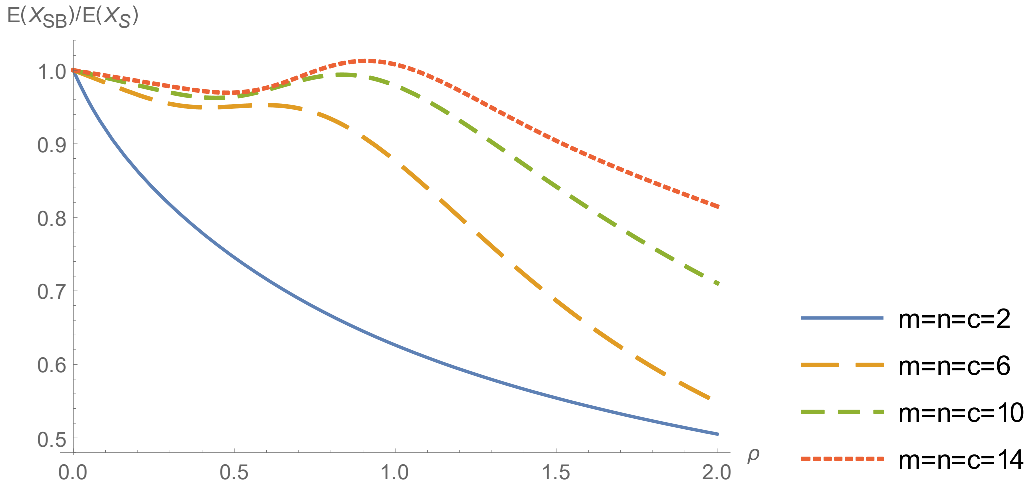

Figure 34. Dependencies of $E(X_{SB})/E(X_S)$ on ρ for a parallel repairable system (case of various values of m=n=c)

6. Conclusion

The conducted study demonstrates the effectiveness of Markov birth–death models in evaluating the reliability of repairable systems with constant failure and repair rates. The developed models, based on the assumption of identical system units and the absence of additional failures in the down state, allowed for deriving analytical expressions for key reliability metrics, including the stationary availability coefficient, MTTF, MBTF, and steady-state probabilities. Furthermore, graphical dependencies of MTTF, MBTF, and the stationary availability coefficient on the system's input parameters were constructed and analyzed, providing valuable insights into the influence of these parameters on system performance.

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