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A synthetic analysis of a recently published book that presents a NUMERICAL method of solving 2nd order Elliptic Partial Differential Equations leading towards ANALYTICAL solutions

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Abstract

The author has published recently (2021) a book having the title [1] **Numerical Method to select an Analytical Polynomial Solution for Linear or Nonlinear Elliptic Partial Differential Equations of second order**

The book can be *freely* downloaded from the **blumenfeld.ro** website. It was written during the isolation period that characterized the COVID pandemic. Because this this circumstance, it was difficult to verify - through discussions with other users interested in PDE integration - all the new ideas and hypotheses introduced by the book. Following the lack of such a collegial communication, some incomplete explanations were observed by the author after the publication of the book. This article aims to facilitate the understanding of the new method, through more detailed explanations of such unclear wording.

1. Linear and nonlinear Partial Differential Equations (PDEs) with constant coefficients

The article aims to facilitate the understanding of a new method to integrate linear and nonlinear elliptic Partial Differential Equations (PDE) that is totally different from the many methods dedicated to this theme. This method is the subject of the recently published book [1], which can be downloaded free of charge from the ***blumenfeld.ro*** website.

It presents the solving of the two-dimensional linear and nonlinear PDEs that include 7 terms: six terms that depend on an unknown function $\phi(x,y)$ (three derivatives of order 2, two derivatives of order 1 and one term depending on $\phi(x,y)$) and a known free term W dependent only on the coordinates x and y.

a. Linear Elliptic PDE with constant coefficients

$$PDE = a\frac{\partial^2\phi}{\partial x^2} + b\frac{\partial^2\phi}{\partial x\partial y} + c\frac{\partial^2\phi}{\partial y^2} + M\frac{\partial\phi}{\partial x} + N\frac{\partial\phi}{\partial y} + P\phi + W(x,y) = 0 \qquad (1.1)$$

In order not to complicate the approach all the coefficients (*a,b,c,M,N,P*) multiplying $\phi(x,y)$ or its derivatives are were considered as constants, while the last term *W(x,y)* is a two-dimensional known function that is supposed different from zero. The unknown solution that has to be found is $\phi = \phi(x,y)$.

Remark. The *PDE* (1.1) is elliptic if $b^2\text{-}4ac < 0$ (1.2)

b.. Nonlinear Elliptic PDE

A more difficult case is represented by the nonlinear elliptic *PDE*

$$\begin{gathered} PDE = (a + a_N\cdot\phi)\frac{\partial^2\phi}{\partial x^2} + (b + b_N\cdot\phi)\frac{\partial^2\phi}{\partial x\partial y} + (c + c_N\cdot\phi)\frac{\partial^2\phi}{\partial y^2} + \\ M\frac{\partial\phi}{\partial x} + N\frac{\partial\phi}{\partial y} + (P + P_N\cdot\phi)\phi + W(x,y) = 0 \end{gathered} \qquad (1.3)$$

where, similar to the *PDE* (1.1), all the coefficients (a,a_N, b,b_N....) are constants.

Developing one of the parenthesis – for instance the first - one obtains

$$T_1 = (a + a_N\cdot\phi)\frac{\partial^2\phi}{\partial x^2} = a\cdot\frac{\partial^2\phi}{\partial x^2} + a_N\cdot\phi\cdot\frac{\partial^2\phi}{\partial x^2} \qquad (1.4)$$

Though *a* and a_N are constants, the last term of this relation, which contains the product $\phi\,.\,\frac{\partial^2\phi}{\partial x^2}$, makes **nonlinear** the second term of (1.4) and therefore the *PDE (1.3)*.

The *PDE*s are integrated on a domain Ω represented by a *rectangular area* having the dimensions **B** (base) and **H** (Height).

2. **The integration method could be ANALYTICAL or NUMERICAL**

The integration of a PDE can be performed using different methods. Any method used can be considered as ANALYTICAL or NUMERICAL. An analytical method looks for the solution in the form of a function $\phi = \phi(x, y)$. The solution must satisfy on the one hand the *PDE* (1.1) or (1.3) from which the integration was started and on the other hand the boundary conditions *imposed by the user on the contour of the integration domain*. If an analytical solution was found, the value of the function can be determined at any point belonging to the domain Ω, by replacing its coordinates in $\phi = \phi(x, y)$.

