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Common Fixed Point Theorems for Generalized (ψ,φ) Weakly Contraction Maps

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Abstract

In this paper, common fixed point theorems of some continuous self-maps under the generalized (ψ,φ) weakly contractions are established in complete metric spaces (X, d). The theorems are proved by subjecting the maps S and T as subspaces of complete metric spaces $(f(X),d)$ and $(g(X),d)$, respectively, and undermining the idea commuting or IT-commuting maps. The results obtained are improvement and unification of some known results as justify with examples. Recent literature are embedded in the present results.

Keywords: *Common fixed point, generalized* (ψ,φ) *weak contraction, complete metric spaces.*

1. Introduction

A large number of contractive-type definitions exist in the literature which generalized the Banach's contraction map for a function or a pair of functions, see [1-3, 5-33]. One of such generalization which is very useful in the present study is due to Ciric [10]:

$$\boldsymbol{d(Tx,Ty) \le rM(x,y)}; \textbf{ for all } \boldsymbol{x,y \in X} \quad (1)$$

where $\boldsymbol{M(x,y) = max\{d(x, y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+ d(y,Tx)}{2}\}}$. Any map T which satisfies (1) is a generalized contraction map and it was similar to the general contraction map in [12]. To unify these numerous definitions, several generalization but weaker conditions also appeared in many literature. At first, the idea of $\boldsymbol{\varphi}$- contraction was introduced as a weaker condition in Hilbert spaces setting, see [2], as follow: A mapping $\boldsymbol{T: X \to X}$ is said to be weakly contractive if

$$\boldsymbol{d(Tx,Ty) \le d(x,y) - \varphi(d(x,y))}; \textbf{ for all } \boldsymbol{x,y \in X} \quad (2)$$

where $\boldsymbol{\varphi}:[\mathbf{0},\infty) \to [\mathbf{0},\infty)$ is semi-continuous and non-decreasing function with $\boldsymbol{\varphi(t) = 0}$ if and only if $\boldsymbol{t = 0}$. Later in [28], Rhoades showed that the weakly contractivecondition was also valid in complete metric spaces if by letting $\boldsymbol{\varphi(t)} =$ at, where $\boldsymbol{a} \in (\mathbf{0},\mathbf{1})$, condition (2) becomes the Banach's contraction condition. The Kannan map and Chatterjea map among other contractive-type maps had also been furnished with the $\boldsymbol{\varphi}$- contraction to prove some fixed point theorems as appeared in [7, 8]. In [21], Khan introduced the idea of altering distance for proving some fixed point theorems of two points while in [15] Dutta and Choudhury introduced $(\psi, \boldsymbol{\varphi})$-weakly contraction for every $\boldsymbol{x,y \in X}$

$$\psi(\boldsymbol{d(Tx,Ty)}) \le \psi(\boldsymbol{d(x,y)}) - \boldsymbol{\varphi(d(x,y))}; \textbf{ for all } \boldsymbol{x,y \in X} \quad (3)$$

where $\psi:[0,\infty) \to [0,\infty)$ is a continuous and monotone non-decreasing function with $\psi(t) = 0$ if and only if $t = 0$. Also in [13], Doric obtained another generalized weakly contractive map by combining the conditions (1) and (3) as follow:

$$\psi(\boldsymbol{d(Tx,Ty)}) \le \psi(\boldsymbol{M(x,y)}) - \boldsymbol{\varphi(M(x,y))}; \textbf{ for all } \boldsymbol{x,y \in X} \quad (4)$$

Another classical generalization was also given in [14] as:

Theorem 1.1. Define a non-increasing function θ from $[0,1)$ onto $(0, 1]$ by

$$\theta(r) = \begin{cases} 1, & 0 \le r < \frac{1}{2} \\ 1 - r, & \frac{1}{2} \le r < 1 \end{cases}$$

