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The counterexample in complex analysis

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Abstract. Complex analysis (CA) may exist some intrinsic errors. In this paper we give a counterexample in CA.

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1. Introduction

The square-root of -1 is denoted by i, it does not exist on the 2-dimensional plane, but Gauss put it on the plane, and later on CA was developed by Cauchy and some others. Nowadays CA is widely used in mathematics. However, in this paper we can construct a counterexample in CA.

2. The necessary preliminaries

Our starting point is a well-known identity, see (7) of p.62 of [D], usually it is used to define $\zeta(s)$ to be an analytic function except at $s = 1$:

$$(s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)) =$$

(1) $$1 + s(s-1)\left(\int_1^\infty \omega(x)\left(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}\right)dx\right),$$

where $\Gamma(.)$ is the gamma function(see [C], Theorem 7.15), $\zeta(.)$ is the zeta function([C], Definition 8.1), and

$$\omega(x) = \frac{1}{2}\theta(x) - \frac{1}{2} = \sum_{n\geq 1} e^{-\pi n^2 x} \text{ (see (5) and (6) of p.62 of [D])},$$

$s = \sigma + it$, $\sigma \geq 1/2$.

Our purpose is to get a contradiction from (1) by giving a new treatment for the integral of the right side of (1).

Lemma 1. For any complex number $s = \sigma + it$, $\sigma \geq -1$, $s \neq 0,1,3,5,7,9$. we have

$$\int_1^\infty \omega(x)\left(x^{\frac{s}{2}-1}\right)dx = A_1 + B_1 + C_1 + D_1 + E_1 + F_1,$$

where

$$A_1 = -\frac{2\alpha}{s}, \alpha = \sum_{n\geq 1} e^{-\pi n^2},$$

$$B_1 = -\frac{4\pi\beta}{s(s+2)}, \beta = \sum_{n\geq 1} n^2 e^{-\pi n^2},$$

$$C_1 = -\frac{8\pi^2\gamma}{s(s+2)(s+4)}, \gamma = \sum_{n\geq 1} n^4 e^{-\pi n^2},$$

$$D_1 = -\frac{16\pi^3\delta}{s(s+2)(s+4)(s+6)}, \delta = \sum_{n\geq 1} n^6 e^{-\pi n^2},$$

$$E_1 = -\frac{32\pi^4\tau}{s(s+2)(s+4)(s+6)(s+8)}, \tau = \sum_{n\geq 1} n^8 e^{-\pi n^2},$$

$$F_1 = \frac{32\pi^5}{s(s+2)(s+4)(s+6)(s+8)}\int_1^\infty \left(\sum_{n\geq 1} n^{10} e^{-\pi x n^2}\right)\left(x^{\frac{s}{2}+4}\right)dx.$$

Proof: Firstly we only consider a real $s > 1/3$. By using the integration by parts for the Riemann-Stieljes integral(see Theorems 7.6 and 7.8 of [A]) consecutively we get

$$\int_1^\infty \omega(x)\left(x^{\frac{s}{2}-1}\right)dx = \frac{2}{s}\int_1^\infty \omega(x)dx^{\frac{s}{2}}$$

$$= \frac{2}{s}\left(-\omega(1) + \int_1^\infty \pi\left(\sum_{n\geq 1} n^2 e^{-\pi x n^2}\right)\left(x^{\frac{s}{2}}\right)\right)dx$$

$$= A_1 + \frac{2\pi}{s}\int_1^\infty \left(\sum_{n\geq 1} n^2 e^{-\pi x n^2}\right)\left(x^{\frac{s}{2}}\right)dx$$

$$= A_1 + \frac{4\pi}{s(s+2)}\left(-\sum_{n\geq 1} n^2 e^{-\pi n^2} + \pi\int_1^\infty \left(\sum_{n\geq 1} n^4 e^{-\pi x n^2}\right)\left(x^{\frac{s}{2}+1}\right)dx\right)$$

$$= A_1 + B_1 + \frac{4\pi^2}{s(s+2)}\int_1^\infty \left(\sum_{n\geq 1} n^4 e^{-\pi x n^2}\right)\left(x^{\frac{s}{2}+1}\right)dx$$

$$= A_1 + B_1 + C_1 + \frac{8\pi^3}{s(s+2)(s+4)}\int_1^\infty \left(\sum_{n\geq 1} n^6 e^{-\pi x n^2}\right)\left(x^{\frac{s}{2}+2}\right)dx$$

$$= \cdots = A_1 + B_1 + C_1 + D_1 + \frac{16\pi^4}{s(s+2)(s+4)(s+6)}\int_1^\infty \left(\sum_{n\geq 1} n^8 e^{-\pi x n^2}\right)\left(x^{\frac{s}{2}+3}\right)dx$$

$$= \cdots = A_1 + B_1 + C_1 + D_1 + E_1 + F_1,$$

which gives the asserted formula. To verify the validity for all $s = \sigma + it$, $\sigma \geq 1/2$, $s \neq 0,1,3,5,7,9$, we use Theorem 3.7 of Chapter 4 of [C]. Let $f: G = \{s = \sigma + it,\ \sigma > -1, s \neq 0,1,3,5,7,9\} \to C$ be defined as

$$f(s) = \int_1^\infty \omega(x)\left(x^{\frac{s}{2}-1}\right)dx - (A_1 + B_1 + C_1 + D_1 + E_1 + F_1).$$

