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Bayes Estimator for inverse Gaussian Distribution with Jeffrey's Prior

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Abstract

This paper presents a Bayesian analysis of the parameters for the inverse Gaussian distribution under the Jeffrey's prior assuming a quadratic loss function. Analysis begins with the parameterization to the parameters in the distribution, then construct the posterior distribution based the likelihood function and prior, while the Bayes estimator is concluded based the posterior mean.

Keywords: *inverse Gaussian distribution, bayes theorem, Jeffrey's prior.*

1. Introduction

The inverse Gaussian distribution is a distribution that has two parameters, this distribution is an important distribution, because it is widely applied in lifetime data analysis in biology, cardiology, hydrology, demography, and economics. The problem in the application of distribution models in general is how to estimate the parameters in the model, in the classical method, the estimation of parameters is only based on sample observations, while the Bayesian method combines information from the sample and initial information is known as priors. In general, priors are derived from past experience of population parameters and also can be obtained from other sources of statistical investigation, therefore the Bayesian approach allows for better decisions.

2. Materials and method

The materials in this paper are several interrelated theories in statistics as inverse Gaussian distribution, the likelihood function, Bayes theorem and Jeffrey's prior.

The inverse Gaussian distribution is a two-parameter family of continuous probability distribution. A continuous random variable, X, is said to have the inverse Gaussian distribution if it has probability density function of the form

$$f(x;\mu,\lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right),\ x>0 \tag{1}$$

for $x > 0$, where $\mu > 0$ and $\lambda > 0$ are parameters.

The joint density function of n random variables $X_1, X_2, \ldots, X_n$ evaluated at $x_1, x_2, \ldots, x_n$ say $f(x_1, x_2, \ldots, x_n; \theta)$, is referred to as a likelihood function. For fixed $x_1, x_2, \ldots, x_n$ the likelihood function is a function of θ and often denoted by $L(\theta)$. If $X_1, X_2, \ldots, X_n$ represents a random sample from $f(x;\theta)$, then the likelihood function is:

$$L(\boldsymbol{\theta}|X) = \prod_{i=1}^{n} f(x_i \mid \theta) \tag{2}$$

Jeffrey's prior is a prior which is constructed mathematically based on the likelihood-function. Let $L(\boldsymbol{\theta}|X) \propto f(X|\boldsymbol{\theta})$ be the likelihood function for $\boldsymbol{\theta}$ based on observation X,

Jeffrey suggests a prior:

$$\pi(\theta) \propto \sqrt{-E_{X|\theta}\left(\frac{\partial^2}{\partial\theta^2} log\, L(\boldsymbol{\theta}|\boldsymbol{X})\right)} \tag{3}$$

The method used is literature study by applying the Bayesian analysis with th Jeffrey's prior. The procedure is done by constructing the likelihood function, the posterior distri-bution and determining the Bayes estimator based the posterior mean.

3. Construction of Estimator

The steps to determine the bayes estimator of parameters for the inverse Gaussian distribution are through parameterization, the likelihood function, the Jeffrey's prior, the posterior distribution, and the the posterior mean.

3.1. Parameterization

Assuming $\mu = \psi^{-1}$ to equation (1), we get:

$$f(x;\psi,\lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2}\ exp\left(-\frac{\lambda(x-\psi^{-1})^2}{2\psi^{-2}x}\right)$$

$$= \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} exp\left(-\frac{\lambda}{2}\frac{(x-\psi^{-1})^2}{\psi^{-2}x}\right)$$

$$= \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} exp\left(-\frac{\lambda}{2}\left(\frac{x-\psi^{-1}}{\psi^{-1}x^{1/2}}\right)^2\right)$$

$$= \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} exp\left(-\frac{\lambda}{2}\left(\psi x^{1/2}-\frac{1}{x^{1/2}}\right)^2\right)$$

$$= \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} exp\left(-\frac{\lambda x}{2}\left(\psi-\frac{1}{x}\right)^2\right) \tag{4}$$