Over time, it has been found that obtaining an analytical solution is very difficult to obtain for many *PDE*s, especially for the nonlinear ones. That is why various NUMERICAL methods have been developed in parallel. Among the numerous NUMERICAL methods, a great expansion has known the *Finite Element Method* (FEM), starting from the second half of the 20th century. FEM was initially based on some simplifying hypotheses or procedures, among which we mention:

a. It was assumed that the solution of the equation is a polynomial, admitting - when the method was put forward - that the polynomial has a degree *Dg = 1* or *Dg = 2*.

b. Due to the low degree of *Dg* used by *FEM*, the results obtained were approximate, the errors being very large. In order to reduce the errors, the domain of integration Ω was divided into subdomains, represented by elements (rectangles or triangles) of small or very small dimensions. The use of small elements has led to improved results, although the low degree of the polynomial has been maintained. In order to obtain the best possible results, the number of elements gradually increased. Towards the end of the 20th century, hundreds of thousands or even millions of elements were used to integrate a *PDE*. However, the advances in computer HARDWARE and SOFTWARE have made it possible to obtain the results in a reasonable time.

c. The integration results using FEM are not obtained as a function, as in the case of the analytical method, but as a great - if not huge - number (sometimes millions) of discrete values. To facilitate the understanding of the results, they are usually presented in the form of graphs.

3. Assumptions and procedures used by the NUMERICAL method described in [1]

A numerical method different from other methods that preceded it has been proposed in two books recently published by author. It started initially [2, published in 2015] using the assumptions and procedures introduced by *FEM*, but later on acquired a different development [1, published in 2021]. The main differences from *FEM* are outlined below.

3.1 New hypothesis: The solution is a complete polynomial of very high degree called the Concordant Function

Assuming that the solution obtained by the integration of a PDE is a polynomial, its degree - instead of Dg = 1 or 2 as considered by FEM - will have values **greater or much greater than *Dg=6***. Unlike FEM which admits that the polynomial may have one or more missing terms, the new hypothesis assumes that the polynomial is COMPLETE, ie it includes all the terms that correspond to the degree declared. For instance, a complete polynomial of degree *Dg = 3* - denoted *CF3* - has 10 terms, namely

$$CF3 = C_1 \quad + C_2 x + C_3 y \quad + C_4 x^2 + C_5 xy + C_6 y^2 \quad + C_7 x^3 + C_8 x^2 y + C_9 xy^2 + C_{10} y^3 \tag{3.1}$$

Except for C_7, C_8, C_9 and C_{10} that multiply the terms of the degree *Dg = 3*, the re[ation (2.1) includes ALL the terms corresponding to the lower degrees, ie: C_1 for *Dg = 0*, C_2 and C_3 for *Dg = 1*, C_4,C_5 and C_6 for *Dg = 2*. Such complete polynomial is called *CONCORDANT FUNCTION* and will be noted further as *CF*.

The function *CF3* (3.1) - having the degree *Dg = 3* - can be written in following matrix form

$$\phi_3 = \phi_3(x, y) = CF3 = \left[X^{(0)}Y^{(0)}\right]_3 * [Cz]_3 \tag{3.2}$$

where, for a complete polynomial of third degree

$$\left[X^{(0)}Y^{(0)}\right]_3 = \left[\, 1 \quad x \;\; y \quad x^2 \;\; xy \;\; y^2 \quad x^3 \;\; x^2 y \;\; xy^2 \;\; y^3 \,\right] \tag{3.3}$$

$$[Cz]_3 = [\, C_1 \quad C_2 \;\; C_3 \quad C_4 \;\; C_5 \;\; C_6 \quad C_7 \;\; C_8 \;\; C_9 \;\; C_{10} \,]^{TR} \tag{3.4}$$

