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Stirling's Formula: Generalization and Improvement

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Abstract. The famous Stirling's formula is generalized and improved, by using the arguments of analytic number theory and mathematical analysis.

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1. Introduction

The well-known Stirling's formula, as can be found in almost every standard textbook of mathematical analysis, states that(see [F]), , for each positive integer n,

$$n! = 1 \times 2 \times 3 \times \cdots \times n = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\theta/(12n)},\ 0 < \theta = \theta_n < 1.$$

There are two objectives in this short paper, the first is to generalize Stirling's formula to an arithmetic progression, and the second is to get an improved version of Stirling's formula. For these we use techniques of analytic number theory. We state the generalization as follows.

Theorem 1. For any positive numbers n_1 and d, and integer $n \geq 1$,

$$n_1(n_1+d)\cdots(n_1+(n-1)d) =$$

$$f(n_1,d)\sqrt{n_1+(n-1)d}\,((n_1+(n-1)d)/e)^{n-1+n_1/d}exp\big(\varphi(n,n_1,d)\big),$$

where $\varphi(n,n_1,d) = \theta(n,n_1,d)/(12(n-1+n_1/d))$, $0 < \theta(n,n_1,d) < 1$, $f(n_1,d)$ is a positive number which only depends on n_1 and d.

It is thus important to determine the value of $f(n_1, d)$ in terms of n_1 and d.

Theorem 2. For any positive numbers n_1 and d, we have

$$f(n_1,d) = \sqrt{2\pi}\left(\Gamma\left(\frac{n_1}{d}\right)\right)^{-1} d^{0.5-n_1/d}.$$

Theorem 2 infers easily the following corollaries, the proofs of which are left to readers as exercises.

Corollary 1. $f(n_1, d)$ has the following properties:

(A) $f(Ln_1,Ld) = L^{1/2-n_1/d}f(n_1,d)$(for any $L > 0$);

(B) $f(n_1+d/2,d) = d^{-1/2}\Gamma(n_1/d)\left(\Gamma(n_1/d+1/2)\right)^{-1} f(n_1,d)$;

(C) $f(k,1) = \sqrt{2\pi}/(k-1)!$(for any positive integer k);

here in (B) the Γ function is defined by

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt\,,\text{for } x > 0.$$

Corollary 2. For a positive integer k (here $(1+2m)!! \equiv 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (1+2m)$)

$$f(k,2) = \begin{cases} \sqrt{2}, & \text{if} \quad k = 1, \\ \dfrac{\sqrt{2}}{(k-2)!!}, & k > 1, \ k \text{ is odd}, \\ \dfrac{\sqrt{2\pi}}{(0.5k-1)!} \cdot 2^{(1-k)/2}, & k > 1, \ k \text{ is even}. \end{cases}$$

Theorem 3. For the original Stirling's formula,

$$\theta_n > 1 - \frac{1}{2}n^{-2} - \frac{7}{8}n^{-3}.$$

2. Lemmas

Lemma 1. (i) For $0 \le x < 1$ we have $\log(1-x) \le -x$; (ii) For $0 \le x \le 1/2$ we have $\log(1-x) \ge -x - x^2$.

Proof. Let $f(x) = log(1-x) + x + x^2,\ 0 \le x \le 1/2$. Then

$$f'(x) = x\left(2 - \frac{1}{1-x}\right) \ge 0 \Rightarrow f(x) \ge f(0) = 0..$$

The first assertion can be deduced similarly.

Lemma 2.(Euler Summation Formula) If **f** has the continuous derivative on the interval [y, x], then

$$\sum_{y<n\le x} f(n) = \int_y^x f(t)dt + \int_y^x \psi(t)f'(t)dt +$$

$$-\psi(x)f(x) + \psi(y)f(y),$$

here [u] is the largest integer not exceeding u, and 中 (u) $= u - [u] - 1/2$..

Proof. This follows from a variant of Theorem 3.1 of [A].

