SCIREA Journal of Mathematics
ISSN: 2995-5823
http://www.scirea.org/journal/Mathematics
September 21, 2025
Volume 10, Issue 4, August 2025
https://doi.org/10.54647/mathematics110551

Finite-time blow-up of classical solutions to the non-isentropic Navier-Stokes system with gravity and some dissipation effects

Wen Wang [1], **Pengfei Zhou** [1]

[1] Department of mathematics, Shandong Normal University, Jinan 250014 China

Abstract:

In this paper, we study the finite time blow up of classical solutions to the Navier-Stokes system under vacuum free boundary conditions with degenerate viscosity, Coriolis force, friction, capillary and gravity. We prove that under certain conditions, the classical solutions of viscous compressible fluids will not exist globally if the initial data admits an isolated mass group.

Keywords: Navier-Stokes system, gravity, vacuum, degenerate viscosity, classical solutions, blow up.

Mathematics Subject Classification 2010: 35B10, 35A01, 35Q31.

1 Introduction

In this paper, we consider a full Navier-Stokes system with Coriolis force, gravity, friction and capillary force:

$$\begin{cases}\rho_t+\text{div}(\rho\mathbf{u})=0\\ \rho\left(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}\right)+\nabla P+f\rho\mathbf{u}^{\perp}+r_0\mathbf{u}+r_1\rho|\mathbf{u}|\mathbf{u}-\kappa\rho\nabla\Delta\rho=\text{div}S-\rho\mathbf{\Phi}\\ S_t+\mathbf{u}\cdot\nabla S=0\end{cases} \tag{1.1}$$

Here space and time variable $(x,t)\in\Omega\times R^+,\Omega\subset R^3$ is a bounded smooth domain, $\rho(x,t)\geq 0$ is the density of the fluid surface, $\mathbf{u}(x,t)=(u_1,u_2,u_3)(x,t)$ is the velocity field, S is the specific entropy, $P=P(\rho,S)$ is the pressure satisfying the equation of state

$$P=A\rho^{\gamma}e^{S}$$

where the constant A>0 and $\gamma\geq 1$ is the adiabatic gas exponent. The constant

$f>0$ is the Coriolis frequency. $r_0\mathbf{u}$ and $r_1\rho|\mathbf{u}|\mathbf{u}$ are the drag terms coming from friction with constants $r_0\geq 0$, $r_1>0$, and $\kappa\geq 0$ is the capillary coefficient.

Without loss of generality, we can assume $f=r_0=r_1=1.\mathbf{u}^{\perp}:=(-u_2,u_1,0)$ for any vector $\mathbf{u}=(u_1,u_2,u_3)$, and the viscous stress tensor S is given by

$$S=2\mu(\rho)D\mathbf{u}+\lambda(\rho)\text{div}\mathbf{u}I,$$

where $D\mathbf{u}=\frac{1}{2}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^t\right)$ is the deformation tensor and I is the identity matrix. $\mu(\rho),\lambda(\rho)$ are the degenerate viscous coefficients which satisfy the restrictions:

$$\mu(\rho)\geq 0,\mu(\rho)+\lambda(\rho)\geq 0,\mu(0)=\lambda(0)=0,\text{ and }\mu(\rho),\lambda(\rho)\lesssim(1+\rho^{\gamma}). \tag{1.2}$$

The term $\mathbf{\Phi}=(0,0,g)$ represents the gravity term with g denoting the gravitational constant. As a core model of free surface flow, (1.1) is widely used to simulate gas-liquid two-phase flows affected by geostrophic effects in the atmospheric boundary layer, such as the interaction between sea spray and air currents in typhoon systems. However, the combination of vacuum initial data, degenerating viscosity, friction and capillary makes the dynamic behavior of the solution of (1.1) become complicated. Some special cases of system (1.1) were concerned in the past. Liu and Yang [9] first showed that compressible Euler equations [i.e. $f=\kappa=r_1=\lambda=\mu=0$ in (1.1) and without gravity] with vacuum and damping inevitably develop singularities if the initial density has compact support. Xin and Yan [14] generalized this to viscous compressible Navier-Stokes equations [i.e. $f=\kappa=r_0=r_1=0$ in (1.1) and without gravity], proving blow-up for solutions with isolated mass groups. When the rotational effect is introduced [i.e. $f>0$], a widely studied model is the rotating shallow water system. A key assumption of this model is that the vertical scale is much smaller than the horizontal scale, so the vast majority of results were generated from two-dimensional model. Hao et al. [6] analyzed the global well-posedness of viscous rotating shallow water systems in Besov spaces, revealing how