This notation allows a very simple writing of the high or very high *CF* function. For instance, if the degree increases from *Dg = 3* to *Dg = 4*, the relation (3.2) is modified by changing only the index 3 by 4

$$\phi_4 = \phi_4(x,y) = CF4 = \left[X^{(0)}Y^{(0)}\right]_4 * [Cz]_4 \tag{3.5}$$

but, where $\left[X^{(0)}Y^{(0)}\right]_4 = \left[1 \;\; x \;\; y \;\; x^2 \;\; xy \;\; y^2 \;\; x^3 \;\; x^2y \;\; xy^2 \;\; y^3 \quad x^4 \;\; x^3y \;\; x^2y^2 \;\; xy^3 \;\; y^4\right]$ (3.6)

$$[Cz]_4 = [\,C_1 \;\; C_2 \;\; C_3 \;\; C_4 \;\; C_5 \;\; C_6 \;\; C_7 \;\; C_8 \;\; C_9 \;\; C_{10} \quad C_{11} \;\; C_{12} \;\; C_{13} \;\; C_{14} \;\; C_{15}\,]^{TR} \tag{3.7}$$

The *oncordant Function CF4* includes 15 terms. It is obvious that both for $\left[X^{(0)}Y^{(0)}\right]_4$ and also for $[Cz]_4$ the first 10 terms are identical with $\left[X^{(0)}Y^{(0)}\right]_3$ and $[Cz]_3$. This simplifies the transition from degree *Dg = 3* to *Dg = 4* and - extending the approach - from degree *n* (*Dg, n*) to *n + 1* (*Dg, n + 1*).

Since in both *PDE*s (1.1) or (1.3) are involved the partial derivatives of first and second order, they can be expressed using the same form of notation. The first derivatives of $\phi_3 = \phi_3(x,y)$ can be written as

$$[\partial\phi_3/\partial x] = \partial\left[\,\left[X^{(0)}Y^{(0)}\right]_3 * [Cz]_3\,\right]/\partial x = \partial\left[X^{(0)}Y^{(0)}\right]_3/\partial x * [Cz]_3 = \left[X^{(1)}Y^{(0)}\right]_3 * [Cz]_3 \tag{3.8}$$

$$[\partial\phi_3/\partial y] = \partial\left[\,\left[X^{(0)}Y^{(0)}\right]_3 * [Cz]_3\,\right]/\partial y = \partial\left[X^{(0)}Y^{(0)}\right]_3/\partial y * [Cz_3] = \left[X^{(0)}Y^{(1)}\right]_3 * [Cz_3] \tag{3.9}$$

where $\left[X^{(1)}Y^{(0)}\right]_3 = \partial\left[X^{(0)}Y^{(0)}\right]_3/\partial x = \left[0 \quad 1 \;\; 0 \quad 2x \;\; y \;\; 0 \quad 3x^2 \;\; 2xy \;\; y^2 \;\; 0\right]$ (3.10)

$$\left[X^{(0)}Y^{(1)}\right]_3 = \partial\left[X^{(0)}Y^{(0)}\right]_3/\partial y = \left[0 \quad 0 \;\; 1 \quad 0 \;\; x \;\; 2y \quad 0 \;\; x^2 \;\; 2xy \;\; 3y^2\right] \tag{3.11}$$

For the relation (3.2), dedicated to a function, it was used the matrix $\left[X^{(0)}Y^{(0)}\right]$. Both exponents (0) indicate that the function CF *is not derived*. In relation (3.8) the notation $\left[X^{(1)}Y^{(0)}\right]$ shows that the CF function is derived only once in relation to the *x* coordinate. The notation $\left[X^{(0)}Y^{(1)}\right]$ used in (3.9) indicates also that the *CF* function is derived only once, but this time in relation to *y*.

An exponent *(2)* , would show a double derivative. The second partial derivative of ϕ_3 versus *x* will be expressed as

$$\left[\partial^2\phi_3/\partial x^2\right] = \partial^2\left[\,\left[X^{(0)}Y^{(0)}\right]*[Cz]\,\right]/\partial x^2 = \left[\partial^2\left[X^{(0)}Y^{(0)}\right]/\partial x^2\,\right]*[Cz] = \left[X^{(2)}Y^{(0)}\right]*[Cz] \tag{3.12}$$

The complete polynomials used in [1] for integration of a *PDE*s include the following number of terms: *Dg* = 6 →28 terms, *Dg* = 7 →36 terms, *Dg* = 8→45 terms, *Dg* = 9 → 55 terms and *Dg* = 10 → 66 terms. The degree of the *CF*s is limited only by the increase in the number of terms, which require modifications of the program. In [1] only 5 polynomials were worked on, because the programming was performed by the author, which has an obviously limited work capacity.

