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Pipes Construction Using Quartic Bézier Curves

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Abstract

A Pipe consists of cross-sectional and longitudinal directions that determine its volume. The objective of this paper is to obtain some formulas for modeling the cross-section and the radius length variation along the center curve of the pipe. The methods are as follows. By using real quartic Bézier functions, we define the cross-section and lengthwise-section radii of the pipe, respectively. Then, we calculate three orthonormal vectors that are the tangent vectors of the center curve and two-unit vectors that are perpendicular to the tangent vectors. After that, we evaluate the formulas to model the pipes. The research results indicate that this proposed method is useful for designing pipe cross-sections and for modifying their volume.

Keywords: *pipes construction, quartic Bézier curve, radius, cross-section, lengthwise directions*

1. Introduction

Several methods have been proposed for pipe design. Su et al. (2015) discussed a method to verify the isometric topology of a pipe that is invariant under pipe elongation. Bizzarri et al.

(2015) introduced a canal surface using an algebraic restriction. Besides, Zhou & Chen (2013) applied the concept of continuous contact among multiple pipes. The study by Moritani et al. (2018) introduced the construction of a piping system using cylinder-based registration and model fitting; furthermore, Bhatt et al. (2015) performed calculations with a B-spline surface that was disconnected. Kusno & Cahyo (2018, 2019) constructed a piping system by connecting several pipe components.

In general, the presented methods use a moving trihedron formula to construct pipes with a homogeneous radius. To develop the method, we propose a novel procedure that models the cross-section and longitudinal directions of pipes. For this reason, the discussion in this article is organized into the following stages. In the first step, it studies the parametric representation to define a pipe shape, and then we introduce the modeling of the pipe cross-section. In the third step, it presents equations for designing the pipes using the quartic Bézier curve. Finally, we make a summary and a conclusion of the results.

2. Parametric Representation of Pipes

The posture of a pipe consists of cross (transversal) and longitudinal sections of the pipe. It can define the cross-sectional shapes of pipes by using the plane parametric curves $\mathbf{C}(v)$ as follows (Kusno; 2018, 2019)

$$\mathbf{C}(v) = r(v)\,(\cos\Omega\,\mathbf{v}_1 + \sin\Omega\,\mathbf{v}_2) \tag{1}$$

with $\Omega = 2\,\pi v$, $0 \leq v \leq 1$. Two unit vetors $\mathbf{v}_1$ and $\mathbf{v}_2$ are mutually orthonormal. In this case, the real function $r(v)$ can be used to design the cross-section shape of pipes by polar representation in the form

$$r_1(v) = a \pm b.\, sin\,\Omega; \qquad r_2(v) = a \pm b.\, cos\,\Omega; \tag{2a}$$

$$r_3(v) = a\, sin\,(n\Omega); \qquad r_4(v) = a\, cos\,(n\Omega); \tag{2b}$$

with a, b, n positive real constants.

Let a regular non-spiral parametric curve $\mathbf{C}(u)$ of class C^2 with the moving trihedron of the curve [$\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$], respectivelly tangent, normal, and binormal vectors (Carmo, 1976; Lischultz, 1969; Kreyzig, 1991; Mortenson, 1996). We can define the circles along the curve $\mathbf{C}(u)$ in which their radius is $\rho(u,v) = r(v)\,\rho(u)$, their centers are on the curve segment $\mathbf{C}(u)$, and they are orthogonal to the vector $\mathbf{t}$ in the interval $0 \leq u \leq 1$. Therefore, for $\Omega = 2\,\pi v$ and $0 \leq u, v \leq 1$, it can define the pipe relatively to the vectors $\mathbf{b}$ and $\mathbf{n}$ as follows

$$\mathbf{P}(u,v) = \mathbf{C}(u) + \rho(u,v)(\cos \Omega\, \mathbf{b} + \sin \Omega\, \mathbf{n}). \tag{3}$$

Based on equations (1) and (3), we propose a new approach for constructing the pipes as follows.