Lemma 3. Let

$$\sigma(x) = \int_0^x \psi(t)dt,$$

Then a(x) is a periodic function of period 1, and |a(x)| ≤ 1/8. Assume that k(x) has continuous derivative function k′(x) on the interval [a, {3], here a and {3 are integers, then

$$\int_\alpha^\beta \psi(x)k(x)dx = -\int_\alpha^\beta \sigma(x)k'(x)dx.$$

Proof. First we need to show a(x + 1) = a(x) for all x. For this we see that

$$\int_x^{x+1} \psi(t)dt = \int_x^{[x]+1} \psi(t)dt + \int_{[x]+1}^{x+1} \psi(t)dt$$

$$= \int_{\{x\}}^1 \psi(t)dt + \int_0^{\{x\}} \psi(t)dt = \int_0^1 \psi(t)dt$$

$$= \int_0^1 (t - 1/2)dt = 0;$$

where we have used the fact [t + n] = [t] + n for any integer n. Thus

$$|\sigma(x)| = |\sigma(\{x\})| = \frac{1}{2}\{x\}(1 - \{x\}) \leq \frac{1}{8}.$$

To verify the remaining assertion, it suffices to show

$$(1) \qquad \int_m^{m+1} \psi(x)k(x)dx = -\int_m^{m+1} \sigma(x)k'(x)dx.$$

for any integer m ∈ [a, {3]. For any sufficiently small E > 0, we use

$\psi(x) = x - m - 1/2$, for $m \leq x \leq m + 1 - \varepsilon$,

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$$\sigma(x) = \left(\int_0^1 + \int_1^2 + \cdots + \int_m^x\right)\psi(u)du = \int_m^x \psi(u)du$$

$$(2) \qquad = \int_0^{x-m} \psi(u)du = \frac{1}{2}(x^2 - 2mx + m^2) - \frac{1}{2}(x - m).$$

Using (2) and the partial integration we get

$$\int_m^{m+1-\varepsilon} \sigma(x)k'(x)dx = \sigma(m+1-\varepsilon)k(m+1-\varepsilon) - \sigma(m)k(m)$$

$$-\int_m^{m+1-\varepsilon} \psi(x)k(x)dx$$

$$= \sigma(1-\varepsilon)k(m+1-\varepsilon) - \int_m^{m+1-\varepsilon} \psi(x)k(x)dx.$$

Letting $\varepsilon \to 0$ we get (1).

Lemma 4. For $s > 0$ we have

$$\Gamma(s) = \lim_{n\to\infty}\left(\frac{n!\cdot n^s}{s(s+1)\cdots(s+n)}\right),$$

and

$$\log(\Gamma(s)) = \left(s + \frac{1}{2}\right)\log(s+1) - s - \log s + \int_1^\infty \frac{\sigma(x)}{x^2}dx$$
$$-\int_1^\infty \frac{\sigma(x)}{(s+x)^2}dx.$$

Proof. For $0 \le t < n$, by Lemma 1 (i) we know that

$$e^{-t/n} \ge 1 - \frac{t}{n} \Rightarrow e^{-t} \ge \left(1 - \frac{t}{n}\right)^{-n}.$$

This obviously holds for $t = n$. By Lemma 1 (ii) we know that for $0 \le t \le n/2$,

$$e^{t/n}(1-t/n) \geq e^{-(\frac{t}{n})^2} \Rightarrow e^t\left(1-\frac{t}{n}\right)^n \geq e^{-t^2/n}.$$

Thus

$$e^{-t}-\left(1-\frac{t}{n}\right)^n = e^{-t}\left(1-e^t\left(1-\frac{t}{n}\right)^n\right) \leq e^{-t}(1-e^{-t^2/n}).$$

If $t<\sqrt{n}$, by Lemma 1 (i) we have

$$e^{-t^2/n} \geq 1-t^2/n,$$

And if $n/2 \geq t \geq \sqrt{n}$, this holds also. Thus

$$e^{-t}-\left(1-\frac{t}{n}\right)^n \leq \frac{t^2}{n}e^{-t},\ \ 0 \leq t \leq \frac{n}{2}.$$

This holds obviously for $\frac{n}{2}<t\leq n$. Thus

$$e^{-t}-\left(1-\frac{t}{n}\right)^n \leq \frac{t^2}{n}e^{-t},\ \ 0 \leq t \leq n.$$

Therefore for $s>0$ we have

$$0 \leq \int_0^n e^{-t}t^{s-1}dt - \int_0^n \left(1-\frac{t}{n}\right)^n t^{s-1}dt$$

$$\leq \frac{1}{n}\int_0^n t^{s+1}e^{-t}dt \leq \frac{1}{n}\Gamma(s+2).$$

Let $n\rightarrow\infty$ we get

(3) $$\Gamma(s) = \lim_{n\rightarrow\infty}\int_0^n \left(1-\frac{t}{n}\right)^n t^{s-1}dt.$$