3.2 The integration is performed using ONE ELEMENT

In the book [2] the author observed that using polynomials of much higher degree than those employed byFEM, the number of elements needed to obtain results with similar accuracy decreases significantly. He merely pointed this out, without using all the advantages of this observation. Only later on did he find that by employing high or very high degree polynomials and **using a single element** (cell) one can obtain integration results that are comparable to those obtained by FEM with hundred of thousands of elements. This is the main novelty introduced by [1].

This new approach has the immediate consequence of substantially reducing the time spent for the integration of a PDE. Instead of solving tens or hundreds of thousands of equations as does FEM, it is sufficient to solve only **28 equations** (if $Dg = 6$) up to **66 equations** (if $Dg = 10$), as in the solutions performed in [1].

This approach leads to a great reduction of the integration time, especially in the case of nonlinear PDEs. For nonlinear equations, the iterative methodology usually used, requires a multiple repetition of solving the system of equations. This is quickly done when the computation concerns a single cell because the number of equations allways is less then 66.

3.3 The integration is initially done using several polynomials with increasing degree from which will be selected ONE assumed as solution

The hypothesis made that the solution is a polynomial was admitted in 3.1, but until now it's degree was not specified. Tests of single-cell solutions have shown that the errors depend on the degree of the polynomial used as a *Concordant Function*. This requires a change in the modality *PDE* is resolved. The starting procedure will be based on the integration performed using **several ascending degree polynomials**. Their number is not limited. Ascending degree has as consequence the increasing the number of terms for each change and requires each time modification of the program. In [1] only 5 polynomials were used, because the programming was done solely by the author.

When starting the calculation it is not known which polynomial should be used to get the best results. As a consequence, in [1] the calculation must be performed initially with ALL the five now available polynomials, which lead to 5 different values of the assumed errors. The choice of the degree of the polynomial that will be considered as solution is made by selecting the one of them who led to the best results.

Remark. By using a multiprocessor computer, the time required for the initial integration will be practically the same no matter the mumber of *CF*s tested, as fhe computatation can be performed simultaneously for all the polynomials used.

3.4 The procedure to obtain the coefficients of each tested polynomial.

Suppose that a 6^{th} degree CF6 is considered as the solution of a PDE. This polynomial includes 28 terms that have to be found. There are considered two sources from which the necessary information to obtain its coefficients comes:

1. The PDE itself that is a differential relation, which can be (1.1) or (1.3).

2. The boundary conditions imposed on the contour of the integration domain. These conditions are in no way dependent on the *PDE*, yet must be met exactly because they are imposed by the user.

How use the two sources of information is an important decision of the method. Based on it, a system of 28 equations will be established for CF6, whose solution will become the 28 terms of the polynomial. This decision must determine how many equations are established based on each source of information.

Source 1 (PDE). The first equation is represented by the *PDE* itself. Other 5 equations can be obtained by deriving *PDE*. In this way the information used penetrates deeper into the structure of the differential equation. It has been used on this purpose the first and second order derivatives of *PDE*. *Two* equations can be obtained from the first order derivatives [$(\partial(PDE)/\partial x, \partial(PDE)/\partial y$] and *three* equations from the second order derivatives [$\partial^2(PDE)/\partial x^2, \partial^2(PDE)/\partial x\partial y, \partial^2(PDE)/\partial y^2$] . They are concretized, connecting them to a central point of the rectangular domain Ω (B × H), which will be called *Target Point*. This point was chosen by the author at the intersection of the two diagonals of the rectangle, having therefore the coordinates x = B / 2, y = H / 2. In this way are established a total of 6 equations based on *Source 1*.

Source 2 (The boundary conditions). Of the 28 required equations 6 were chosen

above, so that 22 more remain to be established. It is assumed that the sides of the rectangle are parallel to the axes. Thus along any side the *two-dimensional* polynomials become *one-*

dimensional, therefore for the degree $Dg=6$ each will have *7* terms. The corners belonging simultaneously to two different polynomials, the number of terms is reduced to 6, namely

in total *4 sides* × *6 terms* = *24 terms*. Since the number of equations to be established is only 22, two boundary conditions are supernumerary and are arbitrarily removed.