2.1. Modeling of Pipe Cross-section

Consider the real function $r(v)$ in equation (1) of the quartic Bézier $r(v) = q(v) = P_o(1- v)^4 + 4 P_1(1-v)^3 v + 6 P_2 (1-v)^2 v^2 + 4 P_3 (1-v) v^3 + P_4 v^4$ where P_o, P_1, P_2, P_3, P_4 are control coefficients of the function $q(v)$ and $0 \leq \ \leq 1$. Therefore, equation (1) can be modified in this way

$$\mathbf{C}(v) = q(v) (\cos \Omega\, \mathbf{v}_1 + \sin \Omega\, \mathbf{v}_2) \tag{4}$$

with $\Omega = 2\pi v$, $0 \leq v \leq 1$ and $[\mathbf{v}_1,\mathbf{v}_2]$ the unit orthonormal vectors. Giving different values for the control coefficients $[P_o, P_1, P_2, P_3, P_4]$ of the function $q(v)$ in equation (4) will obtain various curve shapes of the pipe cross-section. Then, to dilate or move the curve positions $\mathbf{C}(v)$ in the plane $[\mathbf{v}_1,\mathbf{v}_2]$ can use the equation

$$\mathbf{C}(v) = k[\, q(v) (\cos \Omega\, \mathbf{v}_1 + \sin \Omega\, \mathbf{v}_2)] + \tau\, \mathbf{v}_1 + \sigma\, \mathbf{v}_2 \tag{5}$$

with k, τ and σ scalars. Besides, to design the pipe cross-section by using the composition of n different functions $q(v)$ and angle Ω use the equations

$$\mathbf{C}_1(v) = q_1(v) [\cos (\pi v)\, \mathbf{v}_1 + \sin (\pi v)\, \mathbf{v}_2] \tag{6a}$$

$$\mathbf{C}_2(v) = q_2(v) [\cos \pi(v+1)\, \mathbf{v}_1 + \sin \pi(v+1)\, \mathbf{v}_2] \tag{6b}$$

or for $i = 1, 2,\dots, n$

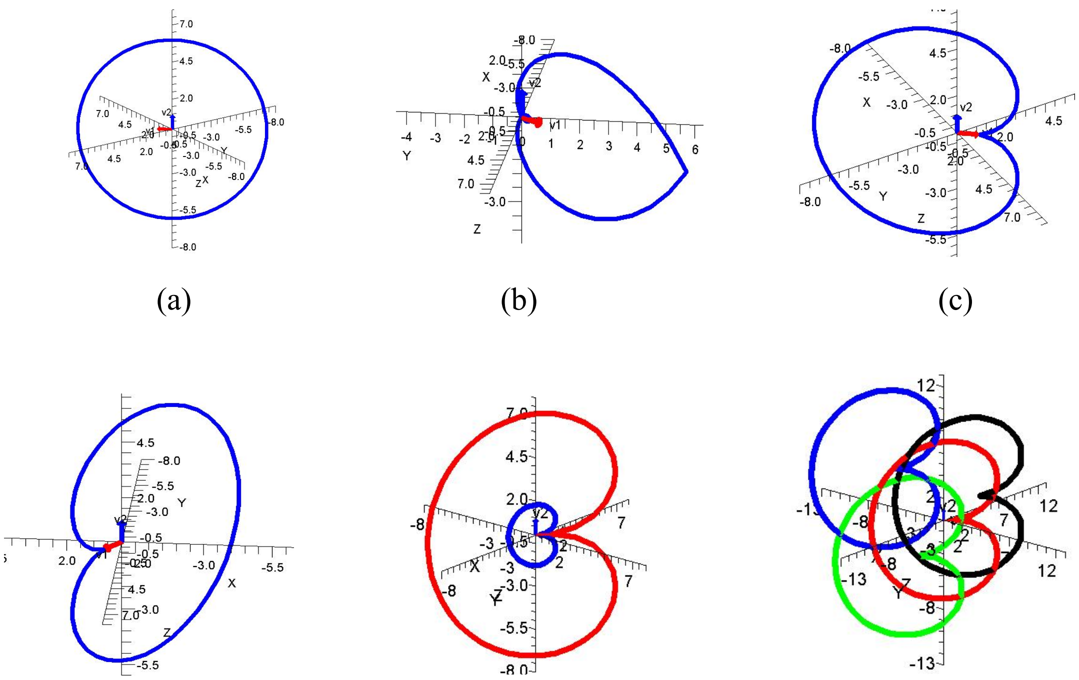
$$\mathbf{C}_i(v) = q_i(v) [\cos (2\pi v/n + (i-1)\, 2\pi/n)\, \mathbf{v}_1 + \sin (2\pi v/n + (i-1)\, 2\pi/n)\, \mathbf{v}_2] \tag{7}$$

and $3 \leq n \leq 8$. For the application of these equations, we evaluate some values of these control coefficients $[P_o, P_1, P_2, P_3, P_4]$ and angle Ω in the following simulations.