Let X be a complete metric space and $T: X \to X$ such that for every $x,y \in X$

$$\boldsymbol{\theta(r)d(Tx,x) \leq d(x,y)} \textbf{ implies } \boldsymbol{d(Tx,Ty) \leq rM(x,y)} \qquad (5)$$

where $M(x, y) = max\{d(x, y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+ d(y,Tx)}{2}\}$. Then T has a unique fixed point. A generalized weak contractive map was also introduced in [29], this result was obtained by combining the conditions (1), (3) and (4) as:

$$\frac{1}{2}\mathbf{d(Tx,x) \leq d(x,y)} \textbf{ implies } \psi(\mathbf{d(Tx,Ty)}) \leq \psi(\mathbf{M(x,y)}) - \varphi(\mathbf{M(x,y)}) \qquad (6)$$

It was remarked there that any map T satisfying (6) is more general than (4). Recently in [16], Fei et al. proved a conjecture posted in [29] and justify the result with an example. Other recent extensions of the weak contractions can be found in the following literature [6, 16, 25]. This present paper is designated to improve, unify and further extend most of these highlighted results in the literature and justify them with examples.

2. Main Results

The following lemma is vital in the proof of the mainresults.

Lemma 2.1. Let (X,d) be a complete metric space and let f and g be two maps such that $f(X)$, $g(X) \subset X$. Let fx_n and gx_n be sequences in $f(X)$ and $g(X)$ such that $d(fx_n,gx_{n+1}) \to 0$ as $n \to 0$. Assume that fx_n is not a Cauchy sequence, then there exists a number $\varepsilon > 0$ and positive integers m_k and n_k with $n_k > m_k >$ k such that $d(fx_{m_k},gx_{n_k}) \geq \varepsilon$ and $d(fx_{m_k},gx_{n_{k-1}}) < \varepsilon$ for each $k \in$ N, then the following hold:

i . $d(fx_{m_{k+1}},gx_{n_k}) = \varepsilon$

i i . $d(fx_{m_k},gx_{n_k}) = \varepsilon$

i i i . $d(fx_{m_{k-1}},gx_{n_k}) = \varepsilon$

The main results are as follows:

Theorem 2.1 Let (X,d) be a complete metric space and let f, g, S and T be four continuous self-maps such that $S(X) \subset g(X)$; $T(X) \subset f(X)$, then for every $x,y \in X$ satisfying

$$\frac{1}{2}\boldsymbol{min}\{\boldsymbol{d}(\boldsymbol{Sx},\boldsymbol{fx}),\boldsymbol{d}(\boldsymbol{Ty},\boldsymbol{gy})\} \leq \boldsymbol{d}(\boldsymbol{fx},\boldsymbol{gy})$$

implies

$$\psi(\boldsymbol{d}(\boldsymbol{Sx},\boldsymbol{Ty})) \leq \psi\left(\boldsymbol{M}_{\boldsymbol{f},\boldsymbol{g}}(\boldsymbol{x},\boldsymbol{y})\right) - \boldsymbol{\varphi}\left(\boldsymbol{M}_{\boldsymbol{f},\boldsymbol{g}}(\boldsymbol{x},\boldsymbol{y})\right) \qquad (7)$$

where $M_{f,g}(x,y) = max\left\{d(fx,g\ y),d(fx,Sx),d(gy,Ty),\frac{d(fx,Ty)+d(gy,Sx)}{2}\right\}$, both ψ and φ are given in (3). Then S,T,g and f have a unique common fixed point.

Proof: Let $x_0 \in X$ be fixed and let $\{x_n\} \in X$ be a sequence satisfying the iterations

$Sx_{n-1} = gx_n$ and $Tx_n = fx_{n+1}$

By the condition (7), the following holds:

$$\frac{1}{2}min\{d(Sx_{n-1},fx_{n-1}),d(Tx_n,gx_n)\} = \frac{1}{2}min\{d(gx_n,fx_{n-1}),d(fx_{n+1},gx_n)\}$$

$$= \frac{1}{2}d(fx_{n+1},gx_n) \leq d(fx_{n-1},gx_n)$$

More so, since

$$\begin{aligned}
&M_{f,g}(Sx_{n-1},Tx_n) \\
&= max\left\{d(fx_{n-1},gx_n),d(fx_{n-1},Sx_{n-1}),d(gx_n,Tx_n),\frac{d(fx_{n-1},Tx_n)+d(gx_n,Sx_{n-1})}{2}\right\} \\
&= max\left\{d(fx_{n-1},gx_n),d(fx_{n-1},gx_n),d(gx_n,fx_{n+1}),\frac{d(fx_{n-1},fx_{n+1})+d(gx_n,gx_n)}{2}\right\} \\
&\leq max\left\{d(fx_{n-1},gx_n),d(fx_{n-1},gx_n),d(gx_n,fx_{n+1}),\frac{d(fx_{n-1},gx_n)+d(gx_n,fx_{n+1})}{2}\right\} \\
&= d(fx_{n-1},gx_n);
\end{aligned}$$