Details on the composition of the system of equations for all 5 polynomials used can be found in the Chapter 2 from [1].

3.5 Using a NUMERICAL method one obtains as result a FUNCTION therefore an ANALYTICAL solution

The integration of an elliptical *PDE* using ONE ELEMENT has a surprising consequence concerning a fundamental change in the way results are obtained.

As shown above, the *FEM* method provides results represented by tens or hundreds of thousands of discrete values. On the contrary, for the method developed in [1] the result of the integration of a PDE is *one selected polynomial*. For example, suppose that the solution obtained by the integration procedure results as a polynomial of 10^{th} degree. This polynomial has 66 coefficients that multiply the x and y coordinates at different powers, therefore a result with ANALYTICAL SOLUTION properties was obtained. This leads to an unusual outcome: an ANALYTICAL solution is obtained using a NUMERICAL method.

4. Evaluation of the accuracy of the results by calculating the RESIDUAL

4.1 The *Residual*

When using a numerical method, the evaluation of the accuracy of the results is also done by numerical methods [5,6]. Such assessments are also used in the case of FEM. As previously seen, the results obtained by the method developed in [1] acquire an analytical character, which modifies the way of approaching the problem of precision evaluation.

For instance, the solution of the second degree equation $E_1(x) = x^2 - 4x + 4 = 0$, obtained with an analytitical procedure, results as $x = 2$. Checking the accuracy of a result obtained with an analytical method can be done by the well-known method of *replacing the result in the given equation*, which leads to a value that will be denoted as R

$$R = E_1(x=2) = 2^2 - 4\times 2 + 4 = 0.$$

The value $R = 0$ allows the user to conclude: "The result is accurate" [1, *Appendix A*].

If the equation to be solved is, $E_2(x) = x^2-4 = 0$ the positive result can be, expressed as $x = \sqrt{2}$. The form used in the practical activity for this result will depend on the number of digits displayed by the computer. Assuming that 4 or 7 or 10 digits are displayed, the value of the solution *x* may be written differently, leading to distinct values for *R,* namely

a. If 4 digits are used → z = 1.414, so $R_{2a}= E_2(z=1,414)= -9.59\times10^{-6}$;

b. If 7 digits are used → z = 1.414213, so $R_{2b}= E_2(z=1.4142135)= -4.41\times10^{-8}$;

c. If 10 digits are used → z = 1.414213562, so $R_{2b}= E_2(z=1.414213562)= -2.63\times10^{-10}$;

As it can be seen, all the values obtained for R are *non-zero*. Therefore *R* will be considered as a *RESIDUAL*. Because all the residuals written above have very small values, all three displayed solutions *x* can be estimated as being accurate. [1, Appendix A].

Due to analytical character of the solution obtained by the method described here, the "replacing method" can be applied also to evaluate the accuracy of the results in case of the integration of a *PDE*. After obtaining the function-solution, the *Residual* will be calculated in the *TARGET POINT* ($xT = B / 2, yT = H / 2$). For this purpose, the values of the function and of all derivatives calculated at this point are replaced in *PDE (1.1)* or *(1.3)*. Since *CF* has a high degree - for instance $Dg = 10$ - the derivatives of first and second order can be considered as having errors close to those of the function. This calculation must be performed for all 5 *CF*s initially used, thus resulting 5 values for *R*. The *Concordant Function* to be selected is the one that corresponds to the *smallest value of R*.

4.2 *Relative Residuals* and the three possibilities to calculate it

The calculation of the *Residual* in a single point (T) is not sufficient for the correct assessment of the result of a *PDE* integration on the whole domain Ω. In view of this observation, in [1] are indicated more complex possibilities for calculating the *Residual*.

Here will not be given details, but only references to [1] that allow deepening the subject. On purpose to eliminate the influence on the *Residual* of the free term $W=W(x,y)$, a relative value - called *Relative Residual* - is calculated according to [1, see(4.12)]. Then [1] defines three ways to calculate the *Relative Residuals*: 1. In a *Single point,* according to [1, see(4.15)], 2. In *Many points,* according to [1, see(4.17)] or 3. *Integral,* according to [1, see(4.19)]. However, their role is the same: to allow the selection of the degree of the polynomial that will be considered as a solution.