By integration by parts we get

$$\int_0^n \left(1-\frac{t}{n}\right)^n t^{s-1}dt = n^s\left(\int_0^1 (1-w)^n w^{s-1}dw\right)$$

$$= \frac{n^{s+1}}{s}\left(\int_0^1 w^s(1-w)^{n-1}dw\right)$$

$$= n^s\cdot\frac{n(n-1)}{s(s+1)}\left(\int_0^1 w^{s+1}(1-w)^{n-2}dw\right)$$

$$= \cdots = \frac{n^s\cdot n!}{s(s+1)\cdots(s+n)}.$$

Thus the first assertion of Lemma 4 follows from (3). Put

$$P_n(s) = (1+s)\left(1+\frac{s}{2}\right)\cdots\left(1+\frac{s}{n}\right).$$

Then

$$\text{(4)} \qquad \log P_n(s) = \log(1+s) + \sum_{1<k\le n}\log\left(1+\frac{s}{k}\right).$$

By Lemma 2 we get

$$\sum_{1<k\le n}\log\left(1+\frac{s}{k}\right) = \int_1^n \log\left(1+\frac{s}{x}\right)dx - \int_1^n \frac{s\cdot\psi(x)}{x(s+x)}dx$$

$$-\frac{1}{2}\log(1+s) + \frac{1}{2}\log\left(1+\frac{s}{n}\right)$$

$$= \left(n+s+\frac{1}{2}\right)\log(s+n) - \left(s+\frac{3}{2}\right)\log(s+1)$$

$$-\left(n+\frac{1}{2}\right)\log n-\int_1^n \frac{s\cdot\psi(x)}{s(s+x)}dx. \tag{5}$$

By Lemma 3 we get

$$\int_1^n \frac{s\cdot\psi(x)}{s(s+x)}dx=\int_1^n \psi(x)\left(\frac{1}{x}-\frac{1}{x+s}\right)dx$$

$$=\int_1^n \frac{\sigma(x)}{x^2}dx-\int_1^n \frac{\sigma(x)}{(x+s)^2}dx. \tag{6}$$

Therefore from (4), (5) and (6) we get

$$\Gamma(s)=\frac{1}{s}(s+1)^{s+1/2}\lim_{n\to\infty}\left\{\left(\frac{n}{s+n}\right)^s\left(\frac{n}{s+n}\right)^{n+1/2}\right.$$
$$\left.\cdot\, exp\left(\int_1^\infty \frac{\sigma(x)}{x^2}dx-\int_1^\infty \frac{\sigma(x)}{(x+s)^2}dx\right)\right\};$$

and the second assertion of Lemma 4 follows.

3. Proof of Theorem 2

By Lemma 2 we have

$$\sum_{1\le k\le n}\log(n_1+(k-1)d)=\log n_1+\sum_{1<k\le n} log(n_1+(k-1)d)$$

$$=\log n_1+\int_1^n \log(n_1+(x-1)d)dx+\frac{1}{2}\log(n_1+(n-1)d)$$
$$-\frac{1}{2}\log(n_1)+\int_1^n \psi(t)\cdot\frac{d}{n_1+(t-1)d}dt,$$

$$\int_1^n \log(n_1+(x-1)d)dx = nlog(n_1+(n-1)d) - \log(n_1)$$

$$-\int_1^n \frac{xd}{n_1+(x-1)d}dx$$

$$= 1-n+\left(n-1+\frac{n_1}{d}\right)\log(n_1+(n-1)d) - \frac{n_1}{d}\log n_1,$$

$$\int_1^n \psi(t)\cdot\frac{d}{n_1+(t-1)d}dt = \int_1^n \sigma(t)\left(t-1+\frac{n_1}{d}\right)^{-2}dt\,;$$

where we have used Lemma 3. Thus

$$\sum_{1\le k\le n}\log(n_1+(k-1)d) = \left(n-\frac{1}{2}+\frac{n_1}{d}\right)\log(n_1+(n-1)d) + \left(\frac{1}{2}-\frac{n_1}{d}\right)\log n_1 + 1 - n + \int_1^n \sigma(t)\left(t-1+\frac{n_1}{d}\right)^{-2}dt. \tag{7}$$

By Theorem 1 we get

$$\sum_{1\le k\le n}\log(n_1+(k-1)d) = \log f(n_1,d) + \left(n-\frac{1}{2}+\frac{n_1}{d}\right)\log(n_1+(n-1)d) - n + 1 - \frac{n_1}{d} + O\left(\frac{1}{n}\right).$$

