SCIREA Journal of Mathematics
http://www.scirea.org/journal/Mathematics
July 31, 2022
Volume 7, Issue 4, August 2022
https://doi.org/10.54647/mathematics11340

Minimax Estimator on Binomial Distribution

Zul Amry and Sisti Nadia Amalia
Department of Mathematics, State University of Medan, Indonesia
Email: zul.amry@gmail.com and sistinadia@unimed.ac.id
Corresponding author: zul.amry@gmail.com

Abstract
This paper discusses the minimax estimator of parameter for binomial distribution. The likelihood function is constructed based on the probability function of the Binomial distribution. The posterior distribution is obtained from the joint of the likelihood function and prior distribution. Furthermore, the Bayes estimator is obtained based on the posterior mean and provide the constancy of the risk of Bayes the minimax estimator can be concluded.

Keywords: *Bayes theorem, binomial distribution, minimax estimator.*

1. Introduction
The binomial distribution is frequently used to model the number of succes in a sample of size n drawn from a population of size N and its application can be used in the business field. The major problem in discussing the binomial distribution and the other distributions is how to estimate the parameters contained in the distribution. In classical statistics, the population parameters will be estimated viewed as a fixed quantity of the unknown, while in the Bayes statistics the parameters will be estimated as a quantity of a random variable which the variation is described by a probability distribution and knowned as the prior distribution. One

way to determine the estimator in Bayes statistics is to use a risk function. Mathematically, if $X_1,\ X_2, \cdots, X_n$ are random samples from $f(x;\theta)$ with the risk function $R(T;\theta)$, then the main principles of estimation theory is how to determine estimator of θ which minimizes of risk function for each value of θ. The estimator of θ that minimizes the risk is called a Bayes estimator and the Bayes estimator that minimizes the maximum risk is called a minimax estimator.

2. Materials and method

The material in this paper is the binomial distribution and several interrelated theories in mathematics and statistics. The literature study method is used by applying a Bayesian analysis based on previous conjugation assumptions.

Definition 2.1. A random variable X has a binomial distribution with parameter n and p $(n = 1,2,\ldots; 0 < p < 1)$ if X has a discrete distribution whose probability function:

$$f(x|n,p) = \begin{cases} \binom{n}{x} p^x q^{n-x}, x = 1,2,\ldots,n \\ 0, \qquad\qquad otherwise \end{cases} \tag{2.1}$$

where $q = 1 - p$, $E(X) = np$ and $Var(X) = npq$

Definition 2.2. A random variable X has a beta distribution with parameter α and β $(\alpha > 0, \beta > 0)$ if X has an absolutely continuous distribution whose probability density function:

$$f(x|\alpha,\beta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, 0 < x < 1 \\ 0, \qquad\qquad otherwise \end{cases} \tag{2.2}$$

with $E(X) = \frac{\alpha}{\alpha+\beta}$ and $Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Definition 2.3. The Bayes theorem be stated as:

$$p(y|x) \propto p(x|y)p_Y(y) \tag{2.3}$$

where $p(y|x)$ is posterior distribution, $p(x|y)$ is likelihood function and $p_Y(y)$ is prior distribution.

Definition 2.4. If T is a Bayes estimator with constant risk $R(T;\theta) = c$ then T is a minimax estimator

3. Results

The likelihood function of p is

$$L(p) = \prod_{i=1}^{n} f(x_i|n,p)$$

$$= \prod_{i=1}^{n} \binom{n}{x_i} p^{x_i} q^{n-x_i}$$

$$\propto \prod_{i=1}^{n} p^{x_i} q^{n-x_i}$$

$$\propto \prod_{i=1}^{n} p^{x_i}(1-p)^{n-x_i}$$

$$\propto p^{\sum x_i}(1-p)^{n^2-\sum x_i}$$

The conjugate prior of p is beta distribution with the parameters $\alpha > 0$ and $\beta > 0$, that is

$$\pi(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}$$

$$\propto p^{\alpha-1}(1-p)^{\beta-1} \tag{3.1}$$

3.1. Bayes estimator

The Bayes estimator is obtained based on the posterior mean.