Remark. The value of the *Residual* has been used in [1, Chapter 6] for the proposal of an engineering method, inspired by the concept of the „allowable stress" used in the *Strength of materials*. It is intended to facilitate the decision of the user *whether a found solution can be accepted or has to be rejected*. On this purpose the *Residual* is compared with a conventional value called as $(Res)_{admissible}$.

5. An example concerning the integration of LINEAR elliptic PDE

5.1 The data that must be furnished by the user

Provided data. As example is taken the following *PDE*

$$PDE = 2\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial x\partial y} + 3\frac{\partial^2\phi}{\partial y^2} + \frac{\partial\phi}{\partial x} - 0.3\frac{\partial\phi}{\partial y} + 0.25\phi + 4 + 2x - y + x^2 + 3xy - 2y^2 + 2 = 0$$

The user has to provide the following information :

1. *The 6 coefficients* of the *PDE* (1.1)

a=2 ; b=1 ; c=3 ; M=1 ; N= -0.3 ; P=0.25

2. *The free term W(x,y)* - here a *second degree* polynomial - which is defined by the following coefficients noted as W_i (here $i=1...6$)

$W_1=4$; $W_2=2$; $W_3= -1$; $W_4=1$; $W_5=3$; $W_6= -2$;

3. *The dimensions B and H of the integration rectangular area.*

B=H=1

4. *The boundary conditions $F_z(x,y)$*, which must be given by the user , are of *Dirichlet* type. The main values of F_z are those corresponding *to the four corners* of the integration domain

$F_{z1}=F_z(x=0, y=0)=2$; $F_{z2}=F_z(x=1, y=0)=4$;$F_{z3}=F_z(x=0, y=1)=5$; $F_{z4}=F_z(x=1, y=H)=4$

The variation of the function F_z between two adjacent corners is LINEAR.

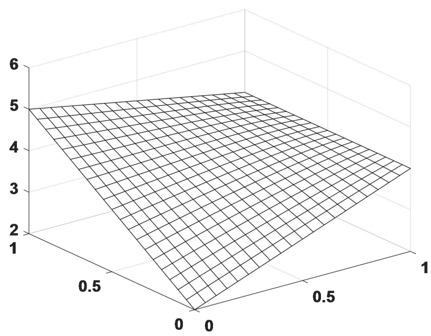
5.2 Results

As previously shawn, the initial computation is done – using a computer without multiprocessor – by supposing that the solution is, successively, one of the five polynomials having *Dg=6,7,8,9,10*. At the end of each computation is can be calculated the *Relative Residual.* The absolute values of Res_{RelI} (4.18) for all *CF*s used are given in *Table 1*

Table 1

CFs	CF6	CF7	CF8	CF9	CF10
$\lvert Res_{RelI} \rvert$	7.4425×10^{-2}	7.2037×10^{-2}	4.2877×10^{-2}	4.2330×10^{-2}	**2.8178×10^{-2}**

The minimum value of Res_{RelI} corresponds to *CF10*. The transposed coefficients *[Cz10]* were used – employing the matrix form – for drawing the graph from Fig,5.1. More examples of linear PDEs are given in [1] (Chapter?)



Fig,5.1 The drawing of the solution based on *CF10*

6. NONLINEAR elliptic PDEs

6.1 The iteration procedure

The strategy for solving a system of nonlinear equations is not unique. We will describe a strategy employing an **iteration procedure** using, as aid, the solution based on *CF6* whose number of unknown coefficients is 28. They result solving a system of 28 equations based on the *boundary conditons* and on the *PDE and its derivatives*, which is obtained similar to that used above for linear PDEs. From them, the 22 equations depending on the boundary conditons – imposed by the user - are not influenced by the nonlinearity, therefore remain unchanged. Only the last 6 equations – bounded to the *Target Point* – take account on the nonlinearity. Consequently, to determine the coefficients of the CF, it has to be established - for a nonlinear *PDE*s - only the last 6 equations. Below will be presented only the special procedure used to obtain the first equation. Details concerning the other 5 equations may be found in [1].