Coriolis forces couple velocity components. Rozanova [11] identiffed conditions for singularity formation in 2D rotational gas dynamics, emphasizing the interplay between pressure and rotation. Degenerate viscosity—where coefficients vanish with density—poses another layer of difficulty. Li et al. [8] introduced the concept of “regular solutions” to tackle 2D shallow water equations with degenerate viscosities, while Duan et al. [4] linked such degeneracy directly to finite-time blow-up in rotating systems. Besides, Duan et al. [4] demonstrated that degenerate viscosity in shallow water equations induces blow-up when initial data contains isolated mass groups, contrasting with the smoothing effects of capillarity and drag. Similarly, Li et al. [7] established finite-time breakdown for degenerate viscous polytropic fluids, even with small initial data. These results highlight the critical role of vacuum and degeneracy in destabilizing solutions. In the study of solution existence for shallow water system and compressible fluids, Ton [13] established short-time existence and uniqueness of classical solutions with Hölder continuous derivatives. For scenarios involving vacuum, Duan et al. [5] proved local existence of classical solutions to the rotating viscous shallow water Cauchy problem. Luo [10] subsequently demonstrated local existence and uniqueness of classical solutions for 2D compressible flows. For global solutions, Sundbye [12] established global existence and uniqueness of strong solutions under small initial data for viscous shallow water equations. Bresch and Desjardins [1] further constructed global weak solutions for 2D viscous shallow water equations. Furthermore, Cho et al. [2, 3] generalized these findings to polytropic fluids. In this paper, we investigate the finite-time blow-up of classical solutions to the non-isentropic compressible Navier-Stokes system with degenerate viscosity and gravity. The system under consideration includes a vacuum free boundary and accounts for effects such as Coriolis force, drag terms, and capillary action. The primary focus is on scenarios where the initial data contains an isolated mass group—a localized region of non-zero density surrounded by vacuum. Under specific conditions, we demonstrate that classical solutions will exhibit finite-time blow-up, meaning the solutions become unbounded within a finite time frame. Our results extend previous blow-up criteria (e.g., Xin and Yan [14]; Duan et al. [4]) to a more comprehensive physical framework. Considering the influence of entropy and gravity, it provides a universal tool for the blasting theory of compressible flows with complex multi-physical fields. The remainder of this paper is organized as follows: In section 2 we introduce the main results and the concept of isolated mass group and energy functional ; In section 3 we first make some prior estimates, and then construct the second-order moment functional $G(t)$, combine with energy inequality and parameter conditions, we prove the main result.

2 Notations and main results

In this section, we state the initial data and vacuum boundary condition of system (1.1) and the main result of this paper.

First, recall the definition of isolated mass group.

Definition 2.1. *(Definition 2.2 in [14]) Let* Ω *be a smooth domain in* R^3*. The pair* (U, V) *is called an isolated mass group of* $\rho_0(x)$*, if both* $U\subset\Omega$ *and* $V\subset\Omega$ *are bounded open sets, U is connected, and satisfy*

$$\begin{cases} V\subset\bar{V}\subset U \\ \rho_0(x)=0, \text{ in } U\setminus V \end{cases}$$

and $\rho_0(x)$ *is not identically equal to zero on V.*

Next, we introduce some notations.

Suppose the bounded domain $\Omega\subset B_R(0)$. Denote by m_0 the initial mass of the isolated mass group V. That is,

$$m_0=\int_V \rho_0(x)\mathrm{d}x>0 \tag{2.1}$$

Set

$$c_0=\int_V |x|^2\rho_0(x)\mathrm{d}x>0,\ c_1=\int_V \rho_0(x)\mathbf{u}_0(x)\cdot x\,\mathrm{d}x, \tag{2.2}$$

$$E_0=\int_V \frac{1}{2}\rho_0(x)|\mathbf{u}_0(x)|^2+\frac{A\rho(x)^\gamma e^{S_0}}{\gamma-1}+\frac{\kappa\left|\nabla\rho_0(x)\right|^2}{2}\,\mathrm{d}x>0. \tag{2.3}$$