Simulation 1

Electing data of two unit orthonormal vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ with $\mathbf{v}_1 = \langle 3/5,4/5,0\rangle$, $\mathbf{v}_2 = \langle 0,0,1\rangle$, and $q(v) = 6$ for Equation (4) presents the circle shape (Figure 1a). **E**mploying the control coefficients $P_0 = 8$, $P_1 = -2$, $P_2 = -1$, $P_3 = -2$, $P_4 = 8$ results Figure 1b and $P_0 = 1$, $P_1 = 8$, $P_2 = 6$, $P_3 = 8$, $P_4 = 1$ gives Figure 1c,d. Furthermore, given $P_0 = 1$, $P_1 = 18$, $P_2 = -8$, $P_3 = 18$, $P_4 = 1$ and $k = \tau = \sigma = 1$. Using Equation (5) for these data results Figure 1e in red color and determining the radius, $k = ¼$, finds Figure 1e in blue color. In case the values $[\tau =2, \sigma =2]$; $[\tau =-3, \sigma =-3]$; and $[\tau =-5, \sigma =5]$, the curve changes the positions in the plane

[$\mathbf{v}_1$,$\mathbf{v}_2$] as shown in Figure 1f, colored in black, green, and blue respectively.



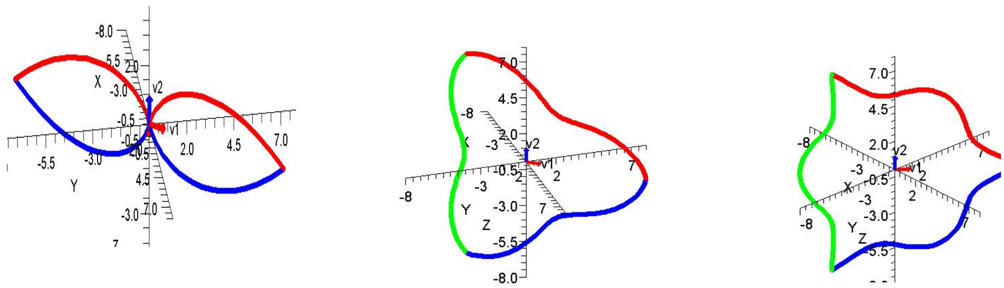
(a) (b) (c)

(d) (e) (f)

Figure 1. Some cross-section shapes of pipe

Simulation 2

Given data of two unity orthonormal vectors $\mathbf{v}_1$ = <3/5,4/5,0>, $\mathbf{v}_2$ = <0,0,1>, and consider the control coefficients of the real function $q(v)$ in Equation (6a,6b) as follows: $P_0 = 8$, $P_1 = -4$, $P_2 = -1$, $P_3 = -4$, $P_4 = 8$. From these data, Figure 2a is obtained, presented as red and blue curves, respectively. If in Equation (7), we set $P_0 = 8$, $P_1 = 7$, $P_2 = -1$, $P_3 = 7$, $P_4 = 8$, and $n = 3$, it presents Figure 2b. Furthermore, we choose $P_0 = 8$, $P_1 = -1/4$, $P_2 = 14$, $P_3 = -1/4$, $P_4 = 8$, and $n = 3$, it gives Figure 2c.