From this and (7) we get, by taking n→ ∞, the equality

$$\log f(n_1,d) = \frac{n_1}{d} + \left(\frac{1}{2}-\frac{n_1}{d}\right)\log(n_1) + \int_1^\infty \sigma(t)\left(t-1+\frac{n_1}{d}\right)^{-2}dt. \tag{8}$$

By Lemma 4 we have

$$\log\Gamma\left(\frac{n_1}{d}\right) = \left(\frac{n_1}{d}+\frac{1}{2}\right)\log\left(\frac{n_1}{d}+1\right) - \frac{n_1}{d} - \log\frac{n_1}{d} + \int_1^\infty \frac{\sigma(x)}{x^2}dx - \int_1^\infty \frac{\sigma(x)}{\left(\frac{n_1}{d}+x\right)^2}dx. \tag{9}$$

Using

$$\int_1^{\infty} \frac{\sigma(x)}{\left(\frac{n_1}{d}+x\right)^2} dx = \int_2^{\infty} \frac{\sigma(x)}{\left(\frac{n_1}{d}+x-1\right)^2} dx$$

$$= \int_1^{\infty} \frac{\sigma(x)}{\left(\frac{n_1}{d}+x-1\right)^2} dx - \int_1^{2} \frac{\sigma(x)}{\left(\frac{n_1}{d}+x-1\right)^2} dx,$$

$$\int_1^{2} \frac{\sigma(x)}{\left(\frac{n_1}{d}+x-1\right)^2} dx = \int_0^{1} \frac{\sigma(x)}{\left(\frac{n_1}{d}+x\right)^2} dx$$

$$= \int_0^{1} \frac{0.5x^2 - 0.5x}{\left(\frac{n_1}{d}+x\right)^2} dx$$

$$= 1 - \left(\frac{1}{2}+\frac{n_1}{d}\right) \log\left(1+\frac{d}{n_1}\right),$$

and thus from "(8)+(9)" (taking summation on both sides respectively) we get

$$\log f(n_1, d) + \log\Gamma\left(\frac{n_1}{d}\right) = \int_1^{\infty} \frac{\sigma(x)}{x^2} dx + 1 + \left(\frac{1}{2} - \frac{n_1}{d}\right) \log d.$$

Using $f(1,1) = \sqrt{2\pi}$ and $\Gamma(1) = 1$ we get

$$\int_1^{\infty} \frac{\sigma(x)}{x^2} dx = log\sqrt{2\pi} - 1,$$

and therefore

$$\log f(n_1, d) = \log\sqrt{2\pi} + \left(\frac{1}{2} - \frac{n_1}{d}\right) \log d - \log\Gamma\left(\frac{n_1}{d}\right).$$

The proof is completed.

4.Proof of Theorem 1

For $0 < X < 1$, we have

$$\log(1+X) = X - \frac{X^2}{2} + \frac{X^3}{3} - \cdots + (-1)^{n-1}\frac{X^n}{n} + \cdots,$$

$$\log(1-X) = -X - \frac{X^2}{2} - \frac{X^3}{3} - \cdots - \frac{X^n}{n} - \cdots,$$

thus

$$(10) \qquad 2X < \log\left(\frac{1+X}{1-X}\right) = 2X\left(1 + \frac{X^2}{3} + \frac{X^4}{5} + \cdots + \frac{X^{2n}}{2n+1} + \cdots\right)$$

$$< 2X\left(1 + \frac{X^2}{3}(1 + X^2 + X^4 + \cdots)\right).$$

Let

$$X = \frac{d}{d(2n-1) + 2n_1},$$

then

$$\frac{1+X}{1-X} = 1 + \frac{d}{d(n-1) + n_1},$$

from (10) we get

$$1 < \left(\frac{n_1}{d} + n - \frac{1}{2}\right)\log\left(1 + \frac{d}{d(n-1) + n_1}\right)$$

$$< 1 + \frac{1}{12}\cdot\frac{1}{\left(\frac{n_1}{d} + n\right)\left(\frac{n_1}{d} + n - 1\right)},$$

therefore

$$(11) \qquad 1 < e^{-d}\left(1 + \frac{d}{d(n-1) + n_1}\right)^{n_1 + \left(n - \frac{1}{2}\right)d} < e^{d/\left(12\left(\frac{n_1}{d}+n\right)\left(\frac{n_1}{d}+n-1\right)\right)}.$$

