SCIREA Journal of Mathematics

http://www.scirea.org/journal/Mathematics

November 9, 2021

Volume 6, Issue 5, October 2021

Comparison principle to the infinity Laplacian equation with lower term

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This work was supported by National Natural Science Foundation of China(No.11501292)

Abstract

In this paper, we establish the comparison principles via the perturbation method for the equation $\Delta_\infty^h u + A(Du) = f(x,u)$ in Ω, where $\Delta_\infty^h u = |Du|^{h-3}\langle D^2uDu,Du\rangle$ is a highly degenerate and h− homogeneous operator associated with the infinity Laplacian. Based on the comparison principle, we obtain the uniqueness of the viscosity solution to the Dirichlet problem

$$\begin{cases} \Delta_\infty^h u + A(Du) = f(x,u), & in\Omega, \\ u = g, & on\ \partial\Omega, \end{cases}$$

where $g \in C\left(\partial\Omega\right)$. During this procedure, we also establish a stability result of the viscosity solution to the inhomogeneous equation.

Keywords: infinity Laplacian; viscosity solutions; comparison principle; stability.

2010 Mathematic Subject Classification. 35D40; 35B51; 34D20; 35J70.

1. Introduction

In this paper, we consider the following inhomogeneous problems for $g \in C\left(\partial\Omega\right)$:

$$\begin{cases} \Delta_\infty^h u + A(Du) = f(x,u), in\, \Omega, \\ u = g, \qquad\qquad on\, \partial\Omega, \end{cases} \tag{1.1}$$

where Ω be a bounded domain in $\mathbb{R}^n (n \geq 2)$, $A(p){:}\mathbb{R}^n \to \mathbb{R}$ is continuous and homogeneous of degree h, $f(x,t) \in C\left(\Omega \times \mathbb{R}\right)$ is non-decreasing in the second variable t, Δ_∞^h is a highly degenerate elliptic operator given by

$$\Delta_\infty^h u = |Du|^{h-3} \sum_{i,j=1}^{n} u_i\, u_j u_{ij}, \;\; h > 1.$$

Notice that the operator Δ_∞^h is not in divergence form. Hence, the notion of the solution is understood here in the viscosity sense introduced by Crandall, Lions [9], and Crandall, Evans, Lions [10].

For the case $h = 1$, $\Delta_\infty^h u$ is the 1-homogeneous normalized infinity-Laplacian operator,

$$\Delta_\infty^N u := |Du|^{-2} \langle D^2 u Du, Du \rangle.$$

And there is a "tug-of-war" game approach to the following normalized infinity Laplacian Dirichlet problem mentioned in [23],

$$\begin{cases} \Delta_\infty^N u = f(x), \;\; in\, \Omega, \\ u = g, \qquad\quad on\, \partial\Omega. \end{cases} \tag{1.2}$$

In [19], Lu and Wang gave a different proof from the PDEs perspective for the existence and uniqueness of the viscosity solution of the problem (1.2). One can see [4, 15, 18, 22, 24], etc. for more about stochastic approach related to the normalized ∞ −Laplacian operator.

For the case $h = 3$, $\Delta_\infty^h u$ is the 3-homogeneous infinity-Laplacian operator,

$$\Delta_\infty u := |Du|^2 \Delta_\infty^N u.$$

The infinity Laplacian Δ_∞ was first introduced by Aronsson [2] in connection with the geometric problem of finding the so-called absolutely minimizing Lipschitz extension. For more properties of the infinity harmonic functions (the viscosity solution to $\Delta_\infty u = 0$), one can see the works of Crandall [7], Crandall, Evans and Gariepy [8], Aronsson, Crandall and Juutinen [3], and the references therein.