(a) (b) (c)

Figure 2. Pipe cross-section defined by several curves

2.2. Pipe Design Using Quartic Bézier Curves

The quartic Bézier curve is represented in the form

$$\mathbf{C}(u) = \mathbf{P}_o(1- u)^4 + 4\,\mathbf{P}_1(1-u)^3\, u + 6\,\mathbf{P}_2(1-u)^2\, u^2 + 4\,\mathbf{P}_3(1-u)\, u^3 + \mathbf{P}_4\, u^4 \tag{8}$$

with u in interval $0 \le u \le 1$ and $\mathbf{P}_o$, $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{P}_4$ control points of the Bézier curve. The first derivative of the curve is

$$\mathbf{C}'(\mathrm{u}) = 4[(\mathbf{P}_1\text{-}\mathbf{P}_o)(1\text{-}u)\,(1\text{-}u)\,(1\text{-}u) + 3(\mathbf{P}_2\text{-}\mathbf{P}_1)(1\text{-}u)\,(1\text{-}u).\,u + 3(\mathbf{P}_3\text{-}\mathbf{P}_2)\,(1\text{-}u).u^2 + (\mathbf{P}_4\text{-}\mathbf{P}_3)\,u^3] \tag{9}$$

$$= < A_x(u), A_y(u), A_z(u) >$$

with

$$A_x(u)= 4[(P_{1x}\text{-}P_{ox})(1\text{-}u)\,(1\text{-}u)\,(1\text{-}u) \;\; +3(P_{2x}\text{-}P_{1x})(1\text{-}u)\,(1\text{-}u).\,u \;\; +3(P_{3x}\text{-}P_{2x})\,(1\text{-}u).u^2 + (P_{4x}\text{-}P_{3x})\,u^3]$$

$$A_y(u)= 4[(P_{1y}\text{-}P_{oy})(1\text{-}u)\,(1\text{-}u)\,(1\text{-}u) \;\; +3(P_{2y}\text{-}P_{1y})(1\text{-}u)\,(1\text{-}u).\,u \;\; +3(P_{3y}\text{-}P_{2y})\,(1\text{-}u).u^2 + (P_{4y}\text{-}P_{3y})\,u^3]$$

$$A_z(u)= 4[(P_{1z}\text{-}P_{oz})(1\text{-}u)\,(1\text{-}u)\,(1\text{-}u) \;\; +3(\mathrm{P}_{2z}\text{-}\mathrm{P}_{1z})(1\text{-}u)\,(1\text{-}u).\,u \;\; +3(P_{3z}\text{-}P_{2z})\,(1\text{-}u).u^2 + (P_{4z}\text{-}P_{3z})\,u^3].$$

It can determine the unit tangent vector $\mathbf{t}$ in the form

$$\mathbf{t} = (1/\xi) < A_x(u), A_y(u), A_z(u) > \text{ and } \xi = [A_x^2(u) + A_y^2(u) + A_z^2(u)]^{1/2} \tag{10}$$

Consider this vector $\mathbf{t}$ as a normal vector of a plane $\mathbf{\Lambda}$ such that every vector in the plane $\mathbf{\Lambda}$ is orthogonal to $\mathbf{t}$. If the tangent vector $\mathbf{t}$ is $\mathbf{t} = <t_x,t_y,t_z> \neq \mathbf{0}$, we can determine two-unit vectors $[\mathbf{u},\mathbf{v}]$ in the plane $\mathbf{\Lambda}$ that meet $\mathbf{t.u} = 0$ and $\mathbf{t} \wedge \mathbf{u} = \mathbf{v}$. A vector $\mathbf{a} = <-t_y,t_x,0>$ verifies $\mathbf{t.a} = -\,t_x.\,t_y + t_y.\,t_x + t_z.0 = 0$. This means that it is be located in the plan $\mathbf{\Lambda}$ and $\mathbf{a} \perp \mathbf{t}$. Therefore, it can define the unity vector $\mathbf{u} = \mathbf{a}/\|\mathbf{a}\| = [1/\sqrt{(\,t_y^2+t_x^2)}] <-t_y,t_x,0>$ and then we calculate $\mathbf{v} = \mathbf{t} \wedge \mathbf{u}$. Thus, we can express Equation (3) in this fashion

$$\mathbf{P}(u,v) \;=\; \mathbf{C}(u) + \rho(u,v).(\cos\Omega\,\mathbf{u} + \sin\Omega\,\mathbf{v}). \tag{11}$$

where $0 \le u,\, v \le 1$.

Based on Equations (5) up to (11), the pipes can be constructed in the following procedure.

a. Define the curve $\mathbf{C}(u)$ of quartic Bézier and calculate the unit orthonormal vectors $[\mathbf{t}, \mathbf{u}, \mathbf{v}]$.

b. Select the real functions $q(v)$ of Equations (5, 6, 7) to construct the cross-section shapes of pipes in direction v.

c. Determine the real functions $q(u)$ to model the pipe volume along the parameter u (lengthwise directions of pipe) such that the radius $\rho(u,v)$ in the interval $0 \le u,\, v \le 1$ is $\rho(u,v) = q(v).\, q(u)$.