Then, there results

$$\psi(d(gx_n,fx_{n+1})) \leq \psi(d(fx_{n-1},gx_n)) - \varphi(d(fx_{n-1},gx_n)) \qquad (8)$$

By the property of $\boldsymbol{\varphi}$, inequality (8) becomes

$$\psi(d(gx_n,fx_{n+1})) \leq \psi(d(fx_{n-1},gx_n)) \qquad (9)$$

Also, by the property of ψ, inequality (9) gives

$$d(gx_n,fx_{n+1}) \le d(fx_{n-1},gx_n)$$

Since $d(gx_n,fx_{n+1})$ is monotonic non-increasing, there is non-negative number p such that

$$\lim_{n\to\infty} d(gx_n,fx_{n+1}) = \lim_{n\to\infty} d(fx_{n-1},gx_n) = p \qquad (10)$$

Applying (10) to inequality (8) gives

$$\psi(p) \le \psi(p) - \varphi(p)$$

As a result, $\boldsymbol{\varphi}(\mathrm{p}) \le 0$ and by the property of $\boldsymbol{\varphi}$, $\varphi(p) = 0$ if and only if $p = 0$

Therefore,

$$\lim_{n\to\infty} d(gx_n,fx_{n+1}) = 0$$

Next is to prove that the sequence gx_n is a Cauchy sequence. Suppose otherwise, for any given $\varepsilon > 0$ there exists $n_k > m_k > k$ such that

$$d(fx_{m_k},gx_{n_k}) \ge \varepsilon \text{ and } d(fx_{m_{k-1}},gx_{n_k}) < \varepsilon$$

By **Lemma 2.1** and condition (7), there results

$$\frac{1}{2}\min\{d(fx_{m_{k-1}},Sx_{m_{k-1}}),d(gx_{n_k},Tx_{n_k})\} = \frac{1}{2}\min\{d(fx_{m_{k-1}},gx_{m_k}),d(gx_{n_k},fx_{n_{k+1}})\}$$

$$= \frac{1}{2}d(gx_{n_k},fx_{n_{k+1}}) = 0 \le d(gx_{n_k},fx_{m_{k-1}})$$

Also,

$$\psi(d(Sx_{m_{k-1}},Tx_{n_k})) \le \psi(\mathrm{M}_{f,g}(Sx_{m_{k-1}},Tx_{n_k})) - \boldsymbol{\varphi}(\mathrm{M}_{f,g}(Sx_{m_{k-1}},Tx_{n_k})) \qquad (11)$$

where

$$\mathrm{M}_{f,g}(Sx_{m_{k-1}},Tx_{n_k})$$
$$= \max\left\{d(fx_{m_{k-1}},gx_{n_k}),d(fx_{m_{k-1}},Sx_{m_{k-1}}),d(gx_{n_k},Tx_{n_k}),\frac{d(gx_{n_k},Sx_{m_{k-1}}) + d(fx_{m_{k-1}},Tx_{n_k})}{2}\right\}$$

$$= \max\left\{d(fx_{m_{k-1}},gx_{n_k}),d(fx_{m_{k-1}},fx_{m_k}),d(gx_{n_k},gx_{n_{k+1}}),\frac{d(gx_{n_k},fx_{m_k}) + d(fx_{m_{k-1}},gx_{n_{k+1}})}{2}\right\}$$

$$= \max\{\varepsilon,0,0,\varepsilon\} = \varepsilon$$

Therefore, inequality (11) becomes

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \boldsymbol{\varphi}(\varepsilon)$$

Consequently, $\boldsymbol{\varphi}(\varepsilon) \leq 0$, and by the property of $\boldsymbol{\varphi}$, $\boldsymbol{\varphi}(\varepsilon) = 0$ if and only if $\varepsilon = 0$. This contradicts the fact that $\varepsilon > 0$. It follows that $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, there exists a point $u \in g(X)$ such that $gx_n \to$ u.