For other h, we have $\Delta_\infty^h u = |Du|^{h-3}\Delta_\infty u = |Du|^{h-1}\Delta_\infty^N u.$

In [20], Lu and Wang proved the comparison principle and existence of the viscosity solution to the infinity Laplace equation

$$\Delta_\infty u = f(x)$$

under Dirichlet condition. Moreover, they also obtained the stability of the viscosity solutions to $\Delta_\infty u = f(x)$. It should be pointed out that they constructed a counterexample to show that the uniqueness does not hold if the inhomogeneous term $f(x)$ changes its sign. A counterexample is given for the normalized infinity Laplacian equation

$$\Delta_\infty^N u = f(x)$$

in [23]. In [5], Bhattacharya and Mohammed considered the comparison principle, existence and nonexistence of the viscosity solutions to a general infinity Laplacian equation with zero order term

$$\Delta_\infty u = f(x, u)$$

with Dirichlet condition. In [17], Liu and Yang established the existence of the viscosity solution to the Dirichlet problem of the h-infinity Laplacian

$$\Delta_\infty^h u = f(x).$$

In this paper, we are interested in the problem (1.1) not only because it is strong degenerate and nonlinear but also it has many applications such as in optimal mass transportation [12, 14], mathematical finance [21], digital image processing [1, 6, 13], etc.

Now we state our main results as follows.

Theorem 1.1 (Comparison principle). Suppose that $f(x,t) \in C\left(\Omega \times \mathbb{R}\right)$ is positive (negative) and non-decreasing in t. Assume that $u \in C\left(\overline{\Omega}\right)$ and $v \in C\left(\overline{\Omega}\right)$ satisfy

$$\Delta_\infty^h u + A(Du) \geq f(x,u), \qquad x \in \Omega$$

and

$$\Delta_\infty^h v + A(Dv) \leq f(x,v),\ x \in \Omega$$

in the viscosity sense. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

Due to the strong degeneracy of the operator Δ_∞^h, we adopt the perturbation method based on the viscosity solutions theory to establish the comparison principle. With the comparison principle in hand, the uniqueness of the Dirichlet problem (1.1) follows immediately.

Theorem 1.2 (uniqueness). If $f(x,t) \in C\left(\Omega \times \mathbb{R}\right)$ is positive (negative) and non-decreasing in t, then there exists at most one viscosity solution to the Dirichlet problem (1.1).

Based on the double variables argument, we can give the following stability result of the viscosity solutions.

Theorem 1.3 (stability). Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of non-negative functions in $C(\Omega)$ such that $f_k \to f$ locally uniformly in Ω for some $f \in C\left(\Omega\right)$. Suppose that for each positive integer k,$u_k \in C\left(\overline{\Omega}\right)$ is a viscosity solution to the problem

$$\begin{cases} \Delta_\infty^h u = f_k, & in\,\Omega, \\ u_k = g & on\,\partial\Omega \end{cases}$$

such that $u_0 \leq u_k \leq u_\infty$ in $\overline{\Omega}$, for some functions u_0 and u_∞ in $C\left(\overline{\Omega}\right)$, with $u_0 = u_\infty = \mathrm{g}$ on $\partial\Omega$. Then $\{u_k\}$ has a subsequence that converges locally uniformly in Ω to a viscosity solution $u \in C\left(\overline{\Omega}\right)$ to the problem

$$\begin{cases} \Delta_\infty^h u = f_k, & in\, \Omega, \\ u_k = g, & on\, \partial\Omega. \end{cases} \tag{1.3}$$

The paper is organized as follows. In Section 2, we state the definition of viscosity solutions to the Dirichlet problem (1.1). In Section 3, we prove the comparison principle to $\Delta_\infty^h u + A(Du) = f(x,u)$, based on the double variables method. And we also use the perturbation method to prove the stability of the viscosity solution.