Obviously $f(s)$ is analytic on G, because each term of the right side is analytic on G. We need to use the part "(b)⇒(a)" of Theorem 3.7 of Chapter 4 of [C](which used properties of open connected topological spaces, see §23 of [M]). If $a = 11$ satisfies the part (b), then the required formula follows, otherwise 11 is a zero of $f(s)$ on G, and locally for some positive integer $n \geq 1$, $f(s)$ can be represented as $f(s) = (s-11)^n g(s)$, $g(11) \neq 0$, $g(s)$ is analytic and continuous at 11, so that there is a sufficiently small positive number ε such that $g(11+\varepsilon) \neq 0$,

$$f(11+\varepsilon) = \varepsilon^n g(11+\varepsilon) \neq 0,$$

but this is impossible, because $f(\sigma) = 0$ for all $\sigma > 10$. Thus the equality of Lemma 2 holds for any complex number $s = \sigma + it, \sigma > -1$, $s \neq 0,1,3,5,7,9$. The proof is finished.

Lemma 2. For complex number $s = \sigma + it$, $\sigma > -1, s \neq 0,1,3,5,7,9$ we have

$$\int_1^\infty \omega(x) \left(x^{-\frac{s}{2}-\frac{1}{2}}\right) dx = A_2 + B_2 + C_2 + D_2 + E_2 + F_2,$$

where

$$A_2 = -\frac{2\alpha}{1-s},$$

$$B_2 = -\frac{4\pi\beta}{(s-1)(s-3)},$$

$$C_2 = \frac{8\pi^2\gamma}{(s-1)(s-3)(s-5)},$$

$$D_2 = -\frac{16\pi^3\delta}{(s-1)(s-3)(s-5)(s-7)}.$$

$$E_2 = \frac{32\pi^4\tau}{(s-1)(s-3)(s-5)(s-7)(s-9)},$$

$$F_2 = \frac{32\pi^4}{(s-1)(s-3)(s-5)(s-7)(s-9)} \int_1^\infty \left(\sum_{n\geq 1} n^{10} e^{-\pi x n^2}\right) \left(x^{-\frac{s}{2}+9/2}\right) dx.$$

Proof: It is similar to Lemma 1. Thus we omit the details.

3. The counterexample

We have (using the expansion $(1+X)^{-1} = 1 - X + X^2 - \cdots$ for $|X| < 0.1$)

$$s(s-1)(A_1 + A_2) = 2\alpha,$$

$$s(s-1)(B_1 + B_2) = -4\pi\beta\left(\frac{s-1}{s+2} + \frac{s}{s-3}\right)$$

$$= -4\pi\beta\big(2 - 15t^{-2} + i(15 - 30\sigma)t^{-3} + O(t^{-4})\big),$$

$$s(s-1)(C_1 + C_2) = 8\pi^2\gamma\left(\frac{s}{(s-3)(s-5)} - \frac{s-1}{(s+2)(s+4)}\right)$$

$$= 8\pi^2\gamma(-15t^{-2} + i(15 - 9\sigma)t^{-3} + O(t^{-4})),$$

$$s(s-1)(D_1 + D_2) = -16\pi^3\delta\left(\frac{s-1}{(s+2)(s+4)(s+6)} + \frac{s}{(s-3)(s-5)(s-7)}\right)$$

$$= 16\pi^3\delta\big(2t^{-2} + i(2 + 4\sigma)t^{-3} + O(t^{-4})\big),$$

$$s(s-1)(E_1 + E_2) = 32\pi^4\tau(-\frac{s-1}{(s+2)(s+4)(s+6)(s+8)}$$

$$+\frac{s}{(s-3)(s-5)(s-7)(s-9)}) + O(t^{-4}) = O(t^{-4}),$$

and thus for all $s = \sigma + it$. $\sigma > -1$ and $|t| > 10$ we get from Lemmas 1, 2 and (1) that (we just trivially estimate F_1, F_2)

$$s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

(2) $$= \rho_1 + \rho_2 t^{-2} + it^{-3}(\rho_3\sigma + \rho_4) + O(t^{-4}),$$

where

$$\rho_1 = 1 + 2\alpha - 8\pi\beta, \rho_2 = 60\pi\beta - 120\pi^2\gamma + 32\pi^3\delta,$$

$$\rho_3 = 120\pi\beta - 72\pi^2\gamma + 64\pi^3\delta,\ \rho_4 = -60\pi\beta + 120\pi^2\gamma + 32\pi^3\delta.$$

If $\rho_1 \neq 0$, we have from (2)

$$s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \rho_1 + O(t^{-2}).$$

if $\rho_1 = 0, \rho_2 \neq 0$, we have

$$s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \rho_2 t^{-2} + O(t^{-3}),$$

and if $\rho_1 = \rho_2 = 0$, we have

$$s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

$$= it^{-3}(\rho_3\sigma + \rho_4) + O(t^{-4}).$$

Thus we can always get a contradiction from (2) for $\sigma \geq 1/2$ and large $|t|$, after multiplying a suitable power of t on both sides of (2), because obviously $\rho_3 >$

0, $\rho_3\sigma + \rho_4 \geq \rho_3/2 + \rho_4 > 0$, and for $s = \sigma + it$. $\sigma = 1/2$ and $|t| \to \infty$ (see (5), § 10, and (3), §12, [D])

$$\Gamma\left(\frac{s}{2}\right) = O(e^{-|t|/5}),\ \ |\zeta(s)| = O(|t|),$$

because the right side then tends to a non-zero constant, while the left side tends to zero.

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