More so, since $S(X) \subset g(X)$, then there is a point $u' \in S(X)$ such that

$$\lim_{n\to\infty} gx_n = gu' = u(= \mathrm{Su})$$

Next is to show that $f u'$ is also a fixed point of S. By condition (7), we have

$$\frac{1}{2}\min\{\mathrm{d}(\mathrm{fu}',\mathrm{S}u'),\mathrm{d}(gx_n,\mathrm{T}x_n)\} = \frac{1}{2}\min\{\mathrm{d}(\mathrm{fu}',gx_n),\mathrm{d}(gx_n,fx_{n+1})\} = 0 \leq \mathrm{d}(\mathrm{fu}',gx_n)$$

But then,

$$\psi(d(Su',Tx_n)) \leq \psi(M_{f,g}(Su',Tx_n)) - \boldsymbol{\varphi}(M_{f,g}(Su',Tx_n))$$

where

$$M_{f,g}(Su',Tx_n) = \max\left\{d(fu',gx_n),\, d(fu',Su'), d(gx_n,Tx_n), \frac{d(gx_n,Su') + d(fu',Tx_n)}{2}\right\}$$

$$= \max\left\{d(fu',gx_n),\, d(fu',gx_n), d(gx_n,fx_{n+1}), \frac{d(gx_n,gx_n) + d(fu',fx_{n+1})}{2}\right\}$$

$$= \max\left\{d(fu',gx_n), d(fu',gx_n), 0, \frac{d(fu',gx_n)}{2}\right\} \qquad = d(fu',gx_n)$$

Therefore,

$$\psi(d(fu',gx_n)) \leq \psi(d(fu',gx_n)) - \boldsymbol{\varphi}(d(fu',gx_n))$$

By applying limit as $n \to \infty$, there gives

$$\psi(d(fu',Su)) \leq \psi(d(fu',Su)) - \varphi(d(fu',Su))$$

As a result, $\boldsymbol{\varphi}\big(d\big(fu^{'}, Su\big)\big) \leq 0$. By the property of $\boldsymbol{\varphi}$, $d\big(fu^{'}, Su\big) = 0$ if and only if $fu^{'} = Su$.

Moreover, since $f(X) \supset T(X)$, then, $u \in f(X)$ implies that

$$\lim_{n\to\infty} fx_n = fu^{'} \ (= Tu^{'})$$

Next is to show that $gu^{'}$ is also a fixed point of mapping T. Again, by condition (7), we have

$$\frac{1}{2}\min\{\mathrm{d}(\mathrm{f}x_n, \mathrm{S}x_n), \mathrm{d}\big(gu^{'}, \mathrm{T}u^{'}\big)\} = \frac{1}{2}\min\{\mathrm{d}(fx_n, gx_{n+1}), \mathrm{d}\big(gu^{'}, fx_n\big)\} = 0 \leq \mathrm{d}\big(gu^{'}, fx_n\big)$$

More so,

$$\psi\big(d(Sx_n, Tu^{'})\big) \leq \psi\big(M_{f,g}(Sx_n, Tu^{'})\big) - \boldsymbol{\varphi}\big(M_{f,g}(Sx_n, Tu^{'})\big)$$

where

$$M_{f,g}\big(Sx_n, Tu^{'}\big) = \max\left\{d\big(fx_n, gu^{'}\big), d(fx_n, Sx_n), d\big(gu^{'}, Tu^{'}\big), \frac{d\big(gu^{'}, Sx_n\big) + d\big(fx_n, Tu^{'}\big)}{2}\right\}$$

$$= \max\left\{d\big(Tu^{'}, gu^{'}\big), d(fx_n, gx_{n+1}), d\big(gu^{'}, Tu^{'}\big), \frac{d\big(gu^{'}, gx_{n+1}\big) + d\big(fx_n, Tu^{'}\big)}{2}\right\}$$

$$= \max\left\{d\big(gu^{'}, Tu^{'}\big), 0, d\big(gu^{'}, Tu^{'}\big), \frac{d\big(gu^{'}, Tu^{'}\big) + d\big(gx_{n+1}, Tu^{'}\big)}{2}\right\}$$

$$= d\big(gu^{'}, Tu^{'}\big)$$

Therefore,

$$\psi\big(d\big(gx_{n+1}, Tu^{'}\big)\big) \leq \psi\big(d\big(gu^{'}, Tu^{'}\big)\big) - \boldsymbol{\varphi}\big(d\big(gu^{'}, Tu^{'}\big)\big)$$