Theorem 3.1 The Bayes estimator of p is

$$\hat{p} = \frac{\sum_{i=1}^{n} x_i + \alpha}{n^2+\alpha+\beta} \tag{3.2}$$

Proof. By applying the Bayes theorem, the posterior distribution of p is

$$\Pi(p) \propto L(p) \times \pi(p) \propto$$

$$\propto p^{\sum_{i=1}^{n} x_i}(1-p)^{n^2-\sum_{i=1}^{n} x_i} \times \; p^{\alpha-1}(1-p)^{\beta-1}$$

$$\propto p^{\sum_{i=1}^{n} x_i+\alpha-1}(1-p)^{n^2-\sum_{i=1}^{n} x_i+\beta-1} \tag{3.3}$$

p has a beta distribution with paramater $\sum_{i=1}^{n} x_i + \alpha$ and $n^2 - \sum_{i=1}^{n} x_i + \beta$ and the Bayes estimator for the parameter p equal to the posterior mean of beta distribution, namely

$$\hat{p} = \frac{\sum_{i=1}^{n} x_i+\alpha}{(\sum_{i=1}^{n} x_i+\alpha)+(n^2-\sum_{i=1}^{n} x_i+\beta)}$$

$$= \frac{\sum_{i=1}^{n} x_i+\alpha}{n^2+\alpha+\beta} \; \blacksquare$$

3.2. Minimax estimator

Let the Bayes estimator in equation (3.2) is written in the form $\hat{p} = ay + b$ where $a = (n^2+\alpha+\beta)^{-1}$, $y = \sum_{i=1}^{n} x_i$ and $b = (n^2+\alpha+\beta)^{-1}$. The minimax estimator be concluded by investigated whether the risk $R(\hat{p}, p)$ is constant or not respect to p.

Theorem 3.2. The minimax estimator of p use the weighted squared error loss function is

$$\hat{p} = \frac{\bar{x}}{n} \tag{3.4}$$

Proof. In this case the loss function is $\frac{(p-\hat{p})^2}{p(1-p)}$, so that the risk function is

$$R(\hat{p},p) = E\left(\frac{(p-\hat{p})^2}{p(1-p)}\right)$$

$$= \frac{1}{p(1-p)} E((p-\hat{p})^2)$$

$$= \frac{1}{p(1-p)} E(p-(ay+b)^2)$$

$$= \frac{1}{p(1-p)} E(p-(ay+b)^2)$$

$$= \frac{1}{p(1-p)} E(p-ay-b)^2$$

$$= \frac{1}{p(1-p)} E\big(-a(y-n^2p)+p-b-an^2p\big)^2$$

$$= \frac{1}{p(1-p)} E\begin{pmatrix}\big(-a(y-n^2p)\big)^2+(p-b-an^2p)^2- \\ 2a(y-n^2p)(p-b-an^2p)\end{pmatrix}$$

$$= \frac{1}{p(1-p)} \begin{pmatrix}E\big(a^2(y-n^2p)^2\big)+E(p-b-an^2p)^2- \\ E\Big(2a(y-n^2p)(p-b-an^2p)\Big)\end{pmatrix}$$

$$= \frac{1}{p(1-p)} \begin{pmatrix}a^2E(y-n^2p)^2+E(p-b-an^2p)^2- \\ 2a(p-b-an^2p)E(y-n^2p)\end{pmatrix}$$

Since $E(y)=n^2p$, $E(y-n^2p)=0$ and $Var(y)=n^2p(1-p)$, then be obtained

$$R(\hat{p},p) = \frac{1}{p(1-p)}\left(a^2Var(y)+(p-b-an^2p)^2\right)$$

$$= \frac{1}{p(1-p)}\left(a^2n^2p(1-p)+(p-b-an^2p)^2\right)$$

$$= a^2n^2+\frac{(p-b-an^2p)^2}{p(1-p)}$$

$$= a^2n^2+\frac{\big(p(1-an^2)-b\big)^2}{p(1-p)}$$

$$= a^2n^2+\frac{p^2(1-an^2)^2}{p(1-p)}+\frac{b^2}{p(1-p)}-\frac{2pb(1-an^2)}{p(1-p)}$$

$$= a^2n^2+\frac{p}{1-p}(1-an^2)^2\frac{b^2}{p(1-p)}-\frac{2b}{(1-p)}(1-an^2)$$

In order to $R(\hat{p},p)$ constant of p, the precondition is $1-an^2=0$ and $b^2=0$ which gives the value of $a=\frac{1}{n^2}$ and $b=0$, so that by substituting to $\hat{p}=ay+b$ can be obtained the minimax estimator of p is $\hat{p}=\frac{1}{n^2}y=\frac{1}{n^2}\sum_{i=1}^{n}x_i=\frac{\bar{x}}{n}$ ■