2. Definition of viscosity solutions

In this section, we first introduce the definition of the viscosity solutions to the problem (1.1), and then establish the comparison principles via the perturbation method for the equation

$$\Delta_\infty^h u + A(Du) = f(x,u), \quad in\, \Omega. \tag{2.1}$$

Since the operator has no divergence structure, we define the viscosity solution by the semi-continuous extension. See for example [11, 16, 19], etc. We remark that the singularity is removable for $h > 1$. Then for $F_h{:}\mathbb{S} \times (\mathbb{R}^n\{0\}) \to \mathbb{R}$ and

$$F_h(M,p){:=}|p|^{h-3}(Mp) \cdot p + A(p),$$

where $\mathbb{S}$ denotes the set of $n \times n$ real symmetric matrices, we can rewrite the equation (2.1) as

$$F_h(D^2u,Du) = f(x,u), \quad x \in \Omega.$$

Due to $A(p) \in C(\mathbb{R}^n)$ is homogeneous of degree h, we obtain $A(0) = 0$. Since $h > 1$, then we have $\lim\limits_{p\to 0} F_h(M,p) = 0$ for arbitrary $M \in \mathbb{S}$. That is, the operator Δ_∞^h is continuous for $h > 1$. Therefore, we can define the continuous extension of F_h as follows,

$$\overline{F_h}(M,p){:} = \begin{cases} F_h(M,p), & if\, p \neq 0, \\ 0, & if\, p = 0. \end{cases}$$

Now we state the definition of viscosity solutions to the problem (1.1).

Definition 2.1. Suppose that $u{:}\overline{\Omega} \to \mathbb{R}$ is an upper semi-continuous function and $u \leqslant g$ on $\partial\Omega$. If for every $x_0 \in \Omega$and $\varphi \in C^2\left(\Omega\right)$ such that $u(x_0) = \varphi(x_0)$ and $u(x) \leqslant \varphi(x)$ for all $x \in \Omega$ near x_0, there holds

$$\overline{F}_h(D^2\varphi(x_0),D\varphi(x_0)) \geqslant f(x_0,\varphi(x_0)).$$

Then we say that u is a viscosity subsolution to (1.1).

Similarly, suppose that $u:\overline{\Omega} \to \mathbb{R}$ is a lower semi-continuous function and $u \geqslant g$ on $\partial\Omega$. If for every $x_0 \in \Omega$ and $\varphi \in C^2\left(\Omega\right)$ such that $u(x_0) = \varphi(x_0)$and $u(x) \geqslant \varphi(x)$ for all $x \in \Omega$ near x_0, there holds

$$\overline{F}_h(D^2\varphi(x_0),D\varphi(x_0)) \leqslant f(x_0,\varphi(x_0)).$$

Then we say that u is a viscosity supersolution to (1.1).

A function $u \in C\left(\Omega\right)$ is a viscosity solution to (1.1) in Ω if it is both a viscosity subsolution and viscosity supersolution of (1.1).

We can also use sub-jets and super-jets (see [11]) to define the viscosity subsolution and the viscosity supersolution equivalently. Now we first recall the definition of sub-jets and super-jets.

The second order super-jet of upper semi-continuous function u at $x_0 \in \Omega$ is the set

$$\mathcal{J}^{2,+}u(x_0) = \{(D\varphi(x_0),D^2\varphi(x_0)):\varphi \in C^2\left(\Omega\right) \ and\ u - \varphi\ has\ a\ local\ maximum\ at\ x_0\},$$

and its closure is

$$\overline{\mathcal{J}}^{2,+}u(x_0) = \{(p,M) \in \mathbb{R}^n \times \mathbb{S}:\exists\ (x_i,p_i,M_i) \in \Omega \times \mathbb{R} \times \mathbb{S}\ such\ that$$

$$(p_i,M_i) \in \mathcal{J}^{2,+}u(x_i)\ \ and\ \ (x_i,p_i,M_i) \to (x_0,p,M)\}.$$