By applying limit as $n \to \infty$, there gives

$$\psi\big(d\big(gu^{'}, Tu^{'}\big)\big) \leq \psi\big(d\big(gu^{'}, Tu^{'}\big)\big) - \boldsymbol{\varphi}\big(d\big(gu^{'}, Tu^{'}\big)\big)$$

Consequently, $\boldsymbol{\varphi}\big(d\big(gu^{'}, Tu^{'}\big)\big) \leq 0$. By the property of $\boldsymbol{\varphi}$, $d\big(gu^{'}, Tu^{'}\big) = 0$ if and only if $gu^{'} = Tu^{'}$.

Hence, $u = gu^{'} = Tu^{'} = fu^{'} = Su$ implies that u is a common fixed point of f, g, S and T.

Next is to prove the uniqueness. Let u and w be two common fixed points of f, g, S and T, then

$$\frac{1}{2}\min\{d(fu,Su),d(gw,Tw)\} = 0 \leq d(fu,gw) \neq 0$$

and

$$\psi(d(Su,Tw)) \leq \psi\big(M_{f,g}(Su,Tw)\big) - \boldsymbol{\varphi}\big(M_{f,g}(Su,Tw)\big)$$

where $M_{f,g}(Su,Tw) = \max\left\{d(fu,gw), d(fu,Su), d(gw,Tw), \frac{d(gw,Su)+d(fu,Tw)}{2}\right\}$

$$= \max\left\{d(u,w), d(u,u), d(w,w), \frac{d(w,u)+d(u,w)}{2}\right\}$$

$$= \max\{d(u,w), 0, 0, d(u,w)\} = d(u,w)$$

Therefore,

$$\psi(d(u,w)) \leq \psi(d(u,w)) - \boldsymbol{\varphi}(d(u,w))$$

But $\boldsymbol{\varphi}(d(u,w)) \leq 0$, by the property of $\boldsymbol{\varphi}$**,** this implies $d(u,w) = 0$. This is a contradiction. Whence, $u = 0$.

Theorem 2.2. Let (X, d) be a complete metric space and let f, g, S and T be four continuous self-maps such that $S(X) \subset g(X)$, $T(X) \subset f(X)$, then for every $x, y \in X$ satisfying

$$\psi(\mathbf{d}(\mathbf{Sx},\mathbf{Ty})) \leq \psi\big(\mathbf{M}_{f,g}(\mathbf{x},\mathbf{y})\big) - \boldsymbol{\varphi}\big(\mathbf{M}_{f,g}(\mathbf{x},\mathbf{y})\big) \qquad (12)$$

Where ψ, $\boldsymbol{\varphi}$ and $M_{f,g}(x,y)$ are defined in **Theorem 2.1**. Then S, T, g and f have unique common fixed point.

Proof: By undermining the initial condition in **Theorem 2.1**. Then, Theorem 4 is easily proved.

Theorem 2.3. Let $(\boldsymbol{X},\boldsymbol{d})$ be a complete metric space and let $\boldsymbol{f},\boldsymbol{S}$ and $\boldsymbol{T}$ be three continuous self-maps such that $\boldsymbol{S}(\boldsymbol{X}), \boldsymbol{T}(\boldsymbol{X}) \subset \boldsymbol{f}(\boldsymbol{X})$, then for every $\boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X}$ satisfying

$$\frac{\mathbf{1}}{\mathbf{2}}\min\{\mathbf{d}(\mathbf{Sx},\mathbf{fx}),\mathbf{d}(\mathbf{Ty},\mathrm{fy})\} \leq \mathbf{d}(\mathbf{fx},\mathrm{fy})$$

implies

$$\psi(\mathbf{d}(\mathbf{Sx},\mathbf{Ty})) \leq \psi(\mathbf{M}_f(\mathbf{x},\mathbf{y})) - \boldsymbol{\varphi}(\mathbf{M}_f(\mathbf{x},\mathbf{y})) \tag{13}$$

Where ψ and $\boldsymbol{\varphi}$ are defined in **Theorem 2.1**, and $\mathrm{M}_f(x,y) = \max\left\{d(fx,fy),d(fx,Sx),d(fy,Ty),\frac{d(fx,Ty)+d(fy,Sx)}{2}\right\}$. Then S, T and f have unique common fixed point.