3.2 Likelihood function

The likelihood function of the parameters (ψ,λ) is:

$$L(\psi,\lambda) = \prod_{i=1}^{n} f(x_i \mid \psi,\lambda)$$

$$= \prod_{i=1}^{n}\left(\frac{\lambda}{2\pi x_i^3}\right)^{1/2} \; exp\left(-\frac{\lambda x_i}{2}\left(\psi-\frac{1}{x_i}\right)^2\right)$$

$$= \left(\frac{\lambda}{2\pi}\right)^{n/2} x^{-3n/2} \; exp\left(-\frac{\lambda}{2}\sum_{i=1}^{n} x_i\left(\psi^2-\frac{2\psi}{x_i}+\frac{1}{x_i^2}\right)\right) \tag{5}$$

because

$$\sum_{i=1}^{n} x_i\left(\psi^2-\frac{2\psi}{x_i}+\frac{1}{x_i^2}\right) = \psi^2\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} 2\psi + \sum_{i=1}^{n}\frac{1}{x_i}$$

$$= \psi^2 n\bar{x} - 2n\psi + \sum_{i=1}^{1}\frac{1}{x_i}; \quad \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$= n\bar{x}\left(\psi^2 - \frac{2\psi}{\bar{x}} + \frac{1}{n\bar{x}}\sum_{i=1}^{1}\frac{1}{x_i}\right)$$

$$= n\bar{x}\left(\psi^2 - \frac{2\psi}{\bar{x}} + \frac{\bar{x}_r}{\bar{x}}\right); \quad \bar{x}_r = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}$$

$$= n\bar{x}\left[\left(\psi^2 - \frac{1}{\bar{x}}\right)^2 - \frac{1}{\bar{x}^2} + \frac{\bar{x}_r}{\bar{x}}\right]$$

$$= n\bar{x}\left(\psi^2 - \frac{1}{\bar{x}}\right)^2 + n\left(\bar{x}_r - \frac{1}{\bar{x}}\right)$$

$$= n\bar{x}\left(\psi^2 - \frac{1}{\bar{x}}\right)^2 + nv, \quad v = \bar{x}_r - \frac{1}{\bar{x}}$$

then the equation (5) become:

$$L(\psi,\lambda) = \left(\frac{\lambda}{2\pi}\right)^{n/2} x^{-3n/2}\, exp\left(-\frac{n\lambda\bar{x}}{2}\left(\psi-\frac{1}{\bar{x}}\right)^2 - \frac{n\lambda v}{2}\right) \tag{6}$$

3.3. Bayes estimator of λ if ψ know.

Based the equation (4), the likelihood function of λ is:

$$L(\lambda) = \left(\frac{\lambda}{2\pi}\right)^{n/2} x^{-3n/2}\, exp\left(-\frac{n\lambda\bar{x}}{2}\left(\psi^2-\frac{2\psi}{\bar{x}}+\frac{1}{\bar{x}^2}\right) - \frac{n\lambda v}{2}\right)$$

$$= \left(\frac{\lambda}{2\pi}\right)^{n/2} x^{-3n/2}\, exp\left(-\frac{n\lambda\bar{x}\psi^2 - 2n\lambda\psi + n\lambda/\bar{x} + n\lambda v}{2}\right)$$

$$= \left(\frac{\lambda}{2\pi}\right)^{n/2} x^{-3n/2}\, exp\left(-\frac{n\lambda(\bar{x}\psi^2 - 2\psi + 1/\bar{x} + v)}{2}\right)$$

$$= \left(\frac{\lambda}{2\pi}\right)^{n/2} x^{-3n/2}\, exp\left(-\frac{n\lambda(\bar{x}\psi^2 - 2\psi + \bar{x}_r)}{2}\right)$$

$$= \left(\frac{\lambda}{2\pi}\right)^{n/2} x^{-3n/2}\, exp\left(-\frac{n\lambda w}{2}\right),\; w = \bar{x}\psi^2 - 2\psi + \bar{x}_r$$

by using “proporsional”, obtained

$$L(\lambda) \propto \lambda^{n/2} exp\left(-\frac{n\lambda w}{2}\right) \quad (7)$$

and the Jeffrey’s prior of λ is:

$$\pi(\lambda) \propto \left|-E\left(\frac{\partial^2}{\partial\lambda^2} log\, L(\lambda)\right)\right|^{1/2}$$

$$\propto \left|-E\frac{\partial^2}{\partial\lambda^2}\left(log\, \lambda^{n/2} exp\left(-\frac{n\lambda w}{2}\right)\right)\right|^{1/2}$$

$$\propto \left|-E\frac{\partial^2}{\partial\lambda^2}\left(\frac{n}{2} log\, \lambda - \frac{n\lambda w}{2}\right)\right|^{1/2}$$

$$\propto \left|-E\frac{\partial}{\partial\lambda}\left(\frac{n\lambda^{-1}}{2} - \frac{nw}{2}\right)\right|^{1/2}$$

$$\propto \left|-E\left(-\frac{n\lambda^{-2}}{2}\right)\right|^{1/2}$$

$$\propto \left|\frac{n\lambda^{-2}}{2}\right|^{1/2}$$

$$\propto \lambda^{-1} \quad (8)$$

Next, by using the bayes theorem to equation (7) and (8), be obtained the posterior distribution

$$\pi(\lambda|\psi, x_1, x_2, \ldots, x_n) \propto L(\lambda) \cdot \pi(\lambda)$$

$$\propto \lambda^{n/2} exp\left(-\frac{n\lambda w}{2}\right) \,.\, \lambda^{-1}$$

$$\propto \lambda^{\left(\frac{n}{2}-1\right)} exp\left(-\frac{nw}{2}\lambda\right) \quad (9)$$