Similarly, the second order sub-jet of lower semi-continuous function u at $x_0 \in \Omega$ is the set

$$\mathcal{J}^{2,-}u(x_0) = \{(D\varphi(x_0),D^2\varphi(x_0)):\varphi \in C^2\left(\Omega\right) \ and\ u - \varphi\ has\ a\ local\ minimum\ at\ x_0\},$$

and its closure is

$$\overline{\mathcal{J}}^{2,-}u(x_0) = \{(p,M) \in \mathbb{R}^n \times \mathbb{S}:\exists\ (x_i,p_i,M_i) \in \Omega \times \mathbb{R} \times \mathbb{S}\ such\ that$$

$$(p_i,M_i) \in \mathcal{J}^{2,-}u(x_i)\ \ and\ \ (x_i,p_i,M_i) \to (x_0,p,M)\}.$$

Definition 2.2. We say that $u \in C\left(\Omega\right)$ is a viscosity subsolution to (2.1) if

$$\overline{F}_h(M,p) \geqslant f(x_0,u(x_0)), \quad \forall\ (p,M) \in \overline{\mathcal{J}}^{2,+}u(x_0), \forall\ x_0 \in \Omega.$$

Similarly, we say that $u \in C\left(\Omega\right)$ is a viscosity supersolution to (2.1) if

$$\overline{F}_h(M,p) \leqslant f(x_0,u(x_0)), \quad \forall\ (p,M) \in \overline{\mathcal{J}}^{2,-}u(x_0), \forall\ x_0 \in \Omega.$$

3. Comparison principle

In this section, we first prove the comparison principle by the perturbation method based on the viscosity solutions theory, and then establish a stability result.

Now we recall the maximum principle for infinity harmonic functions which can be deduced from Harnack's inequality (see for example [3,7]).

Lemma 3.1 Let $u \in C\left(\overline{\Omega}\right)$ satisfy $\Delta_{\infty} u \geqslant 0$ in the viscosity sense. Then

$$\sup_{\Omega} u = \sup_{\partial\Omega} u\,.$$

Moreover, the supremum occurs only on the boundary $\partial\Omega$ unless u is a constant.

Next, we use the double variables method to prove the comparison principle.

Proof of Theorem 1.1. We consider the case for $f(x,t)$ is positive. The case for $f(x,t)$ is similar and we leave it to the reader.

We define

$$u_\varepsilon {:=} u + \varepsilon\left(u - \sup_{\partial\Omega} u\,.\right), \quad \varepsilon > 0.$$

By Lemma 3.1, we have $u \leqslant \sup_{\partial\Omega} u\,.$ and $u_\varepsilon \leqslant u$ in Ω. Since $f(x,t) > 0$ in $\Omega \times \mathbb{R}$ and $A(p) \in C(\mathbb{R}^n,\mathbb{R})$ is homogeneous of degree h, it is easy to verify

$$\Delta_{\infty}^{h} u_\varepsilon + A(Du_\varepsilon) = (1+\varepsilon)^h\left[\Delta_{\infty}^{h} u + A(Du)\right] \geqslant (1+\varepsilon)^h f(x,u) \geqslant (1+\varepsilon)^h f(x,u_\varepsilon) \geqslant f(x,u_\varepsilon),$$

that is, u_ε is a viscosity subsolution to the equation (2.1).

We claim that $u_\varepsilon \leqslant v$ in Ω. We argue by contradiction. Suppose that $u_\varepsilon > v$ somewhere in Ω.

Set

$$M = \sup_{\Omega} (u_\varepsilon - v) = u_\varepsilon(x_0) - v(x_0) > 0. \tag{3.1}$$

Using the arguments in [11], we double the variables

$$w_j(x,y) = u_\varepsilon(x) - v(y) - j|x-y|^4/4,\ (x,y) \in \Omega \times \Omega, j = 1, 2, \cdots.$$

We denote the maximum point of w_j over $\overline{\Omega} \times \overline{\Omega}$ by (x_j,y_j). According to Proposition 3.7 in [11], we have

$$\lim_{j\to\infty} M_j = \lim_{j\to\infty} \left(u_\varepsilon(x_j) - v(y_j) - j\left|x_j - y_j\right|^4/4\right) = M$$

and

$$\lim_{j\to\infty} j\left|x_j - y_j\right|^4/4 = 0.$$

It is obvious $x_j \to x_0,\ y_j \to x_0$ as $j \to \infty$. Since $M > 0 \geqslant \sup_{\partial\Omega} (u_\varepsilon - v)$, there is an open set Ω_0 such that x_0, x_j and $y_j \in \Omega_0 \subseteq \Omega$ for $j \to \infty$.