Denote by $X(\xi,t)$ the particle path starting from ξ when t=0, thus

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}X(\xi,t)=\mathbf{u}(X(\xi,t),t) \\ X(\xi,0)=\xi \end{cases} \tag{2.4}$$

Set

$$U(t)=\{X(\xi,t) \mid \xi\in U\} \text{ and } V(t)=\{X(\xi,t) \mid \xi\in V\}.$$

Then, the initial data and vacuum boundary condition of system (1.1) are given by

$$\rho(x,0)=\rho_0(x),\mathbf{u}(x,0)=\mathbf{u}_0(x),S(x,0)=S_0(x), \text{ on } \Omega\times\{t=0\}. \tag{2.5}$$

and

$$\rho(x,t)=0, \text{ on } \partial V(t)\times R^+, \tag{2.6}$$

where $\partial V(t)$ is a free boundary separating fluid from vacuum and (2.6) is the boundary condition corresponding to continuous density.

On the vacuum free boundary $\partial V(t)$, we give more explanations.

• The pressure $P=A\rho^\gamma e^S$ continuously approaches zero.

• The velocity u and entropy S remain continuously differentiable.

We say that $\rho(x,t)$, $u(x,t)$, $S(x,t)$ is a classical solution to the initial-boundary-value problem for shallow water system (1.1), (2.5), (2.6) on $\Omega\times(0,T)$ for some positive T, if

• $\rho\in C^1(\Omega\times[0,T)),\mathbf{u}\in C^1([0,T),C^2(\Omega)),S\in C^1(\Omega\times[0,T))$ and satisfies system (1.1) point-wisely on $\Omega\times(0,T)$;

• $(\rho(x,t),\mathbf{u}(x,t),S(x,t))$ satisfies the initial and boundary conditions (2.5), (2.6) continuously. Finally, our main result of this paper is stated as follows.

Theorem 2.1. *Suppose* $(\rho(x,t),\boldsymbol{u}(x,t),S(x,t))$ *is a classical solution to the initial-boundary-value problem for shallow water system (1.1), (2.5), (2.6) on*

$\Omega\times(0,T)$*, and satisffes the following conditions:*

(A1) The initial height $\rho_0(x)$ *admits an isolated mass group* (V,U).

(A2) The entropy at the initial state is non-negative, i.e. $s_0:=\inf_{x\in\Omega} S_0(x)\geq 0$.

(A3) $\frac{3Ae^{s_0}}{|\Omega|^{\gamma-1}}>gR$ *and* $u_3>0$.

Then $(\rho(x,t),\boldsymbol{u}(x,t),S(x,t))$ *will blow up in finite time, i.e. there exists a positive time* $T^*<+\infty$*, such that* $T\leq T^*<+\infty$.

3 Proof of theorem

In this section, we prove the main results. Since $\rho_0(x)=0$ in $U\setminus V$, mass equation shows that

$$\rho(x,t)=0, \text{ in } U(t)\setminus V(t). \tag{3.1}$$

Lemma 3.1. *Suppose* $(\rho(x,t),\boldsymbol{u}(x,t),S(x,t))$ *is a classical solution to system (1.1) on* $\Omega\times(0,T)$*, then*

$$\int_{U(t)} \rho(x,t)dx=m_0>0, \tag{3.2}$$

and

$$\int_{U(t)} \frac{1}{2}\rho|\boldsymbol{u}|^2+\frac{A\rho^\gamma e^S}{\gamma-1}+\frac{\kappa|\nabla\rho|^2}{2}\, dx$$

$$+\int_0^t \int_{U(t)} 2\mu(\rho)|D\boldsymbol{u}|^2+\lambda(\rho)|div\boldsymbol{u}|^2+|\boldsymbol{u}|^2+\rho|\boldsymbol{u}|^3+\rho g u_3\, dxdt=E_0, \tag{3.3}$$

where E_0 *is defined in (2.3).*

Proof. A straight calculation shows that for any $f(x,t)\in C^1(R^3\times R^+)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{U(t)} f(x,t)\mathrm{d}x=\int_{U(t)} \partial_t f(x,t)\mathrm{d}x+\int_{\partial U(t)} f(x,t)(\mathbf{u}(x,t)\cdot\ \mathbf{r})\mathrm{d}t. \tag{3.4}$$

where $\mathbf{r}$ is the unit out normal to $\partial U(t)$. Integrating the mass equation, then (3.2) holds by (1.1)1 , (3.1) and (3.4). Note that ρ^γ satisfies

$$(\rho^\gamma)_t+\gamma\rho^\gamma\mathrm{div}\mathbf{u}+\nabla(\rho^\gamma)\cdot\ \mathbf{u}=0. \tag{3.5}$$