Based the equation (9) can be concluded that the posterior distribution have the gamma distribution with parameters $\frac{n}{2}$ and $\frac{nw}{2}$ and the poasterior mean is

$$\hat{\lambda} = \frac{n/2}{nw/2} = w^{-1} \quad (10)$$

3.4. Bayes estimator of ψ if λ know.

Based the equation (6), the likelihood function of ψ is:

$$L(\psi) \propto \; exp\left(-\frac{n\lambda\bar{x}}{2}\left(\psi-\frac{1}{\bar{x}}\right)^2\right) \tag{11}$$

and the Jeffrey's prior of ψ is:

$$\boldsymbol{\pi}(\psi) \propto \left|-\boldsymbol{E}\left(\frac{\partial^2}{\partial\psi^2}\,\boldsymbol{log\,exp}\left(-\frac{n\lambda\bar{x}}{2}\left(\psi-\frac{1}{\bar{x}}\right)^2\right)\right)\right|^{1/2}$$

$$\propto \left|-\boldsymbol{E}\frac{\partial^2}{\partial\psi^2}\left(\left(-\frac{n\lambda\bar{x}}{2}\left(\psi-\frac{1}{\bar{x}}\right)^2\right)\right)\right|^{1/2}$$

$$\propto \left|-\boldsymbol{E}\frac{\partial}{\partial\psi}\left(-\,\boldsymbol{n\lambda\bar{x}}\left(\psi-\frac{1}{\bar{x}}\right)\right)\right|^{1/2}$$

$$\propto |\boldsymbol{E}(\boldsymbol{n\lambda\bar{x}})|^{1/2}$$

$$\propto |\boldsymbol{constant}|^{1/2}$$

$$\propto 1 \tag{12}$$

Next, by using the bayes theorem to equation (11) and (12), be obtained the posterior distribution

$$\pi(\psi|\lambda,x_1,x_2,\dots,x_n) \propto L(\psi)\cdot\pi(\psi)$$

$$\propto exp\left(-\frac{n\lambda\bar{x}}{2}\left(\psi-\frac{1}{\bar{x}}\right)^2\right)$$

ψ normal distributed with mean $\frac{1}{\bar{x}}$ and variance $(n\lambda\bar{x})^{-1}$, denote:

$$\psi\sim\left(\frac{1}{\bar{x}},(n\lambda\bar{x})^{-1}\right) \tag{13}$$

To discuss the equation (13) further, we need the following definition and theorems

Theorem 1

If $X\sim N(\mu,\tau^{-1})$, then $\Phi(\mu\tau^{\frac{1}{2}}) = \int_0^\infty f(x;\mu,\tau^{-1})\,dx$

Proof

Suppose $I = \int_0^\infty f(x;\mu,\tau^{-1})\,dx = \int_0^\infty \left(\frac{\tau}{2\pi}\right)^{\frac{1}{2}} exp\left(\frac{-\tau(x-\mu)^2}{2}\right)dx$

The transformation $z = \tau^{\frac{1}{2}}(x-\mu)$ give

$I=\int_{-\mu\tau^{\frac{1}{2}}}^{\infty}\left(\frac{1}{2\pi}\right)^{\frac{1}{2}} exp\left(\frac{-z^2}{2}\right)dz = \int_{-\infty}^{\mu\tau^{\frac{1}{2}}}\left(\frac{1}{2\pi}\right)^{\frac{1}{2}} exp\left(\frac{-z^2}{2}\right)dz = \Phi\left(\mu\tau^{\frac{1}{2}}\right)$ ■

Definition 1

A random variable $X \sim N(\mu, \tau^{-1})$; $x > 0$, $\mu \in R$, and $\tau > 0$, if it has the probability density function

$$f(x;\mu,\tau^{-1}) = \frac{1}{\Phi(\mu\tau^{\frac{1}{2}})}\left(\frac{\tau}{2\pi}\right)^{\frac{1}{2}} exp\left(\frac{-\tau(x-\mu)^2}{2}\right)$$