Set

$$\varphi(x) = j\left|x - y_j\right|^4/4, \quad \phi(y) = - j\left|x_j - y\right|^4/4.$$

It is clear that the functions $u_\varepsilon - \varphi$ and $v - \phi$ have a local maximum at x_j and a local minimum at y_j respectively. We consider the two cases: either $x_j \neq y_j$ or $x_j = y_j$ for $j \to \infty$.

Case 1: If $x_j = y_j$, we have $D\varphi(x_j) = 0$ and $D^2\varphi(x_j) = 0$. Since $A(p) \in C(\mathbb{R}^n)$ is homogeneous of degree h, we get $A(0) = 0$. And since u_ε is a viscosity subsolution, we have

$$f(x_j,\varphi(x_j)) = f(x_j,u_\varepsilon(x_j)) \leqslant 0$$

which is contradictory to $f > 0$ in Ω.

Case 2: If $x_j \neq y_j$, we shall use jets and maximum principle for semi-continuous functions, see [11]. There exist $n \times n$ symmetric matrices X_j and Y_j such that $Y_j - X_j \geqslant 0$ and

$$\left(p_j,X_j\right) \in \overline{\mathcal{J}}^{2,+}u_\varepsilon\left(x_j\right), \quad \left(p_j,Y_j\right) \in \overline{\mathcal{J}}^{2,-}v\left(y_j\right),$$

where $p_j = j\left|x_j - y_j\right|^2\left(x_j - y_j\right)$. Again since $\Delta_\infty^h u_\varepsilon + A(Du_\varepsilon) \geqslant (1+\varepsilon)^h f(x,u_\varepsilon) \geqslant f(x,u_\varepsilon)$ and $\Delta_\infty^h v + A(Dv) \leqslant f(x,v)$ in the viscosity sense, we have

$$\begin{aligned} 0 &\leqslant \left|p_j\right|^{h-3}\langle X_j p_j,p_j\rangle + A\left(p_j\right) - (1+\varepsilon)^h f\left(x_j,u_\varepsilon\left(x_j\right)\right) \\ &\leqslant \left|p_j\right|^{h-3}\langle Y_j p_j,p_j\rangle + A\left(p_j\right) - f\left(y_j,v\left(y_j\right)\right) + f\left(y_j,v\left(y_j\right)\right)(1+\varepsilon)^h f\left(x_j,u_\varepsilon\left(x_j\right)\right) \\ &\leqslant f\left(y_j,v\left(y_j\right)\right) - (1+\varepsilon)^h f\left(x_j,u_\varepsilon\left(x_j\right)\right), \end{aligned}$$

where we have used $Y_j - X_j \geqslant 0$. Letting $j \to \infty$, we get

$$f(x_0,v(x_0)) - (1+\varepsilon)^h f(x_0,u_\varepsilon(x_0)) \geqslant 0. \tag{3.3}$$

Since $f(x,t)$ is non-decreasing in t and $u_\varepsilon(x_0) > v(x_0)$, we have $f(x_0,u_\varepsilon(x_0)) \geqslant f(x_0,v(x_0))$ which contradicts to (3.3). Hence, we have $u_\varepsilon \leqslant v$ in Ω. Letting $\varepsilon \to 0$, we get $u \leqslant v$ in Ω.

Now, we prove the stability of the problem (1.3).