We simulate the pipe' construction as follows.

Simulation 3

Electing data of the quartic Bézier curve $\mathbf{C}(u)$ with control points $\mathbf{P}_o$= <-10,-30,15>, $\mathbf{P}_1$ =<0,-32,32>, $\mathbf{P}_2$ = <0,30,15>, $\mathbf{P}_3$ = <0,-10,8>, $\mathbf{P}_4$ = <20,26,-30> and $\rho(u,v) = 6$ gives the pipe shape in Figure 3a. In case $\rho(u,v) = 2[(1- u)^4 + 12(1-u)^3 u +18(1-u)^2 u^2 +8 (1-u) u^3 + u^4]$, it finds Figure 3b. Afterthat, we set $\mathbf{P}_o$= <-10,-35,10>, $\mathbf{P}_1$ =<-5,-12,35>, $\mathbf{P}_2$ = <0,30,15>, $\mathbf{P}_3$ = <5,-10,8>, $\mathbf{P}_4$ = <20,26,-30> and $\rho(u,v) = q(v) = 8(1- v)^4 -8(1-v)^3 v - 6(1-v)^2 v^2 -8 (1-v) v^3 + 8 v^4$, it presents Figure 3c. Furthermore, if $\rho(u,v) = [8(1- v)^4 -8(1-v)^3 v - 6(1-v)^2 v^2 -8 (1-v) v^3 + 8 v^4][(1- u)^4 + 12(1-u)^3 u +18(1-u)^2 u^2 +8 (1-u) u^3 + u^4]$, we obtain Figure 3d.

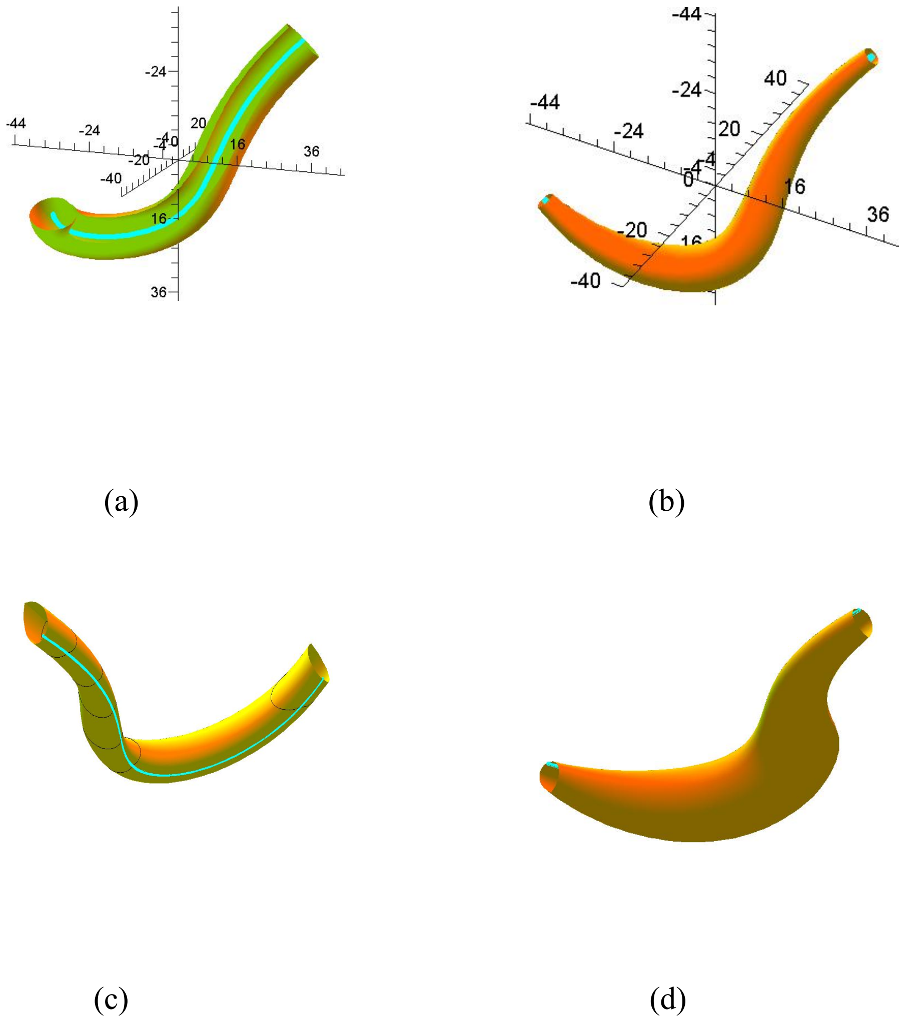
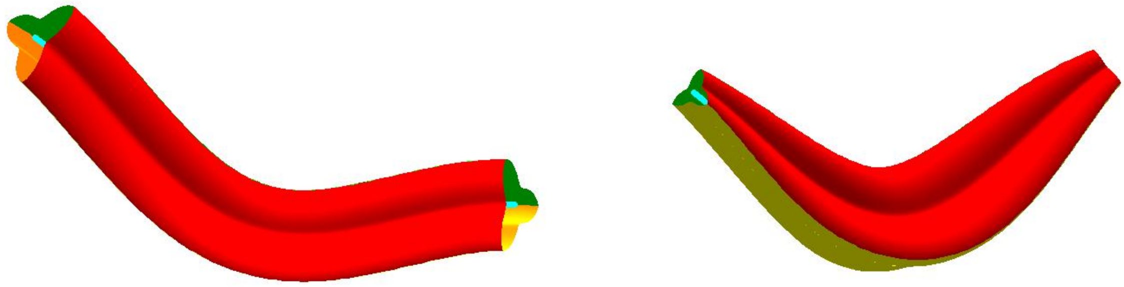