To explain the methodology for solving nonlinear *PDE*s it will be chosen a simpler type, like (1.3). If the function $\phi(x, y)$ is replaced in (1.3) by a *Concordant Function z(x,y)*, the *PDE* - noted now as PDE_{NonLin} - becomes

$$\begin{aligned} PDE_{NonLin} &= (a + a_n z)\frac{\partial^2 z}{\partial x^2} + (b + b_n z)\frac{\partial^2 z}{\partial x \partial y} + (c + c_n z)\frac{\partial^2 z}{\partial y^2} + \\ &+ M\frac{\partial z}{\partial x} + N\frac{\partial z}{\partial y} + (P + P_n z)z + W(x, y) = 0 \end{aligned} \tag{6.1}$$

To highlight the nonlinearity, let's develop (6.1)

$$\begin{aligned} PDE_{NonLin} &= a\frac{\partial^2 z}{\partial x^2} + a_n z\frac{\partial^2 z}{\partial x^2} + b\frac{\partial^2 z}{\partial x \partial y} + b_n z\frac{\partial^2 z}{\partial x \partial y} + c\frac{\partial^2 z}{\partial y^2} + c_n \phi\frac{\partial^2 z}{\partial y^2} + \\ &+ M\frac{\partial z}{\partial x} + N\frac{\partial z}{\partial y} + P\,z + P_n z * z + W(x, y) = 0 \end{aligned} \tag{6.2}$$

or

$$\begin{aligned} PDE_{NonLin} &= PDE + a_n z\frac{\partial^2 z}{\partial x^2} + b_n z\frac{\partial^2 z}{\partial x \partial y} + c_n z\frac{\partial^2 z}{\partial y^2} + P_n z \times z = \\ &= PDE + z\left(a_n\frac{\partial^2 z}{\partial x^2} + b_n\frac{\partial^2 z}{\partial x \partial y} + c_n z\frac{\partial^2 z}{\partial y^2} + P_n z\right) = 0 \end{aligned} \tag{6.3}$$

If it must be solved a nonlinear equation with one unknown using an iteration procedure, its start value is taken sometimes at random. Here, behind *z(x,y)* are hiding 28 coefficients that have to be chosen - for the first iteration - to get as close to the nonlinear solution as the entry data of the problem allows. This make unusable the selection „at random" for this case. Instead, the 28 coefficients may be „guided" in the good direction by transforming - for starting the iteration procedure - the *nonlinear PDE* into a *linear PDE* and solving it. The coefficients of $[Cz6]_{linear}$ thus obtained represent a linear solution that may be far from the nonlinear one, but at least they respect the boundary conditions. This transformation – nonlinear to linear - can be obtained by annulling the coefficients in the nonlinear PDE (6.3)

$$a_n = b_n = c_n = P_n = 0 \tag{6.4}$$

The solving of the fake PDE thus resulted will be considered the *„iteration {0}"*, which unfolds before starts the actual iterative procedure. It results the solution $(z)_{ITER\{0\}}$, which is computed at the *Target Point*, therefore it will be noted as $((z)_{ITER\{0\}})_T$. To „guide" the choosing of the coefficients obtained for $[Cz6]_{linear}=[Cz6]_{Iter\{0\}}$ towards the nonlinear solution, a first iteration noted as *ITER {1}* will be accomplished based on (6.3) and (6.4). This first iteration will be done by replacing in the nonlinear part (the second term of (6.3)), the function $z(x,y)$ that multiply the paranthesis by $(z)_{ITER\{0\}}$ already computed and known. Conseqently, (6.3) may be written

$$(PDE_{NonLin})_{ITER\{1\}} = PDE + ((z)_{ITER\{0\}})_T \times \left(a_n \frac{\partial^2 z}{\partial x^2} + b_n \frac{\partial^2 z}{\partial x \partial y} + c_n \frac{\partial^2 z}{\partial y^2} + P_n z \right) \tag{6.5}$$

The parenthesis remains dependant on $z(x,y)$, but (6.5) continues to be a linear equation because $(z)_{ITER\{0\}})_T$ is a *number*. By solving (6.5) the PDE_{NonLin} receives some information (possibly quite remote) concerning the nonlinear part. The iteration procedure continues according with a generalisation of (6.5), which for ITER {i} may be written as

$$(PDE_{NonLin})_{ITER\{i\}} = PDE + ((z)_{ITER\{i-1\}})_T \times \left(a_n \frac{\partial^2 z}{\partial x^2} + b_n \frac{\partial^2 z}{\partial x \partial y} + c_n \frac{\partial^2 z}{\partial y^2} + P_n z \right) \tag{6.6}$$