Multiplying the second equation of (1.1) by $\mathbf{u}$ and integrating the result, we have

$$\int_{U(t)} \rho\mathbf{u}_t\cdot\ \mathbf{u}+\rho\mathbf{u}\cdot\ \nabla\mathbf{u}\cdot\ \mathbf{u}+\nabla P\cdot\ \mathbf{u}+\rho\mathbf{u}^\perp\cdot\ \mathbf{u}+|\mathbf{u}|^2+\rho|\mathbf{u}|^3-\kappa\rho\nabla\Delta\rho\cdot\ \mathbf{u}\mathrm{d}x$$

$$-\int_{U(t)} \mathrm{div}S\cdot\ \mathbf{u}\mathrm{d}x+\int_{U(t)} \rho\Phi\cdot\ \mathbf{u}\mathrm{d}x$$

$$=\int_{U(t)} \left(\frac{1}{2}\rho|\mathbf{u}|^2\right)_t+\left(\frac{A\rho^\gamma e^S}{\gamma-1}\right)_t+\left(\frac{\kappa|\nabla\rho|^2}{2}\right)_t \mathrm{d}x$$

$$+\int_{U(t)} 2\mu(\rho)|D\mathbf{u}|^2+\lambda(\rho)|\mathrm{div}\mathbf{u}|^2+|\mathbf{u}|^2+\rho|\mathbf{u}|^3+\rho g u_3\,\mathrm{d}x=0, \tag{3.6}$$

where one has used

$$\int_{U(t)} \nabla(A\rho^\gamma e^S)\cdot\ \mathbf{u}\mathrm{d}x=\int_{U(t)} Ae^S\nabla(\rho^\gamma)\cdot\ \mathbf{u}+A\rho^\gamma\nabla(e^S)\cdot\ \mathbf{u}\mathrm{d}x$$

$$=\int_{U(t)} Ae^S[-(\rho^\gamma)_t-\gamma\rho^\gamma\mathrm{div}\mathbf{u}]+A\rho^\gamma\nabla(e^S)\cdot\ \mathbf{u}\mathrm{d}x$$

$$=\int_{U(t)} -Ae^S(\rho^\gamma)_t+\gamma\nabla(Ae^S\rho^\gamma)\cdot\ \mathbf{u}+A\rho^\gamma\nabla(e^S)\cdot\ \mathbf{u}\mathrm{d}x,$$

which leads to

$$\int_{U(t)} \nabla(A\rho^\gamma e^S)\cdot \mathbf{u}\mathrm{d}x = \frac{1}{\gamma-1}\int_{U(t)} Ae^S(\rho^\gamma)_t - A\rho^\gamma e^S \nabla S\cdot \mathbf{u}\,\mathrm{d}x$$
$$= \frac{1}{\gamma-1}\int_{U(t)} Ae^S(\rho^\gamma)_t + A\rho^\gamma e^S S_t\,\mathrm{d}x$$
$$= \frac{1}{\gamma-1}\int_{U(t)} (A\rho^\gamma e^S)_t\,\mathrm{d}x.$$

Integrate (3.6) with respect to t, we obtain

$$\int_{U(t)} \frac{1}{2}\rho|\mathbf{u}|^2 + \frac{A\rho^\gamma e^S}{\gamma-1} + \frac{\kappa|\nabla\rho|^2}{2}\,\mathrm{d}x$$
$$+\int_0^t \int_{U(t)} 2\mu(\rho)|D\mathbf{u}|^2 + \lambda(\rho)|\mathrm{div}\mathbf{u}|^2 + |\mathbf{u}|^2 + \rho|\mathbf{u}|^3 + \rho g u_3\,\mathrm{d}x\,\mathrm{d}t = E_0.$$

This completes the proof of Lemma 3.1.