Theorem 3.3 The minimax estimator of p use the squared error loss function is

$$\hat{p}=\frac{(n^2+\alpha+\beta-2)\sum_{i=1}^{n}x_i+\alpha n^2}{n^2(n^2+\alpha+\beta)} \tag{3.5}$$

Proof. $\hat{p}=\frac{(n^2+\alpha+\beta-2)\sum_{i=1}^{n}x_i+\alpha n^2}{n^2(n^2+\alpha+\beta)}$

$$= \frac{(n^2+\alpha+\beta-2)}{n^2(n^2+\alpha+\beta)} \sum_{i=1}^{n} x_i + \frac{\alpha}{(n^2+\alpha+\beta)}$$

$$= ay + b$$

where $a = \frac{(n^2+\alpha+\beta-2)}{n^2(n^2+\alpha+\beta)}$, $y = \sum_{i=1}^{n} x_i$ and $b = \frac{\alpha}{(n^2+\alpha+\beta)}$

The squared error loss function is $(p - \hat{p})^2$, so that the risk function is

$$R(\hat{p}, p) = E(p - \hat{p})^2$$

$$= E\big(p - (ay + b)\big)^2$$

$$= E(p - ay - b)^2$$

$$= E\big(-a(y - n^2p) + p - b - an^2p\ \big)^2 2$$

$$= E\big(-a(y - n^2p) + p - b - an^2p\ \big)^2 \mathrm{b}$$

$$= E\left(\left(-a(y - n^2p)\right)^2 + (p - b - an^2p)^2 - 2a(y - n^2p)(p - b - an^2p)\right)$$

$$= E\big(a^2(y - n^2p)^2\big) + E(p - b - an^2p)^2 - E\left(2a(y - n^2p)(p - b - an^2p)\right)$$

$$= a^2E(y - n^2p)^2 + E(p - b - an^2p)^2 - 2a(p - b - an^2p)E(y - n^2p)$$

$$= a^2Var(y) + E(p - b - an^2p)^2 - 2a(p - b - an^2p)E(y - n^2p)$$

Since $Var(y) = n^2p(1 - p)$ and $E(y - n^2p) = 0$, then be obtained

$$R(\hat{p}, p) = a^2\big(n^2p(1 - p)\big) + (p - b - an^2p)^2$$

$$= a^2n^2p(1 - p) + (p - b - an^2p)^2$$

$$= a^2n^2p - a^2n^2p^2 + \big(p(1 - an^2) - b\big)^2$$

$$= a^2n^2p - a^2n^2p^2 + p^2(1 - an^2)^2 + b^2 - 2pb(1 - an^2)$$

$$= p^2\big((1 - an^2)^2 - a^2n^2\big) + p\left(a^2n^2 - 2b(1 - an^2)\right) + b^2$$

In order to $R(\hat{p}, p)$ constant of p, the precondition is $(1 - an^2)^2 - a^2n^2 = 0$ and $a^2n^2 - 2b(1 - an^2) = 0$ which gives $2b(1 - an^2) = (1 - an^2)^2$ or $a = \frac{1-2b}{n^2}$, so that by substituting to $\hat{p} = ay + b$ can be obtained

$$\hat{p} = \left(\frac{1-2b}{n^2}\right) y + b$$

$$= \left(\frac{1-2(n^2+\alpha+\beta)^{-1}}{n^2}\right) y + \alpha(n^2 + \alpha + \beta)^{-1}$$

$$= \left(\frac{1-.\frac{2}{n^2+\alpha+\beta}}{n^2}\right) y + \frac{\alpha}{n^2+\alpha+\beta}$$