Proof of Theorem 1.3. Set $M := \sup_\Omega u_\infty - \inf_\Omega u_0$. Clearly, we have $\sup_\Omega u_k - \inf_\Omega u_k \leq M$, for every $k = 1, 2,\cdots$. Let Ω_0 be any compact subset of Ω and d:=dist$\left(\Omega_0,\partial\Omega\right)$. We take $R > 0$ such that $4R < d$. Since $\Delta_\infty^h u_k \geqslant 0$ in Ω, we have $\Delta_\infty u_k \geqslant 0$ in Ω. By Lemma 2.9 of [3], we get

$$|u_k(x) - u_k(y)| \leqslant M\frac{|x-y|}{R}, \quad \forall z \in \Omega_0,\, x,y \in B_{R/2}(z).$$

By compactness, we get $\{u_k\}$ is equicontinuous in Ω_0. On taking an exhaustion of Ω by subdomains compactly contained in Ω, we apply the standard method of Cantor diagonalization to extract a subsequence of $\{u_k\}$ that converges uniformly on compact subsets of Ω. For simplicity we will continue to denote such subsequence by $\{u_k\}$. Set

$$u(x) := \lim_{k\to\infty} u_k(x), \quad x \in \Omega.$$

We extend this definition to the closure $\overline{\Omega}$ by defining $u = g$ on $\partial\Omega$. By the assumption, we have $u_0 \leqslant u \leqslant u_\infty$ in $\overline{\Omega}$. This means that $u \in C\left(\overline{\Omega}\right)$.

Next, we show that $\Delta^h_\infty u = f$ in the viscosity sense. Suppose that $\varphi \in C^2\left(\Omega\right)$ and $u - \varphi$ has a local maximum at some $x_0 \in \Omega$, i.e.

$$u(x) - \varphi(x) \leqslant u(x_0) - \varphi(x_0), \quad x \in B_r(x_0) \subseteq \Omega$$

for some $r > 0$. Suppose that x_k is a point of maximum of

$$u_k(x) - \left(\varphi(x) + \frac{\varepsilon}{2}|x - x_0|^2\right), \quad \varepsilon > 0, x \in \overline{B}_r(x_0).$$

Particularly,

$$u_k(x_k) - \left(\varphi(x_k) + \frac{\varepsilon}{2}|x_k - x_0|^2\right) \geqslant u_k(x_0) - \varphi(x_0).$$

Since $x_k \in \overline{B}_r(x_0)$, by passing to a subsequence, $x_k \to \hat{x}$, for some $\hat{x} \in \overline{B}_r(x_0)$. Letting $k \to \infty$ in (3.4), we have

$$u(\hat{x}) - \left(\varphi(\hat{x}) + \frac{\varepsilon}{2}|\hat{x} - x_0|^2\right) \geqslant u(x_0) - \varphi(x_0),$$

i.e.

$$\frac{\varepsilon}{2}|\hat{x} - x_0|^2 \leqslant u(\hat{x}) - \varphi(\hat{x}) - (u(x_0) - \varphi(x_0)) \leqslant 0.$$

Then we have $\hat{x} = x_0$ Thus, $x_k \in B_{r/2}(x_0)$ for sufficiently large k. Since u_k is a viscosity subsolution and x_k is a point of local maximum of $u_k(x) - \left(\varphi(x) + \frac{\varepsilon}{2}|x - x_0|^2\right)$ in $B_r(x_0)$, we have

$$\Delta^h_\infty \varphi(x_k) + O(\varepsilon) \geqslant f_k(x_k). \tag{3.5}$$

Taking the limit in (3.5) and recalling that $f_k \to f$ locally uniformly in Ω, we find that

$$\Delta^h_\infty \varphi(x_0) + O(\varepsilon) \geqslant f(x_0).$$

Letting $\varepsilon \to 0$, we have $\Delta^h_\infty u \geqslant f$ in the viscosity sense. Similarly, we can prove that u is a viscosity supersolution.

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