Because $((z)_{ITER\{i-1\}})_T$ is a number, it results the following **linear** equation

$$\begin{aligned} (PDE_{NonLin})_{ITER\{i\}} &= \left(a + a_n ((z)_{ITER\{i-1\}})_T\right) \frac{\partial^2 z}{\partial x^2} + \left(b + b_n ((z)_{ITER\{i-1\}})_T\right) \frac{\partial^2 z}{\partial x \partial y} + \\ &+ \left(c + c_n ((z)_{ITER\{i-1\}})_T\right) \frac{\partial^2 z}{\partial y^2} + M \frac{\partial z}{\partial x} + N \frac{\partial z}{\partial y} + \left(P + P_n ((z)_{ITER\{i-1\}})_T\right) z + W(x,y) = 0 \end{aligned} \tag{6.7}$$

Each iteration is preceded by the calculation of the modified coefficients represented by the parentheses involving $((z)_{ITER\{i-1\}})_T$, after which (6.7) becomes similar to any linear *PDE* (1.1).

The same procedure is applied to the other 5 equations connected with the $(PDE_{NonLin})_{ITER\{i\}}$ that comprise its 2 first order and 3 second order derivatives. The 6 equations thus obtained are added to the boundary conditions, leading to an equation system with a small number of unknowns that can be very quickly solved. The iteration procedure has to be repeated until the convergence meets the condition specified in [1].

The strategy to reduce at each step the nonlinear problem to a linear one allows a quick solution of a nonlinear PDE. Consequently, the complete duration of the iteration procedure is drastically reduced.

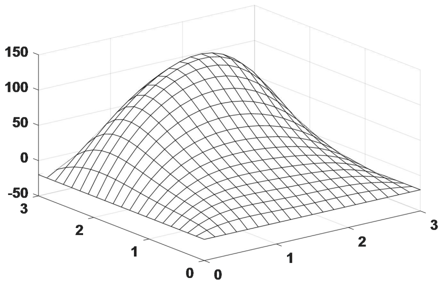
Remark. The iteration procedure described above was used firstly in the article [4]. The article was received with great interest by a large number of specialists. As a result, the author was invited to publish other articles in different journals or to participate in the elaboration of a part of some OPEN BOOKS. So far he has not been able to attend these invitations.

6.2 Example of solving a nonlinear *PDE*

The following example shows how effective the method used is. A nonlinear *PDE* (Poisson type) is the following

$$PDE = (4+0.5\phi)\frac{\partial^2\phi}{\partial x^2} + (3+0.4\phi)\frac{\partial^2\phi}{\partial y^2} \quad +16+8x-4y+4x^2+12xy-8y^2+$$
$$+8x^3+4x^2y-12xy^2+16y^3+4x^4-8x^3y+12x^2y^2+20xy^3+8y^4=0$$

The imposed boundary conditions have equal values (F_Z= -20) all along the the integration contour. The nonlinear *PDE* has to be solved for *B=H=3*.



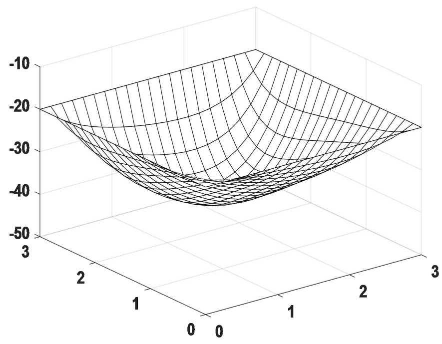


Fig. 6.1 *Iteration{0}* ← (***CF10***) → **Fig. 6.2 Nonlinear convergent solution**

Using *CF10*, the graph of the *Iteration {0}* is given in Fig.6.1, while the graph of the nonlinear convergent solution is represented in Fig.6.2. The example has been chosen because the appearance changes from a “hill” with positive values of the function *z(x,y)* at *Iteration {0}*, to a “pit” with negative values *z(x,y)* after the complete iteration procedure. This modification appears after 38 iterations, which lasts (on an out of date computer) in 0.483 seconds. The rapidity of the solution shows the efficiency of the method.

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