Figure 3. Pipes defined by quartic Bézier curves

Simulation 4

We set data $q(v)$ in Equation (7) in the form $q(v) = 8(1- v)^4 +28(1-v)^3 v - 6(1-v)^2 v^2 +28 (1-v) v^3 + 8 v^4$ and $n = 3$. Implementing these data in Equation (11) with the control point $\mathbf{P}_o$=

<-40,-0,10>, $\mathbf{P}_1$ =<-15,0,0>, $\mathbf{P}_2$ = <0,20,-15>, $\mathbf{P}_3$ = <15,0,8>, $\mathbf{P}_4$ = <35,0,30> and $\rho(u,v) = q(v)$ presents Figure 4a. If $\rho(u,v) = q(v)[\,(1-u)^4 + 12(1-u)^3 u + 18(1-u)^2 u^2 + 8(1-u) u^3 + u^4]$, it obtaints Figure 4b.



(a) (b)

Figure 4. Volume modification of the pipe

This paper establishes a procedure for constructing various pipe shapes with a center curve of the quartic Bézier $\mathbf{C}(u)$. The choice $q(v)$ constant will find the cross-section form of a circle (Figure 1a). The value $\rho(u,v)$ is constant, yielding a pipe with constant cross-section and volume along the interval $0 \le u \le 1$ (Figure 3a), but if $\rho(u,v)$ is not constant, the pipe volume will be modified (Figure 3b). In case $q(v)$ and $\rho(u,v)$ are not constant, the radius of cross-section curves and the volume of the pipes will be modified (Figure 1b,1c,1d,3d).

When $q(v)$ is defined from the composition of the formula as shown in Equation (6,7) and $q(u)$ is constant, we find the pipes' cross-section of two or more leaf forms (Figure 2) and the constant volume (Figure 4a). When $q(v)$ and $q(u)$ are not constant, it constructs pipes with cross-sectional shapes of different radii and volumes (Figure 4b).

3. Conclusion

It presents parametric equations for designing the cross-section and different pipe volumes using quartic Bézier curves. Evaluating different values for the control coefficients of the radius function obtains various shapes of the pipes' cross-section. Besides, selecting certain control points of the quartic Bézier curve and the control coefficients of the fluctuation function modify the shapes and volumes of the pipes. Therefore, these equations give the desired pipe forms for multiple thicknesses and various volume fluctuations. It is expected that the pipes' construction is efficient and useful. The pipe design method has been discussed; it is now important to introduce a method for building an effective pipeline network.

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