$$= \left(\frac{\frac{n^2+\alpha+\beta-2}{n^2+\alpha+\beta}}{n^2} \right) y + \frac{\alpha}{n^2+\alpha+\beta}$$

$$= \frac{(n^2+\alpha+\beta-2)\sum_{i=1}^{n} x_i}{n^2(n^2+\alpha+\beta)} + \frac{\alpha n^2}{n^2(n^2+\alpha+\beta)}$$

$$= \frac{(n^2+\alpha+\beta-2)}{n^2(n^2+\alpha+\beta)} \sum_{i=1}^{n} x_i + \frac{\alpha n^2}{n^2(n^2+\alpha+\beta)}$$

$$= \frac{(n^2+\alpha+\beta-2)\sum_{i=1}^{n} x_i + \alpha n^2}{n^2(n^2+\alpha+\beta)} \blacksquare$$

4. Conclusion

Based on the discussions that have been made, when the weighted squared error loss function is used, the minimax estimator is $\hat{p} = \frac{\bar{x}}{n}$ but when the squared error loss function is used, the minimax estimator is $\hat{p} = \frac{(n^2+\alpha+\beta-2)\sum_{i=1}^{n} x_i + \alpha n^2}{n^2(n^2+\alpha+\beta)}$

References

[1] Abushal, T. A. (2019). Bayesian Estimation of the Reliabelity Characteristic of Shanker Distribution. *Journal of the Egyptian Mathematical Society,* 1–15.

[2] Ayed, F. et al, (2021). Consistent Estimation of Small Masses in Feature Sampling. *Journal of Machine Learning Research*, 22, 1–28.

[3] Bain, L. J. and Engelhardt, M., (2006). *Introduction to Probability and Mathematical Statistics*, 2nd, Belmont, California: Duxbury Press.

[4] Borisov, A., (2021). Minimax Estimation in Regression under Sample Conformity Constraints. *Mathematics MDPI*, 1–21.

[5] Debnath, M. R. et al, (2021). Minimax Estimation of the Scale Parameters of the Laplace Double Exponential Distribution. *International Journal of Statistical Sciences*, 21(1), 105–116.

[6] Fatima, K., Ahmad, S.P., (2018). Bayesian Approach in Estimation of Shape Parameter of the Exponentiated Moment Exponential Distribution. *Journal of Statistical Theory and Applications,* 17(2), 359–374.

[7] Hasan, M. R., (2019). Minimax Estimation of the Scale Parameter of Laplace Distribution under Modified Linear Exponential (MLINEX) Loss Function. *J. Sci. Res.*, 11 (3), 273–284.

[8] Jeon, Y.E., Kang, S. B., (2020). Bayesian Estimation for the Exponential Distribution Based on Generalized Multiply Type-II Hybrid Cencoring. *Communications for Statistical Applications and Methods*, 27(4), 413–430.

[9] Li, L., (2016). Minimax Estimation of the Parameter of Maxwell Distribution Under Different Loss Functions. *American Journal of Theoretical and Applied Statistics*, 5(4), 202–207.

[10] Okasha, H. M., (2019). E-Bayesian Estimation for the Exponential Model Based on Record Statistics. *Journal of Statistical Theory and Applications*, 18(3), 236–243.

[11] Osu, B. O, Eggege, S. O, Ekpeyong, E. J., (2017). Application of Generalizad Binomial Distribution Model for Option pricing. *American Journal of Applied Mathematics and Statistics*, 5(2), 62–71.

[12] Podder, C. K., (2020). Minimax Estimation of Scale Parameter in a CLass of Life Time Distributions for Different Loss Functions. *International Journal of Statistical Sciences*, 20(2), 85–98.

[13] Rasheed, H. A., Khalifa, Z. N., (2016). Semi-Minimax Estimators of Maxwell Distribution under New Loss Function. *Mathematics and Statistics Journal*, 2(3), 16–22.

[14] Zul Amry, (2020). Bayesian Estimate of Parameters for ARMA Model Forecasting. *Tatra Mountains Mathematical Publication*, Vol.75, 23-32.

[15] Zul Amry, (2021). Bayes Estimator for inverse Gaussian Distribution with Jeffrey's Prior. *SCIREA Journal of Mathematics*, Vol. 6, Issue 4, 44-50.