Theorem 2

If $X \sim N(\mu, \tau^{-1})$, then $E(X) = \frac{1}{\Phi(\mu\tau^{\frac{1}{2}})\,(2\tau\pi)^{\frac{1}{2}}}\, exp\left(\frac{-\mu^2\tau}{2}\right) + \mu$

Proof

$$E(X) = \int_0^\infty x f(x;\eta,\tau^{-1})dx = \int_0^\infty x \frac{1}{\Phi(\eta\tau^{\frac{1}{2}})}\left(\frac{\tau}{2\pi}\right)^{\frac{1}{2}} exp\left(\frac{-\tau(x-\eta)^2}{2}\right) dx$$

with transformation of $z = \tau^{\frac{1}{2}}(x-\mu)$, obtained

$$E(X) = \int_{-\mu\tau^{\frac{1}{2}}}^{\infty} \frac{(z+\mu\tau^{\frac{1}{2}})/\tau^{\frac{1}{2}}}{\Phi(\mu\tau^{\frac{1}{2}})}\left(\frac{\tau}{2\pi}\right)^{\frac{1}{2}} exp\left(\frac{-z^2}{2}\right) \frac{\boldsymbol{dz}}{\boldsymbol{\tau^{\frac{1}{2}}}}$$

$$= \int_{-\mu\tau^{\frac{1}{2}}}^{\infty} \frac{\left(\frac{z}{\tau^{\frac{1}{2}}}\right)+\mu}{\Phi(\mu\tau^{\frac{1}{2}})}\left(\frac{1}{2\pi}\right)^{\frac{1}{2}} exp\left(\frac{-z^2}{2}\right) dz$$

$$= \int_{-\mu\tau^{\frac{1}{2}}}^{\infty} \frac{z}{\Phi(\mu\tau^{\frac{1}{2}})\tau^{\frac{1}{2}}}\left(\frac{1}{2\pi}\right)^{\frac{1}{2}} exp\left(\frac{-z^2}{2}\right) dz + \int_{-\mu\tau^{\frac{1}{2}}}^{\infty} \frac{\mu}{\Phi(\mu\tau^{\frac{1}{2}})}\left(\frac{1}{2\pi}\right)^{\frac{1}{2}} exp\left(\frac{-z^2}{2}\right) dz$$

$$= \frac{1}{\Phi(\mu\tau^{\frac{1}{2}})\,(2\tau\pi)^{\frac{1}{2}}}\int_{-\mu\tau^{\frac{1}{2}}}^{\infty} z\; exp\left(\frac{-z^2}{2}\right) dz + \frac{\mu}{\Phi(\mu\tau^{\frac{1}{2}})}\int_{-\mu\tau^{\frac{1}{2}}}^{\infty}\left(\frac{1}{2\pi}\right)^{\frac{1}{2}} exp\left(\frac{-z^2}{2}\right) dz$$

$$= -\frac{1}{\Phi(\mu\tau^{\frac{1}{2}})\,(2\tau\pi)^{\frac{1}{2}}}\int_{-\mu\tau^{\frac{1}{2}}}^{\infty} exp\left(\frac{-z^2}{2}\right) d\left(\frac{-z^2}{2}\right) + \frac{\mu}{\Phi(\mu\tau^{\frac{1}{2}})}\Phi(\mu\tau^{\frac{1}{2}})$$

$$= \frac{1}{\Phi(\mu\tau^{\frac{1}{2}})\,(2\tau\pi)^{\frac{1}{2}}}\, exp\left(-\frac{\mu^2\tau}{2}\right) + \mu \blacksquare$$

Furthermore, based on theorem 5 and equation (13), the Bayes estimator for ψ is

$$\widehat{\psi} = E(\psi) = \frac{1}{\Phi\left(n^{\frac{1}{2}}\lambda^{\frac{1}{2}}(\bar{x})^{-\frac{1}{2}}\right)\,(2n\lambda\bar{x}\pi)^{\frac{1}{2}}}\, exp\left(-\frac{n\lambda(\bar{x})^{-1}}{2}\right) + (\bar{x})^{-1} \qquad (14)$$

4. Conclusion

The main purpose of this paper to finding the mathematical expression of the Bayes estimator of parameters for the inverse Gaussian distribution under the Jeffrey's prior assuming a quadratic loss function so that the estimator is obtained based on the posterior mean. The Bayes estimator of μ is $\hat{\mu} = \widehat{\psi}^{-1}$, where $\widehat{\psi}$ in equation (14), while the Bayes estimator of λ is $\hat{\lambda} = \widehat{w}^{-1}$, where $\widehat{w} = \bar{x}\widehat{\psi}^2 - 2\widehat{\psi} + \bar{x}_r$ and $\bar{x}_r = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}$.

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