Let

$$a_n = \left(n_1(n_1+d)\cdots(n_1+(n-1)d)\right)^d e^{n_1+(n-1)d}$$

$$\times (n_1 + (n-1)d)^{-n_1-(n-0.5)d},$$

then (11) gives

$$(12) \qquad 1 < \frac{a_n}{a_{n+1}} = e^{-d}\left(1 + \frac{d}{d(n-1) + n_1}\right)^{n_1 + \left(n - \frac{1}{2}\right)d}$$

$$< e^{d/\left(12\left(\frac{n_1}{d}+n\right)\left(\frac{n_1}{d}+n-1\right)\right)} = e^{-d/\left(12\left(\frac{n_1}{d}+n\right)\right)} \cdot e^{d/\left(12\left(\frac{n_1}{d}+n-1\right)\right)},$$

therefore $a_n > a_{n+1}$, a_n is monotonically decreasing and $a_n > 0$, thus

$$\lim_{n\to\infty} a_n = A \geq 0$$

exists. Also, using (12) we have

$$b_n = a_n e^{-d/\left(12\left(\frac{n_1}{d}+n-1\right)\right)}$$

is monotonically increasing and $b_n < a_n \le a_1$, and moreover

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = A \ge b_1 > 0,$$

Thus we can write $A = (f(n_1, d))^d$, here $f(n_1, d)$ is a positive number depending only on n_1 and d. From

$$b_n = a_n e^{-d/\left(12\left(\frac{n_1}{d}+n-1\right)\right)} < (f(n_1, d))^d < a_n,$$

the stated formula follows.

5. Proof of Theorem 3

Let $n_1 = d = 1$ in Theorem 1. Using (10) we get

$$2X\left(1+\frac{X^2}{3}\right) < \log\left(\frac{1+X}{1-X}\right),$$

thus using the notation of Theorem 1, similarly to (12) we get

$$e^{\theta_n/(12n)-\theta_{n+1}/(12(n+1))} = \frac{a_n}{a_{n+1}} > e^{1/\left(3(2n+1)^2\right)},$$

$$\frac{\theta_n}{4n} > \frac{\theta_{n+1}}{4(n+1)} + \frac{1}{(2n+1)^2},\ n \ge 1.$$

Repeating this inequality, for any larger integer N, we get

$$(13) \qquad \frac{\theta_n}{4n} > \frac{\theta_N}{4N} + \sum_{n<m\le N} \frac{1}{(2m-1)^2} > \sum_{n<m\le N} \frac{1}{(2m-1)^2}.$$

Using Lemma 2 we get

$$\sum_{n<m\le N}\frac{1}{(2m-1)^2}=\int_n^N\frac{1}{(2w-1)^2}dw-\frac{1}{2}\left(\frac{1}{(2n-1)^2}-\frac{1}{(2N-1)^2}\right)$$

$$-4\int_n^N\frac{\psi(u)}{(2u-1)^3}du,$$

and by Lemma 3 we get (using $|\sigma(u)|\le 0.5$)

$$\int_n^N\frac{\psi(u)}{(2u-1)^3}du=6\int_n^N\frac{\sigma(u)}{(2u-1)^4}du,$$

$$\left|4\int_n^N\frac{\psi(u)}{(2u-1)^3}du\right|\le 3\int_n^N(2u-1)^{-4}du<0.5(2n)^{-3},$$

and consequently, for a suitable positive constant C, we get

$$\sum_{n<m\le N}\frac{1}{(2m-1)^2}\ge\frac{1}{2(2n-1)}-\frac{1}{2(2n-1)^2}-\frac{1}{4}n^{-3}-CN^{-1}, \tag{14}$$

provided N is sufficiently. Taking $N\to\infty$ we obtain from (13) and (14)

$$\frac{\theta_n}{4n}\ge\frac{1}{2(2n-1)}-\frac{1}{2(2n-1)^2}-\frac{1}{16}n^{-3}. \tag{15}$$

We have

$$\frac{1}{2n-1}>\frac{1}{2n}\left(1+\frac{1}{2n}+\frac{1}{4n^2}+\frac{1}{8n^3}\right),$$

$$\frac{1}{(2n-1)^2}=\frac{1}{4n^2}\left(1+\sum_{k\ge1}\frac{k+1}{2^kn^k}\right)<\frac{1}{4n^2}\left(1+\frac{1}{n}+\frac{2}{n^2}\right),$$

and thus (15) yields the required estimate.

Reference

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