Lemma 3.2. *Suppose $(\rho(x,t),\boldsymbol{u}(x,t),S(x,t))$ is a classical solution to system (1.1) on $\Omega\times(0,T)$. As $u_3>0$ on $\Omega\times(0,T)$, the following estimates*

$$\int_0^t \left|\int_{U(t)} (2\mu(\rho)+3\lambda(\rho))\,div\boldsymbol{u}dx\right| dt \le Ct^{\frac{1}{2}}, \tag{3.7}$$

and

$$\int_0^t |I(t)|dt \le Ct^{\frac{1}{2}} + Ct^{\frac{1}{3}} + Ct^{\frac{2}{3}} \tag{3.8}$$

hold, where $I(t)=\int_{U(t)}\left(\rho\boldsymbol{u}^\perp+\boldsymbol{u}+\rho|\boldsymbol{u}|\boldsymbol{u}\right)\cdot x\,dx$, C is a positive constant depending on E_0,s_0,m_0,γ and the diameter of Ω.

Proof. According to (1.2) and (3.3) , we have

$$\int_0^t \left|\int_{U(t)} (2\mu(\rho)+3\lambda(\rho))\mathrm{div}\mathbf{u}\mathrm{d}x\right| \mathrm{d}t$$

$$\le \int_0^t \int_{U(t)} (2\mu(\rho)+3\lambda(\rho))|\mathrm{div}\mathbf{u}|\mathrm{d}x\,\mathrm{d}t$$

$$\le 3\left(\int_0^t \int_{U(t)} (\mu(\rho)+\lambda(\rho))|\mathrm{div}\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^t \int_{U(t)} (\mu(\rho)+\lambda(\rho))\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}$$

$$\le CE_0^{\frac{1}{2}}\left(\int_0^t \int_{U(t)} (\mu(\rho)+\lambda(\rho))\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} \tag{3.9}$$

It is easy to see that ordinary differential equations (2.4) has a unique solution

$$x=\varphi(t)\in C^1(\Omega\times(0,T)).$$

Then by (1.1) 3 we have

$$\frac{\mathrm{d}}{\mathrm{d}t}S(\varphi(t),t)=0$$

Integrating the above equation we get

$$S(x,t)=S(\varphi(0),0)=S_0(\varphi(0))\geq 0 \tag{3.10}$$

Since $\mu(\rho),\lambda(\rho)\lesssim(1+\rho^\gamma)$ and (3.3), (3.10),

$$\int_{U(t)}(\mu(\rho)+\lambda(\rho))\mathrm{d}x\leq C\int_{U(t)}(1+\rho^\gamma)\mathrm{d}x\leq C\left(1+\int_{U(t)}\frac{A\rho^\gamma e^S}{\gamma-1}\,\mathrm{d}x\right)\leq C(1+E_0),$$

which together with (3.9) leads to

$$\int_0^t\left|\iint_{U(t)}(2\mu(\rho)+3\lambda(\rho))\mathrm{div}\mathbf{u}\mathrm{d}x\right|\mathrm{d}t\leq Ct^{\frac{1}{2}},$$

where $C>0$ depends on the diameter of Ω,E_0,s_0 and γ.

Next, we will estimate

$$I(t):=\int_{U(t)}\left(\rho\mathbf{u}^{\perp}+\mathbf{u}+\rho|\mathbf{u}|\mathbf{u}\right)\cdot\ x\,\mathrm{d}x$$

According to (3.3),

$$\int_0^t |I(t)|\mathrm{d}t$$

$$\le \int_0^t \int_{U(t)} \rho\left|\mathbf{u}^{\perp}\cdot x\right|\mathrm{d}x\,\mathrm{d}t + \int_0^t \int_{U(t)} |\mathbf{u}\cdot x|\mathrm{d}x\,\mathrm{d}t + \int_0^t \int_{U(t)} \rho|\mathbf{u}||\mathbf{u}\cdot x|\mathrm{d}x\,\mathrm{d}t$$

$$\le CR\int_0^t \int_{U(t)} \rho|\mathbf{u}|\mathrm{d}x\,\mathrm{d}t + CR\int_0^t \int_{U(t)} |\mathbf{u}|\mathrm{d}x\,\mathrm{d}t + CR\int_0^t \int_{U(t)} \rho|\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$\le CR\left(\int_0^t \int_{U(t)} \rho|\mathbf{u}|^3\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{3}}\left(\int_0^t \int_{U(t)} \rho\mathrm{d}x\,\mathrm{d}t\right)^{\frac{2}{3}}$$

$$+CR\left(\int_0^t \int_{U(t)} r_0|\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^t 1\,\mathrm{d}t\right)^{\frac{1}{2}}$$