Proof: The proof is analog of the proof of **Theorem 2.1** if $X \supset f(X) \equiv g(X) \supset S(X);\ T(X)$

Theorem 2.4. Let (X,d) be a complete metric space and let f,S and T be three continuous self-maps such that $S(X), T(X) \subset f(X)$, then for every $x,y \in X$ satisfying

$$\psi(\mathbf{d}(\mathbf{Sx},\mathbf{Ty})) \leq \psi(\mathbf{M}_f(\mathbf{x},\mathbf{y})) - \boldsymbol{\varphi}(\mathbf{M}_f(\mathbf{x},\mathbf{y})) \tag{14}$$

Where ψ and $\boldsymbol{\varphi}$ are defined in **Theorem 2.1**, and $\mathrm{M}_f(x,y) = \max\left\{d(fx,fy),d(fx,Sx),d(fy,Ty),\frac{d(fx,Ty)+d(fy,Sx)}{2}\right\}$. Then S, T and f have unique common fixed point.

Proof: The proof follows from **Theorem 2.3**

Corollary 2.1. Let $(\boldsymbol{X},\boldsymbol{d})$ be a complete metric space and let $\boldsymbol{S}$ and $\boldsymbol{T}$ be two continuous self-maps such that for every $\boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X}$ satisfying

$$\frac{\mathbf{1}}{\mathbf{2}}\min\{\mathbf{d}(\mathbf{x},\mathbf{Sx}),\mathbf{d}(\mathrm{x},\mathrm{Ty})\} \leq \mathbf{d}(\mathbf{x},\mathrm{y})$$

implies

$$\psi(\mathbf{d}(\mathbf{Sx},\mathbf{Ty})) \leq \psi(\mathrm{M}(\mathbf{x},\mathbf{y})) - \boldsymbol{\varphi}(\mathrm{M}(\mathbf{x},\mathbf{y})) \tag{15}$$

Where ψ and $\boldsymbol{\varphi}$ are defined in **Theorem 2.1**, and $\mathrm{M}(x,y) = \max\left\{d(x,y),d(x,Sx),d(y,Ty),\frac{d(x,Ty)+d(y,Sx)}{2}\right\}$. Then S and T have unique common fixed point.

Corollary 2.2. Let $(\boldsymbol{X}, \boldsymbol{d})$ be a complete metric space and let $\boldsymbol{S}$ and $\boldsymbol{T}$ be two continuous self-maps such that for every $\boldsymbol{x,y} \in \boldsymbol{X}$ satisfying

$$\psi(\mathbf{d}(\mathbf{Sx},\mathbf{Ty})) \leq \psi(\mathrm{M}(\mathbf{x},\mathbf{y})) - \boldsymbol{\varphi}(\mathrm{M}(\mathbf{x},\mathbf{y})) \qquad (16)$$

Where ψ and $\boldsymbol{\varphi}$ are defined in **Theorem 2.1**, and $\mathrm{M}(x,y)$ is given in **Corollary 2.1.** Then S and T have unique common fixed point.

Proof: The proof follows from **Corollary 2 . 1**

Corollary 2.3. Let (X, d) be a complete metric space and let $T{:}X \to X$ be a continuous self-map such that for every $x,y \in X$ satisfying

$$\frac{1}{2}\mathbf{min}\{\mathrm{d}(x,\mathbf{T}x)\} \leq \mathrm{d}(x,\mathbf{y}) \text{ implies } \psi(\mathrm{d}(\mathrm{T}x,\mathrm{T}y)) \leq \psi(\mathbf{M}(x,y)) - \boldsymbol{\varphi}(\mathbf{M}(x,y)) \quad (17)$$

Where ψ and $\boldsymbol{\varphi}$ are defined in **Theorem 2.1**, and $\mathrm{M}(x,y)$ is given by (1). Then $\boldsymbol{T}$ has unique fixed point.