$$+CR\left(\int_0^t \int_{U(t)} \rho|\mathbf{u}|^3\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{2}{3}}\left(\int_0^t \int_{U(t)} \rho\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{3}}$$

$$\le CRE_0^{\frac{1}{3}}m_0^{\frac{2}{3}}t^{\frac{2}{3}} + CRE_0^{\frac{1}{2}}t^{\frac{1}{2}} + CRE_0^{\frac{2}{3}}m_0^{\frac{1}{3}}t^{\frac{1}{3}}$$

$$\le C\left(t^{\frac{1}{3}}+t^{\frac{1}{2}}+t^{\frac{2}{3}}\right) \tag{3.11}$$

where $C>0$ depends on E_0, m_0 and the diameter of $\Omega\subset B_R(0)$.

Proof of Theorem 2.1. Set

$$G(t)=\int_{U(t)} |x|^2\rho(x,t)\mathrm{d}x \tag{3.12}$$

Then, (1.1), (3.1) and (3.4) give that

$$G'(t)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{U(t)} |x|^2\rho(x,t)\mathrm{d}x\right)=-\int_{U(t)} |x|^2\mathrm{div}(\rho\mathbf{u})\mathrm{d}x=2\int_{U(t)} \rho\mathbf{u}\cdot x\,\mathrm{d}x.$$

And

$$\frac{1}{2}G''(t)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{U(t)} \rho\mathbf{u}\cdot x\,\mathrm{d}x\right)=\int_{U(t)} (\rho\mathbf{u})_t\cdot x\,\mathrm{d}x$$

$$=\int_{U(t)} \left[-\mathbf{u}\mathrm{div}(\rho\mathbf{u})-\rho\mathbf{u}\cdot\nabla\mathbf{u}-\nabla P-\left(\rho\mathbf{u}^{\perp}+\mathbf{u}+\rho|\mathbf{u}|\mathbf{u}\right)+\kappa\rho\nabla\Delta\rho+\mathrm{div}S-\rho\Phi\right]\cdot x\,\mathrm{d}x$$

$$=-\int_{U(t)} [\mathbf{u}\mathrm{div}(\rho\mathbf{u})+\rho\mathbf{u}\cdot\nabla\mathbf{u}]\cdot x\,\mathrm{d}x-\int_{U(t)} \nabla P\cdot x\,\mathrm{d}x-I(t)$$

$$+\int_{U(t)} \kappa\rho\nabla\Delta\rho\cdot x\,\mathrm{d}x+\int_{U(t)} \mathrm{div}S\cdot x\,\mathrm{d}x-\int_{U(t)} \rho g x_3\,\mathrm{d}x$$

$$=\int_{U(t)} \rho|\mathbf{u}|^2\,\mathrm{d}x+3A\int_{U(t)} \rho^{\gamma}e^{S}\,\mathrm{d}x-I(t)+\int_{U(t)} \kappa\rho\nabla\Delta\rho\cdot x\,\mathrm{d}x$$

$$-\int_{U(t)} (2\mu(\rho)+3\lambda(\rho))\mathrm{div}\mathbf{u}\,\mathrm{d}x-\int_{U(t)} \rho g x_3\,\mathrm{d}x, \tag{3.13}$$

where the vectors $\mathbf{u}=(u_1,u_2,u_3)$ and $x=(x_1,x_2,x_3)$.

According to condition (A2) and Jensen's inequality, we know that in a bounded domain,

$$3A\int_{U(t)} \rho^{\gamma}e^{S}\,\mathrm{d}x\geq 3Ae^{s_0}\int_{U(t)} \rho^{\gamma}\mathrm{d}x\geq C_1\int_{U(t)} \rho\mathrm{d}x=C_1 m_0 \tag{3.14}$$

where the constant $C_1:=\frac{3Ae^{s_0}}{|U(t)|^{\gamma-1}}>\frac{3Ae^{s_0}}{|\Omega|^{\gamma-1}}>0$. And

$$\int_{U(t)} \kappa\rho\nabla\Delta\rho\cdot x\,\mathrm{d}x=-\int_{U(t)} \kappa\Delta\rho\,\mathrm{div}(\rho x)\mathrm{d}x$$

$$=-\int_{U(t)} \kappa\Delta\rho\nabla\rho\cdot x+3\kappa\rho\Delta\rho\,\mathrm{d}x$$