Proof: By letting $S \equiv T$ in **Corollary 2 . 1**, then the proof of this corollary is obvious.

Remark 2. Corollary 2.3 is the main result in [29].

The following Corollary is due to [14]:

Corollary 2.4. Let (X, d) be a complete metric space and let $T{:}X \to X$ be a continuous self-map such that for every $x,y \in X$ satisfying

$$\psi(\mathrm{d}(\mathrm{T}x,\mathrm{T}y)) \leq \psi(\mathbf{M}(x,y)) - \boldsymbol{\varphi}(\mathbf{M}(x,y)) \qquad (18)$$

Where ψ and φ are defined in **Theorem 2.1**, and $\mathrm{M}(x,y)$ is given by (1). Then T has unique fixed point.

Proof: The proof follows from **Corollary 2.2**.

The following examples are used to compare the above Theorems and Corollaries.

Example 1. Let $X = \Re^+$ be endowed with the usual metric $d(x,y) = |x - y|$.

Define $fx: = \frac{3}{2}x$, $gy: = \frac{4}{3}y$, $Sx: = \frac{9}{10}x$ and $Ty: = y$, for all $x,y \in \Re^+$.

Also, define $\psi:[0,\infty) \rightarrow [0,\infty)$ by $\psi(t) = \frac{1}{2}t$ and

$\boldsymbol{\varphi}:[0,\infty)^5 \rightarrow [0,\infty)$ by $\boldsymbol{\varphi}(t) = \frac{1}{6}\max\{t_1,t_2,t_3,\frac{t_4+t_5}{2}\}$.

Clearly, both $S(X)$ and $T(X)$ are subspaces of $f(X)$ and $g(X)$, respectively. The map T does not satisfy conditions (16) and (17) in **Corollaries 2.3** and **2.4**, respectively. More so, the pair (S,T) does not satisfy condition (13) of Theorem **2.3**. However, **Theorem 2.1, Theorem 2.2, Corollary 2 . 1** and **Corollary 2.2** satisfy all the hypotheses of **Example 1.** In fact, the left inequality of condition (7) is $\frac{9x-10y}{20}$ while the right inequality is $\frac{1}{10}x$ for each $x,y \in \Re^+$.

Example 2. Let $X = \{(1,1,1),(4,1,0),(1,4,0),(4,5,1),(5,4,1)\} \equiv \{a,b,c,d,e\}$ be endowed with the taxicab metric $d(x,y) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|$, where x and y are partially ordered vectors in X. Define

$S:X \rightarrow X$ by $Sa = Sb = Sc = a$, $Sd = b$ and $Se = c$;

$T:X \rightarrow X$ by $Ta = Tb = Tc = a$, $Td = c$ and $Te = b$;

$$f:X \rightarrow X \text{ by } fx = \frac{3}{2}x; \quad \text{and} \quad g:X \rightarrow X \text{ by } gy = 2y;$$

For given $\psi(t) = \frac{9}{10}t$ and $\boldsymbol{\varphi}(t) = \frac{1}{2}t$, all aforementioned results are compared in the Table below:

Table 1: Results for **Example 2**

Results	(x,y)			
	(a,b)	(b,c)	(c,d)	(d,e)
Theorem 2.1	Valid	Valid	Valid	Valid
Theorem 2.2	Valid	Valid	Valid	Valid
Theorem 2.3	Valid	Valid	Valid	Not valid

Corollary 2.1	Valid	Valid	Not valid	Not valid
Corollary 2.2	Valid	Valid	Not valid	Valid
Corollary 2.3	Valid	Valid	Not valid	Valid
Corollary 2.4	Valid	Valid	Not valid	Not valid

In **Table 1**, it is observed that **Theorem 2.3** and **Corollaries 2.1-2.4** fail to satisfy atleast one of the hypotheses of **Example 2**. On the other hand, the condition (7) and condition (8) of **Theorem 2.1** and **Theorem 2.2**, respectively, satisfy all the hypotheses of Example 2. This clearly shows the improvement and generality of **Theorem 2.1** and **2.2** over other conditions in the literature.