$$=\int_{U(t)} \kappa\nabla\rho\cdot\nabla(\nabla\rho\cdot x)+3\kappa\nabla\rho\cdot\nabla\rho\,\mathrm{d}x$$

$$=\int_{U(t)} \frac{\kappa}{2}\nabla\left(|\nabla\rho|^2\right)\cdot x+4\kappa|\nabla\rho|^2\,\mathrm{d}x$$

$$=\int_{U(t)} -\frac{3}{2}\kappa|\nabla\rho|^2+4\kappa|\nabla\rho|^2\,\mathrm{d}x$$

$$=\int_{U(t)} \frac{5}{2}\kappa|\nabla\rho|^2\,\mathrm{d}x\geq 0, \tag{3.15}$$

$$\int_{U(t)} \rho g x_3\,\mathrm{d}x\leq gR\int_{U(t)} \rho\mathrm{d}x=gRm_0. \tag{3.16}$$

Integrate (3.13) with respect to t,together with (A3),(3.5) - (3.8) and (3.14)- (3.16), we have

$$\frac{1}{2}G'(t)\geq\frac{1}{2}c_1+(C_1-gR)m_0 t-Ct^{\frac{1}{2}}-Ct^{\frac{1}{3}}-Ct^{\frac{2}{3}}.$$

Therefore,

$$G(t)=\int_{U(t)} |x|^2\rho(x,t)\mathrm{d}x\leq R^2\int_{U(t)} \rho(x,t)\mathrm{d}x=m_0R^2,$$

where c_0 is defined in (2.2).

It is obvious that $G(t)\to+\infty$ as $t\to+\infty$, while

$$G(t)=\int_{U(t)}|x|^2\rho(x,t)\mathrm{d}x\le R^2\int_{U(t)}\rho(x,t)\mathrm{d}x=m_0R^2$$

where $\Omega\subset B_R(0)$. This leads to a finite bound on T, That is, there exists a positive time T, that is, there exists a positive time $T^*<+\infty$ such that $T\le T^*<+\infty$. We finish the proof of Theorem 1.

References

[1] D. Bresch, B. Desjardins, Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostrophic model, Comm. Math. Phys. 238(2003), 211-223. 10

[2] Y. Cho, H. Choe, and H.Kim, Unique solvability of the initial boundary value problems for compressible viscous fluids, J. Math. Pures Appl. 83(3)(2004), 243–275.

[3] Y. Cho, H. Kim, Existence results for viscous polytropic fluids with vacuum, J. Differential Equations. 228(2006), 377-411.

[4] B. Duan, Z. Luo, and W. Yan, Finite-time blow-up of classical solutions to the rotating shallow water system with degenerate viscosity, Z. Angew. Math. Phys. 70(2)(2019), 54.

[5] B. Duan, Z. Luo, and Y. Zheng, Local existence of classical solutions to shallow water equations with Cauchy data containing vacuum, SIAM J. Math. Anal. 44(2)(2012), 541-567.

[6] C. Hao, L. Hsiao and H. Li, Cauchy problem for viscous rotating shallow water, J. Differential Equations 247(2009), 3234–3257.

[7] Y. Li, R. Pan and S. Zhu, On classical solutions for viscous polytropic fluids with degenerate viscosities and vacuum, Arch. Rational Mech. Anal. 234(2019), 1281-1334.

[8] Y. Li, R. Pan and S. Zhu, On classical solutions to 2D shallow water equations with degenerate viscosities, J. Math. Fluid Mech. 19(2017), 151-190.

[9] T. Liu, T. Yang, Compressible Euler equations with vacuum, J. Differential Equations. 140(2)(1997), 223-237.

[10] Z. Luo, Local existence of classical solutions to the two-dimensional viscous compressible flows with vacuum, Commun. Math. Sci. 10(2)(2012), 527- 554.

[11] O. Rozanova, Singularity formation for rotational gas dynamics, J. Math. Anal. Appl. 492(2020), 124405.

[12] L. Sundbye, Global existence for the Cauchy problem for the viscous shallow water equations, Rocky Mountain J. Math. 28(3)(1998), 1135–1152.

[13] B. Ton, Existence and uniqueness of a classical solution of an initial-boundary value problem of the theory of shallow waters, SIAM J. Math. Anal. 12(2)(1981), 229-241.

[14] Z. Xin, W. Yan, On blowup of classical solutions to the compressible Navier–Stokes equations, Comm. Math. Phys. 321(2013), 529-541.