The following example shows the advantage of Theorem 3 over other results without initial conditions.

Example 3. Let $X = \{(1,1),(1,5),(5,1),(5,6),(6,5)\}$ be endowed with the taxicab metric $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$, where $x = (x_1,x_2)$ and $y = (y_1,y_2)$ are partially ordered in X.

Define

$$f:X \to X \text{ by } f\,x = x; \quad g:X \to X \text{ by } g\,y = \frac{4}{3}y.$$

Also, $S:X \to X$ and $T:X \to X$, respectively by

$$S(x_1,x_2) = \begin{cases} (x_1,1), & x_1 \le x_2 \\ (1,x_2), & x_1 > x_2 \end{cases}$$

and

$$T(y_1,y_2) = \begin{cases} (y_1,1), & y_1 \le y_2 \\ (1,y_2), & y_1 > y_2 \end{cases}$$

For given $\psi(t) = \frac{3}{4}t$ and $\boldsymbol{\varphi}(t) = \frac{1}{8}t$, the list of observations are presented as follow:

Table 2: Results for **Example 3**

Results	(x,y)				
	$[(1,1),(1,5)]$	$[(1,5),(5,1)]$	$[(5,1),(5,6)]$	$[(5,6),(6,5)]$	$[(6,5),(5,6)]$
Theorem 2.1	Valid	Valid	Valid	Valid	Valid
Theorem 2.2	Valid	Valid	Valid	Valid	Valid
Theorem 2.3	Valid	Valid	Valid	Valid	Valid
Theorem 2.4	Valid	Valid	Valid	Not valid	Not valid

Corollary 2.1	Valid	Valid	Valid	Valid	Valid
Corollary 2.2	Valid	Valid	Valid	Not valid	Not valid
Corollary 2.3	Valid	Valid	Valid	Valid	Valid
Corollary 2.4	Valid	Valid	valid	Not valid	Not valid

In **Table 2**, observe that **Theorem 2.4**, **Corollary 2.2** and **Corollary 2 . 4** are without initial conditions and all fail to satisfy the hypotheses of **Example 3** at either the point [(5,6),(6,5)] or [(6,5),(5,6)]

Corollary 2.5. Let (X, d) be a complete metric space and let f,g,S and T be four continuous self-maps such that for every $x,y \in X$ satisfying

$$\mathbf{d(Sx,Ty)} \leq \mathbf{M}_{f,g}\mathbf{(x,y)} - \boldsymbol{\varphi}\left(\mathbf{M}_{f,g}\mathbf{(x,y)}\right) \qquad (19)$$

Where $\boldsymbol{\varphi}$ and $\mathrm{M}_{f,g}(\mathrm{x,y})$ are defined in **Theorem 2.2**. Then S, T, g and f have unique common fixed point.

Proof: Let $\psi(s) = s,\ s \geq 0$ in **Theorem 2.2**. Then the proof is immediate.

Corollary 2.6. Let (X, d) be a complete metric space and let f,g,S and T be four continuous self-maps such that for every $x,y \in X$ satisfying

$$\mathbf{d(Sx,Ty)} \leq \boldsymbol{r}\mathbf{M}_{f,g}\mathbf{(x,y)} \qquad (20)$$

where $r \in (0,1)$ and $\mathrm{M}_{f,g}(\mathrm{x,y})$ are defined in **Theorem 2.2**. Then S, T, g and f have unique common fixed point.

Proof: By letting Let $\boldsymbol{\varphi}(t) = \alpha t$, where $\alpha \in (0,1)$ in **Corollary 2.5**. The condition (20) is easily deduced, where $r = 1-\alpha$, then the proof follows.

Remark 4. If $f \equiv g$ and $S \equiv T$ in condition (20), then **Corollary 2 . 6** reduces to the result in [11].

3. Concluding Remarks

The study has established some results concerning generalized (ψ,φ) weakly contraction in complete metric spaces to prove unique common fixed point theorems of some continuous self -maps. Example 1, 2 and 3 clearly show the advantage of the present result over other existing results for some continuous self maps satisfying the aforementioned conditions given the pair (ψ,φ). Therefore, this study is an improvement and unification of other known results in the literature and could be further used to generalize results in admissible mappings or comparable mappings.

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